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STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF COMPLEX UNCERTAIN VARIABLES

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Abstract. This paper introduces the statistical convergence concepts of double sequences of complex uncertain variables: statistical convergence almost surely(a.s.), statistical convergence in measure, statistical convergence in mean, statistical convergence in distribution and statistical convergence uniformly almost surely(u.a.s.).

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1. Introduction

Convergence of sequence as a fundamental theory of mathematics, plays a very important role in probability theory. There are many convergence concepts of random sequences such as convergence in probability, convergence in mean, convergence almost surely, convergence in distribution and so on. The sequences of uncertain variables was introduced and studied by You [22].

In our daily life, we often encounter the case that there are lacking in the data about the events, not only for economic reasons or technical difficulties, but also for influence of unexpected events. Due to non-availability enough data to get probability distribution of events, we have to consult with some domain experts to give belief degree that each event would happen while making decisions. Daniel Kahneman and Amos Tversky showed that human beings usually overweight unlikely events. Liu [9], showed that human beings usually estimate a much wider range of values than the object actually takes. Thus the belief degree may have much larger variance than the real frequency. In this case, if we

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insist on dealing with the belief degree by using probability theory, counterintuitive results may occur. Interested readers may refer to Liu [9] for an example.

In order to deal with belief degree, an uncertainty theory was founded by Liu [8], and refined by him in 2010 which is based on an uncertain measure which satisfies normality, duality, subadditivity and product axioms.

B. Liu introduced the concept of complex uncertain variables. The notion on complex uncertain variables was applied in introducing the different notion sequences of complex uncertain variables by Chen et al. [3]. There after many researcher applied this concept and studied different concepts of the sequences from the point of view of complex uncertain sequences. It was investigated by Datta and Tripathy [4], Nath and Tripathy [10], Roy et al. [11], Saha et al. [12], Tripathy and Dowari [14], Tripathy and Nath [16, 17] and others.

The following notion of uncertain space was introduced by Liu [9].

Definition 1.1. [9] Let L be a σ algebra on a nonempty set Γ. A set function M is called an uncertain measure if it satisfies the following axioms:
Axiom 1(Normality Axiom) $M\{\Gamma\} = 1$;
Axiom 2 (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in L$;
Axiom 3 (Subadditivity Axiom) For every countable sequence of $\{\Lambda_j\} \in L$, we have

$$M\{\cup_{j=1}^{\infty}\Lambda_j\} \leq \Sigma_{j=1}^{\infty}M\{\Lambda_j\}.$$

The triplet (Γ, L, M) is called an uncertainty space, and each element Λ in L is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [9] as follows:

Axiom 4 (Product Axiom) Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1$, 2, 3, . . . The product uncertain measure M is an uncertain measure satisfying

$$M\{\Pi_{k=1}^{\infty}\Lambda_k\} = \wedge_{k=1}^{\infty}M\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, . . .$, respectively.

Definition 1.2. [9] An uncertain variable ξ is a measurable function from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$ is an event.

Definition 1.3. [9] The uncertainty distribution ϕ of an uncertain variable ξ is defined by $\phi(x) = M\{\xi \leq x\}$, for all $x \in R$.

A double sequence x is a double infinite array of numbers $x = \{x_{n,k}\}$. The notion was introduced by A. Pringsheim in the year 1900. Some preliminary works on double sequences is found in Browmich [2]. There after it followed by Hardy [6], who introduced the concept of regular convergence, Basarir and Sonalcan [1], Dutta et al. [5], Tripathy and Sarma [18, 19, 20] , Tripathy and Sen [21] and others. Tripathy and Goswami [15] studied triple sequences probabilistic normed spaces. The double sequence of complex uncertain variables has been studied by Datta and Tripathy [4].

Throughout the article ${}_2\omega(\Lambda)$, ${}_2\ell_\infty(\Lambda)$, ${}_2c(\Lambda)$, ${}_2c_0(\Lambda)$ denote the spaces of all *bounded*, *convergent in Pringsheim's sense*, *null in Pringsheim's sense* double sequences of complex uncertain variables.

2. Preliminaries

In this section we introduce different types of statistical convergence of double sequences in Pringsheim's sense.

