J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 283 - 288*
https://doi.org/10.14317/jami.2022.283

ABSOLUTE SUMMABILITY FACTORS FOR CESÀRO AND RIESZ MEANS

MEHMET ALI SARIGÖL

ABSTRACT. In this paper we characterize the sets of summability factors $(|C,0|_k, |R,p_n|_s)$ and $(|R,p_n|_k, |C,0|_s)$, $1 < k \leq s < \infty$, which also extends some known results.

AMS Mathematics Subject Classification : 40C05, 40D25, 40F05, 46A45.
Key words and phrases : Absolute Cesàro and Riesz summability, summability factors, equivalance theorems, matrix transformations.

1. Introduction

One of research areas in the theory of summability is absolute summability factors and comparison of the methods, which plays an important roles in Fourier analysis and Approximation theory, and has been widely examined by many authors up to now. On this topic, Bosanquet and Das [4] therein is an important resource. First we recall related definitions. Let Σx_n be an infinite series with partial sum s_n. By (t_n^α) denote the n-th Cesàro means of order α with $\alpha > -1$ of the sequences (s_n). By Flett's notation (see [5]), the series Σx_n is called summable $|C,\alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| t_n^\alpha - t_{n-1}^\alpha \right|^k < \infty.$$

Also, we note that the method $|C,0|_k$ reduces to

$$\sum_{n=1}^{\infty} n^{k-1} |x_n|^k < \infty.$$

The sequence-to-sequence transformation

$$V_n = \frac{1}{R_n} \sum_{n=0}^{n} r_n s_n. \tag{1.1}$$

Received November 1, 2020. Revised March 7, 2021. Accepted August 4, 2021.

defines the sequence (V_n) of the (R, r_n) Riesz means of the sequence (s_n), generated by the sequence of coefficients (r_n), where (r_n) be a sequence of positive real constants with $R_n = r_0 + r_1 + \cdots + r_n \to \infty$ as $n \to \infty$. The series Σx_n is called summable $|R, r_n|_k$, $k \geq 1$, if (see [14])

$$\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k < \infty.$$

Let X and Y be two summability methods. The set of summability factors (X, Y) is defined by the set of all sequences $\lambda = (\lambda_v)$ such that $\Sigma \lambda_v x_v$ is summable Y, whenever Σx_v is summable X. The set (X, Y) was studied by various authors. For more information, we refer to Bosanquet and Das [4] therein. Further, for the special case $\lambda = e = (1, 1, ...)$, this set reduces to inclusion relation $X \Rightarrow Y$, which means that Y includes X.

Throughout the paper, k^* denotes the conjugate of $k > 1, i.e., 1/k^* + 1/k = 1$.

Inclusion problems dealing with absolute Cesàro and absolute Riesz mean summabilities of infinite series were studied by many authors (see, for instance, [2-14]). Hereof, the following result was established by Bor [2].

Theorem 1.1. *Let* $1 < k < \infty$ *and*

$$\sum_{n=v}^{\infty} \frac{n^{k-1} r_n^k}{R_n^k R_{n-1}} = O\left(\frac{v^{k-1} r_{v-1}^{k-1}}{R_{v-1}^k}\right). \tag{1.2}$$

If

$$R_{n+1} \geq d R_n \tag{1.3}$$

where d *is a constant such that* $d > 1$, *then* $|C, 0|_k \Leftrightarrow |R, r_n|_k$.

We note that (1.2) is equivalent to $P_n = O(p_n)$. In fact, $P_n = O(p_n)$ holds iff there exists a constan $M > 0$ such that, for all $n \geq 1$, $P_n/p_n \leq M$, or, equivalently, $P_n \geq dP_{n-1}$, where $d = \frac{M}{M-1} > 1$.

It is obvious for $k = 1$ that this result is satisfied and also gives a Tauber condition (1.3) for the summability method $|R, r_n|$.

Further, by omitting the condition (1.2), it has recently been shown in [12] that the condition (1.3) is not only sufficient but also necessary for the conclusion of Theorem 1.1 to hold as follows.

Theorem 1.2. *Let* $1 < k \leq s < \infty$. *Then,* $|C, 0|_k \Rightarrow |R, r_n|_s$ *if and only if*

$$\left(\sum_{v=1}^{m} \frac{R_{v-1}^{k^*}}{v}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} r_n}{R_n R_{n-1}}\right)^s\right)^{1/s} = O(1).$$

Theorem 1.3. *Let* $1 < k \leq s < \infty$. *Then,* $|R, r_n|_k \Rightarrow |C, 0|_s$ *if and only if*

$$\left(\sum_{v=m-1}^{m} \frac{1}{v} \left(\frac{R_{v-1} R_v}{r_v}\right)^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{m+1} \frac{n^{s-1}}{R_n^s}\right)^{1/s} = O(1).$$

Theorem 1.4. *Let* $1 \leq k < \infty$ *Then,* $|C,0|_k \Leftrightarrow |R,r_n|_k$ *if and only if the condition* (1.3) *is satisfied.*

2. Main Results

The purpose of this paper is to characterize the summability sets $(|C,0|_k, |R,r_n|_s)$ and $(|R,r_n|_k, |C,0|_s)$, for the case $1 < k \leq s < \infty$, which also extend some well known results.

The set of all sequences consisting k- absolutely convergent series is denoted by ℓ_k.

A factorable matrix T is defined by

$$t_{nv} = \begin{cases} b_n a_v, 0 \leq v \leq n, \\ 0, \qquad v > n. \end{cases}$$

where (b_n) and (a_n) are sequences of real or complex numbers.

Now we prove the following theorems.

Theorem 2.1. *Let* $1 < k \leq s < \infty$ *and* $\lambda = (\lambda_v)$ *be a sequence of numbers. Then,* $\lambda \in (|C,0|_k, |R,p_n|_s)$ *if and only if*

$$\left(\sum_{v=1}^{m} \frac{P_{v-1}^{k^*}}{v} |\lambda_v|^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} \left(\frac{n^{1/s^*} p_n}{P_n P_{n-1}}\right)^{s}\right)^{1/s} = O(1). \tag{2.1}$$

Theorem 2.2. *Let* $1 < k \leq s < \infty$ *and* $\lambda = (\lambda_n)$ *be a sequence of numbers. Then,* $\lambda \in (|R,p_n|_k, |C,0|_s)$ *if and only if*

$$\left(\sum_{v=m-1}^{m} \frac{1}{v} \left(\frac{P_{v-1} P_v}{p_v}\right)^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{m+1} \left|\frac{n^{1/s^*} \lambda_n}{P_n}\right|^{s}\right)^{1/s} = O(1). \tag{2.2}$$

It may be noticed that Theorem 2.1- 2.2 are, in the special case $u = e$, reduced to Theore 1.2-1.3, respectively.

Also, if we take $r = \lambda = e$, then $|R,r_n|_k = |C,1|_k$ and $R_n = n+1$. Further, since

$$\sum_{n=m}^{\infty} \frac{1}{n(n+1)^s} = O\left(\frac{1}{m^s}\right),$$

condition (2.1) holds but not condition (2.2) . Therefore, the following result of Flett [4] is immediately deduced.

Corollary 2.3. *Let* $1 < k \leq s < \infty$. *Then,* $|C,0|_k \Rightarrow |C,1|_s$, *but* $|C,1|