Definition 2.1. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *statistically convergent almost surely* (*s.a.s.*) to $\zeta(\gamma)$ in *Pringsheim's sense*, if for every $\varepsilon > 0$, there exists an event Λ with $M\{\Lambda\}$=1, such that

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \le n, j \le k : \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon\}| = 0,$$

for every $\gamma \in \Lambda$, where the vertical bars denote the cardinality of the set embodied by $|.|$.

Definition 2.2. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *statistically convergent in measure* to $\zeta(\gamma)$ in *Pringsheim's sense*, if

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \le n, j \le k : M\left(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right) \ge \delta\}| = 0,$$

for every ε, $\delta > 0$.

Definition 2.3. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *statistically convergent in mean* to $\zeta(\gamma)$ in *Pringsheim's sense*, if

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \le n, j \le k : E[||\zeta_{i,j}(\gamma) - \zeta(\gamma)||] \ge \varepsilon\}| = 0,$$

for every $\varepsilon > 0$.

Definition 2.4. Let ϕ, ϕ_1, ϕ_2... be the complex uncertainty distributions of complex uncertain variables ζ, ζ_1, ζ_2....respectively. We say the complex uncertain double sequence $\{\zeta_{n,k}(\gamma\})$ *statistically converges in distribution* to $\zeta(\gamma)$ in *Pringsheim's sense*, if for every $\varepsilon > 0$,

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \le n, j \le k : ||\phi_{i,j}(c) - \phi(c)|| \ge \varepsilon\}| = 0,$$

for all c at which $\phi(c)$ is continuous.

Definition 2.5. The complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be *statistically convergent u.a.s* to $\zeta(\gamma)$ in *Pringsheim's sense*, if for every $\varepsilon > 0$, There exists $\delta > 0$ and a sequence of events $\{E'_{n,k}\}$, such that

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : ||M(E'_{n,k}) - 0|| \geq \varepsilon\}| = 0.$$

$$\Rightarrow \lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : ||\zeta_{i,j}(\gamma) - \zeta(\gamma)|| \geq \varepsilon\}| = 0.$$

Definition 2.6. A complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is said to be statistically bounded if there exists a real number $A > 0$, such that

$$\delta[\{n, k \in N : ||\zeta_{n,k}(\gamma)|| > A\}] = 0.$$

3. Main Results

In this section, we establish the result of this article.

Theorem 3.1. *If the complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges in mean to $\zeta(\gamma)$, then $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to $\zeta(\gamma)$.*

Proof. It follows from the Markov inequality that for any given $\varepsilon, \delta > 0$, we have,

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : M(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}|$$

$$\leq \lim_{n,k\to\infty} \frac{1}{nk}\left|\left\{i \leq n, j \leq k : \left(\frac{E[||\zeta_{i,j}(\gamma) - \zeta(\gamma)||}{\varepsilon}\right) \geq \delta\right\}\right|.$$

Thus, $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to ζ and the theorem is proved. □

Remark 3.1. The Converse of above theorem is not true in general i.e. statistical convergence in measure does not imply statistical convergence in mean. Following example illustrates this.

Example 3.2. Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, ...$ with

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \dfrac{1}{n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda} \dfrac{1}{n+1} < 0.5; \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1}{n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1}{n+1} < 0.5; \\ 0.5, & \text{otherwise}, \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_{n,k}(\gamma) = \begin{cases} (n+k+1)i, & \text{if } \gamma = \gamma_{n+k}; \\ 0, & \text{otherwise}, \end{cases}$$

for $n = 1, 2, \ . \ . \ .$ and $\zeta \equiv 0$. We have, for $\varepsilon, \delta > 0$ and $n \geq 2$,

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : M(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}| =$$

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : M(\gamma : \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}|$$

$$= \lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : M\{\gamma_{i+j}\} \geq \delta\}| = 0.$$

Thus, the sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to ζ. However, for each $n, k \geq 1$, we have the uncertainty distribution of uncertain variable $\|\zeta_{n,k} - \zeta\|$ is

$$\phi_{n,k}(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{n+k+1}, & \text{if } 0 \leq x < n + k + 1; \\ 1, & \text{if } x \geq n + k + 1, \end{cases}$$

So, for each $n, k \geq 1$, we have,

$$\lim_{n,k\to\infty} \frac{1}{nk}|\{i \leq n, j \leq k : E(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon)|$$

$= \int_0^{n+k+1}[1 - (1 - \frac{1}{n+k+1})]dx = 1.$

That is, the double sequence $\{\zeta_{n,k}(\gamma)\}$ does not statistically converge in mean to $\zeta(\gamma)$.

Theorem 3.3. *Let $\{\zeta_{n,k}(\gamma)\}$ be a double sequence of complex uncertain variables with real part $\{\xi_{n,k}(\gamma)\}$ and imaginary part $\{\eta_{n,k}(\gamma)\}$ respectively, for $n, k \in N$. If the uncertain double sequences $\{\xi_{n,k}(\gamma)\}$ and $\{\eta_{n,k}(\gamma)\}$ statistically converge in measure to ξ and η, respectively, then complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to $\zeta = \xi + i\eta$.*

Proof. It follows from the definition of statistical convergence in measure of uncertain double sequence that for given $\varepsilon > 0$ and $\delta > 0$,

$$\lim_{n,k\to\infty} \frac{1}{nk}\left|\left\{i \leq n, j \leq k : M(\|\xi_{i,j}(\gamma) - \xi(\gamma)\| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\right\}\right| = 0$$

and

$$\lim_{n,k\to\infty} \frac{1}{nk}\left|\left\{i \leq n, j \leq k : M(\|\eta_{i,j}(\gamma) - \eta(\gamma)\| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\right\}\right| = 0.$$

We have, $\|\zeta_{n,k} - \zeta\| = \sqrt{|\xi_{n,k} - \xi|^2 + |\eta_{n,k} - \eta|^2}$. Thus we have,
$\{\|\zeta_{n,k} - \zeta\| \geq \varepsilon\} \subset \{\{|\xi_{n,k} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\} \cup \{|\eta_{n,k} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\}\}.$

Using subadditivity axiom of uncertain measure, we obtain

$$\lim_{n,k\to\infty}\frac{1}{nk}|\{i\le n, j\le k : M(\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\|\ge\varepsilon)\ge\delta\}|$$

$$\le \lim_{n,k\to\infty}\frac{1}{nk}\left|\left\{i\le n, j\le k : M(\|\xi_{i,j}(\gamma)-\xi(\gamma)\|\ge\frac{\varepsilon}{\sqrt{2}})\ge\delta\right\}\right|$$

$$+\lim_{n,k\to\infty}\frac{1}{nk}\left|\left\{i\le n, j\le k : M(\|\eta_{i,j}(\gamma)-\eta(\gamma)\|\ge\frac{\varepsilon}{\sqrt{2}})\ge\delta\right\}\right|=0.$$

This implies,

$$\lim_{n,k\to\infty}\frac{1}{nk}|\{i\le n, j\le k : M(\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\|\ge\varepsilon)\ge\delta\}|\le 0.$$

That is,

$$\lim_{n,k\to\infty}\frac{1}{nk}|\{i\le n, j\le k : M(\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\|\ge\varepsilon)\ge\delta\}|= 0.$$

Hence, $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to $\zeta(\gamma)$. □

Theorem 3.4. *Let the double sequence of complex uncertain variables $\{\zeta_{n,k}(\gamma)\}$ have real part $\{\xi_{n,k}(\gamma)\}$ and imaginary part $\{\eta_{n,k}(\gamma)\}$ respectively, for $n,k\in N$. If the uncertain double sequences $\{\xi_{n,k}(\gamma)\}$ and $\{\eta_{n,k}(\gamma)\}$ statistically converge in measure to ξ and η respectively, then the complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges in distribution to $\zeta=\xi+i\eta$.*

Proof. Let $c=a+ib$ be a given continuity point of the complex uncertainty distribution Φ . On the one hand, for any $\alpha>a,\beta>b$, we have,
$\{\xi_{n,k}\le a,\eta_{n,k}\le b\}=\{\xi_{n,k}\le a,\eta_{n,k}\le b,\xi\le\alpha,\eta\le\beta\}\cup\{\xi_{n,k}\le a,\eta_{n,k}\le b,\xi>\alpha,\eta>\beta\}\cup\{\xi_{n,k}\le a,\eta_{n,k}\le b,\xi\le\alpha,\eta>\beta\}\cup\{\xi_{n,k}\le a,\eta_{n,k}\le b,\xi>\alpha,\eta\le\beta\}\subset\{\xi\le\alpha,\eta\le\beta\}\cup\{\|\xi_{n,k}-\xi\|\ge\alpha-a\}\cup\{\|\eta_{n,k}-\eta\|\ge\beta-b\}$.
It follows from the subadditivity axiom that
$\Phi_{n,k}(c)=\Phi_{n,k}(a+ib)\le\Phi(\alpha+i\beta)+M\{\|\xi_{n,k}-\xi\|\ge\alpha-a\}+M\{\|\eta_{n,k}-\eta\|\ge\beta-b\}$.
Since, $\{\xi_{n,k}(\gamma)\}$ and $\{\eta_{n,k}(\gamma)\}$ are statistically convergent in measure to ξ and η respectively, so for a given $\varepsilon>0$ we have,

$$\lim_{n,k\to\infty}\frac{1}{nk}|\{i\le n, j\le k : M(\|\xi_{i,j}-\xi\|\ge\alpha-a)\ge\varepsilon\}|=0$$

and

$$\lim_{n,k\to\infty}\frac{1}{nk}|\{i\le n, j\le k : M(\|\eta_{i,j}-\eta\|\ge\beta-b)\ge\varepsilon\}|=0.$$

Thus, we obtain $\sup \Phi_{n,k}(c) \leq \Phi(\alpha + i\beta)$ for any $\alpha > a, \beta > b$. On taking $\alpha + i\beta \to a + ib$, we get,

$$\lim_{n,k\to\infty} \sup \Phi_{n,k}(c) \leq \Phi(c). \tag{1}$$

On the other hand, for any $x < a, y < b$ we have,
$\{\xi \leq x, \eta \leq y\} = \{\xi_{n,k} \leq a, \eta_{n,k} \leq b, \xi \leq x, \eta \leq y\} \cup \{\xi_{n,k} \leq a, \eta_{n,k} \leq b, \xi \leq x, \eta \leq y\} \cup \{\xi_{n,k} > a, \eta_{n,k} \leq b, \xi \leq x, \eta \leq y\} \cup \{\xi_{n,k} > a, \eta_{n,k} > b, \xi \leq x, \eta \leq y\} \subset \{\xi_{n,k} \leq a, \eta_{n,k} \leq b\} \cup \{\|\xi_{n,k} - \xi\| \geq a - x\} \cup \{\|\eta_{n,k} - \eta\| \geq b - y\}$.

Which implies,
$\Phi(x + iy) \leq \Phi_{n,k}(a + ib) + M\{\|\xi_{n,k} - \xi\| \geq a - x\} + M\{\|\eta_{n,k} - \eta\| \geq b - y\}$.
Since,

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \leq n, j \leq k : M(\|\xi_{i,j} - \xi\| \geq a - x) \geq \varepsilon\}| = 0$$

and

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \leq n, j \leq k : M(\|\eta_{i,j} - \eta\| \geq b - y) \geq \varepsilon\}| = 0.$$

We obtain, $\Phi(x + iy) \leq \inf \Phi_{n,k}(a + ib)$, for any $x < a, y < b$. On taking $x + iy \to a + ib$, we get,

$$\Phi(c) \leq \lim_{n,k\to\infty} \inf \Phi_{n,k}(c). \tag{2}$$

It follows from (1) and (2) that, $\Phi_{n,k}(c) \to \Phi(c)$, as $n \to \infty$. That is, the complex uncertain double sequence $\{\zeta_{n,k}\}$ is statistically convergent in distribution to $\zeta = \xi + i\eta$.

□

Remark 3.2. The Converse of the above theorem is not necessarily true in general, i.e. statistical convergence in distribution does not imply statistical convergence in measure. Following example illustrates this.

Example 3.5. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}$. Define a complex uncertain variable $\zeta(\gamma)$ by

$$\zeta(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1; \\ -i, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define, $\zeta_{n,k} = -\zeta$ for $n, k \in N$. Then, $\zeta_{n,k}$ and ζ have the same distribution, given as follows:

$$\Phi_{n,k}(c) = \Phi_{n,k}(a + ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty; \\ 0, & \text{if } a \geq 0, b < -1; \\ \frac{1}{2}, & \text{if } a \geq 0, -1 \leq b < 1; \\ 1, & \text{if } a \geq 0, b \geq 1. \end{cases}$$

Then, it can be verified that $\{\zeta_{n,k}\}$ is statistical convergent in distribution to ζ. However, for a given $\varepsilon > 0$, we have,

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \le n, j \le k : M(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon) \ge 1\}|$$

$$= \lim_{n,k\to\infty} \frac{1}{nk} |\{i \le n, j \le k : M(\gamma : \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon) \ge 1\}| = 0.$$

That is, the sequence $\{\zeta_{n,k}(\gamma)\}$ does not statistically converge in measure to ζ. By Theorem 3.3, the real part and imaginary part of $\{\zeta_{n,k}(\gamma)\}$ are not statistically convergent in measure. In addition, since $\zeta_{n,k}(\gamma) = -\zeta$, for $n, k \in N$, the sequence $\{\zeta_{n,k}(\gamma)\}$ does not statistically converge a.s. to ζ.

Remark 3.3. Statistically convergence a.s. does not imply statistically convergence in measure in general.

Example 3.6. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, ...\}$ with

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \dfrac{n+k}{2(n+k)+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda} \dfrac{n+k}{2(n+k)+1} < 0.5; \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n+k}{2(n+k)+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n+k}{2(n+k)+1} < 0.5; \\ 0.5, & \text{otherwise.} \end{cases}$$

Then, define the sequence of complex uncertain variable $\{\zeta_{n,k}(\gamma)\}$ by

$$\zeta_{n,k}(\gamma) = \begin{cases} (n+k+1)i, & \text{if } \gamma = \gamma_{n+k}; \\ 0, & \text{otherwise,} \end{cases}$$

for $n, k \in N$ and $\zeta \equiv 0$. Then the double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges a.s. to ζ. However, for a given $\varepsilon > 0$, we have,

$$\lim_{n,k\to\infty} \frac{1}{nk} \left|\left\{i \le n, j \le k : M(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon) \ge \frac{1}{2}\right\}\right|$$

$$= \lim_{n,k\to\infty} \frac{1}{nk} \left|\left\{i \le n, j \le k : M(\gamma : \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon) \ge \frac{1}{2}\right\}\right|$$

$$= \lim_{n,k\to\infty} \frac{1}{nk} \left|\left\{i \le n, j \le k : M\{\gamma_{i+j}\} \ge \frac{1}{2}\right\}\right| = 0.$$

That is, the double sequence $\{\zeta_{n,k}(\gamma)\}$, does not statistically converge in measure to ζ.

In addition, the complex uncertainty distributions of the complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ are given by

$$\Phi_{n,k}(c) = \Phi_{n,k}(a+ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1 - \frac{n+k}{2(n+k)+1}, & \text{if } a \geq 0, 0 \leq b < n+k; \\ 1, & \text{if } a \geq 0, b \geq n+k, \end{cases}$$

for $n, k \in N$ respectively. The complex uncertain distribution of $\zeta(\gamma)$ is given by

$$\Phi(c) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1, & \text{if } a \geq 0, b \geq 0. \end{cases}$$

Clearly, the double sequence of distribution functions $\{\Phi_{n,k}(c)\}$ does not converge to $\Phi(c)$ when $a \geq 0, b \geq 0$. That is, the double sequence $\{\zeta_{n,k}(\gamma)\}$ does not statistically converges to ζ in distribution.

Remark 3.4. Statistically convergence of double sequences of complex uncertain variables in measure does not imply statistically convergence a.s in general.

Example 3.7. Consider the uncertainty space (Γ, L, M) to be $[0,1]$ with Borel algebra and Lebesgue measure. For any positive integer n, k there is an integer m such that $n = 2^m \pm p, k = 2^m \pm p$ where p is an integer between 0 and $2^m - 1$. Define the double sequence of complex uncertain variables $\{\zeta_{n,k}(\gamma)\}$ by

$$\zeta_{n,k}(\gamma) = \begin{cases} 2i, & \text{if } \frac{p}{2^m} \leq \gamma \leq \frac{p+1}{2^m}; \\ 0, & \text{otherwise}, \end{cases}$$

for $n, k \in N$ and $\zeta \equiv 0$. We have for given $\varepsilon, \delta > 0$ and $n, k \geq 1$,

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \leq n, j \leq k : M(\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}|$$

$$= \lim_{n,k\to\infty} \frac{1}{nk} |\{i \leq n, j \leq k : M(\gamma : \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}|$$

$$= \lim_{n,k\to\infty} \frac{1}{nk} |\{i \leq n, j \leq k : M\{\gamma_{i+j}\} \geq \delta\}| = 0.$$

So, the double sequence of complex uncertain variables $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to ζ. In addition, for every $\varepsilon > 0$, we have,

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \leq n, j \leq k : E[\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\|] \geq \varepsilon\}| = 0.$$

Thus, the double sequence $\{\zeta_{n,k}(\gamma)\}$ also statistically converges in mean to ζ. However, for any $\gamma \in [0,1]$ there is an infinite number of intervals of the form $[\frac{p}{2^m}, \frac{p+1}{2^m}]$ containing γ. Thus, $\{\zeta_{n,k}(\gamma)\}$ does not statistically converge to 0. In other words, the double sequence $\{\zeta_{n,k}(\gamma)\}$ does not statistically converge a.s. to ζ.

Remark 3.5. Statistically convergence of double sequences of complex uncertain sequences a.s. does not imply statistically converge in mean in general.

Example 3.8. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, ...\}$ with

$$M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n}.$$

Define the double sequence of complex uncertain variables $\{\zeta_{n,k}(\gamma)\}$ by

$$\zeta_{n,k}(\gamma) = \begin{cases} i2^{n+k}, & \text{if } \gamma = \gamma_{n+k}; \\ 0, & \text{otherwise}, \end{cases}$$

for $n, k \in N$ and $\zeta \equiv 0$. Then, the double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges a.s. to ζ. However, the uncertainty distributions of the complex uncertain double sequences $\{\zeta_{n,k}(\gamma)\}$ are given by

$$\Phi_{n,k}(\gamma) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{2^{n+k}}, & \text{if } 0 \le x < 2^{n+k}; \\ 1, & \text{if } x \ge 2^{n+k}, \end{cases}$$

for $n, k \in N$. Then we have,

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \le n, j \le k : E[\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\|] \ge 1\}| = 0.$$

So, it can be verified that the double sequence complex uncertain variables $\{\zeta_{n,k}(\gamma)\}$ does not statistically converge in mean to ζ.

Theorem 3.9. *If $\{\zeta_{n,k}(\gamma)\}$ is a double sequence of complex uncertain variables, then the double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges a.s. to ζ if and only if, for any δ, $\varepsilon > 0$, we have,*

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \le n, j \le k : M\left(\bigcap_{n=k=1}^{\infty} \bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon \right) \ge \delta \right\} \right| = 0.$$

Proof. By the definition of statistical convergence a.s., we have, there exists an event Λ with $M\{\Lambda\} = 1$, such that

$$\lim_{n,k\to\infty} \frac{1}{nk} |\{i \le n, j \le k : \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon\}| = 0$$

for every $\varepsilon > 0$. Then, for any $\varepsilon > 0$, there exist n, k such that $\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon$, where $i > n, j > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \leq n, j \leq k : M\left(\bigcap_{n=k=1}^{\infty} \bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| < \varepsilon \right) \geq 1 \right\} \right| = 0.$$

It follows from the duality axiom of uncertain measure that

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \leq n, j \leq k : M\left(\bigcap_{n=k=1}^{\infty} \bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

□

Theorem 3.10. *If $\{\zeta_{n,k}(\gamma)\}$ is a double sequence of complex uncertain variables, then the double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges a.s. to ζ if and only if, for any δ, $\varepsilon > 0$, we have,*

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \leq n, j \leq k : M\left(\bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

Proof. Let the double sequence $\{\zeta_{n,k}(\gamma)\}$ be statistically convergent uniformly a.s. to ζ, then for any $\delta > 0$, there exists B such that $M\{B\} < \delta$ and $\{\zeta_{n,k}(\gamma)\}$ statistically uniformly converges to ζ on $\Gamma - B$. Thus, for any $\varepsilon > 0$, there exists $i, j > 0$ such that $\|\zeta_{n,k} - \zeta\| < 0$, where $n \geq i, k \geq j$ and $\gamma \in \Gamma - B$. That is,

$$\bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \{\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \subset B.$$

It follows from the subadditivity axiom that

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \leq n, j \leq k : M\left(\bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right) \right\} \right| \leq \delta(M\{B\}) < \delta.$$

Then,

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \leq n, j \leq k : M\left(\bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

On the contrary, if

$$\lim_{n,k\to\infty} \frac{1}{nk} \left| \left\{ i \leq n, j \leq k : M\left(\bigcup_{i=n}^{\infty} \bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

We have for a given $\varepsilon > 0$, $\delta > 0$ and $m \geq 2$, there exists $\{m_k\}$ such that

$$\delta\left(M\left(\bigcup_{i=j=m_k}^{\infty}\left\{\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\| \geq \frac{1}{m}\right\}\right)\right) < \frac{\delta}{2^m}.$$

Let,

$$B = \bigcup_{m=1}^{\infty}\bigcup_{i=j=m_k}^{\infty}\left\{\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\| \geq \frac{1}{m}\right\}.$$

Then,

$$\delta(M\{B\}) \leq \sum_{m=1}^{\infty}\delta\left(M\left(\bigcup_{i=j=m_k}^{\infty}\left\{\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\| \geq \frac{1}{m}\right\}\right)\right) \leq \sum_{m=1}^{\infty}\frac{\delta}{2^m}.$$

Furthermore, we have

$$\sup_{\gamma\in\Gamma-B}\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\| < \frac{1}{m},$$

for any $i, j \in N$ and $i, j > m_k$. The theorem is proved.

$\square$

Theorem 3.11. *If the complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ is statistically converges uniformly a.s. to ζ, then $\{\zeta_{n,k}(\gamma)\}$ statistically converges a.s. to ζ.*

Proof. It follows from above theorem, if $\{\zeta_{n,k}(\gamma)\}$ statistically converges uniformly a.s. to ζ, then

$$\lim_{n,k\to\infty}\frac{1}{nk}\left|\left\{i \leq n, j \leq k : M\left(\bigcup_{i=n}^{\infty}\bigcup_{j=k}^{\infty}\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\| \geq \varepsilon\right) \geq \delta\right\}\right| = 0.$$

Since,

$$\begin{aligned}&\delta\left(M\left(\bigcap_{n=k=1}^{\infty}\bigcup_{i=n}^{\infty}\bigcup_{j=k}^{\infty}\{\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\|\} \geq \varepsilon\right)\right)\\ &\leq \delta\left(M\left(\bigcup_{i=n}^{\infty}\bigcup_{j=k}^{\infty}\{\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\| \geq \varepsilon\}\right)\right),\end{aligned}$$

on taking the limit as $n, k \to \infty$ on both side of the above inequality, we obtain

$$\delta\left(M\left(\bigcap_{n=k=1}^{\infty}\bigcup_{i=n}^{\infty}\bigcup_{j=k}^{\infty}\{\|\zeta_{i,j}(\gamma)-\zeta(\gamma)\|\} \geq \varepsilon\right)\right) = 0.$$

By theorem 3.10, the double sequence $\{\zeta_{n,k}(\gamma)\}$ is statistically convergent a.s. to ζ. □

Theorem 3.12. *If a complex uncertain double sequence $\{\zeta_{n,k}(\gamma)\}$ statistically converges uniformly a.s. to ζ, then $\{\zeta_{n,k}(\gamma)\}$ statistically converges in measure to ζ.*

Proof. Let $\{\zeta_{n,k}(\gamma)\}$ be statistically convergent uniformly a.s. to ζ, then from theorem 3.11, we have,

$$\lim_{n,k\to\infty} \frac{1}{nk}\left|\left\{i \le n, j \le k : M\left(\bigcup_{i=n}^{\infty}\bigcup_{j=k}^{\infty} \|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right) \ge \delta\right\}\right| = 0.$$

Since,

$$\delta\left(M\left\{\|\zeta_{n,k}(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right\}\right) \le \left(M\left(\bigcup_{i=n}^{\infty}\bigcup_{j=k}^{\infty}\left\{\|\zeta_{i,j}(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right\}\right)\right).$$

On taking the limit as $n, k \to \infty$, we get, $\{\zeta_{n,k}(\gamma)\}$ is statistically convergent in measure to ζ. □

4. Conclusion

In this article we have introduced the notion of statistically convergent double sequences of complex uncertain variables. We have established the relationship between different types of concepts of convergence such as mean, measure, almost sure, general convergence etc. We have provided and discussed counter examples in this article.

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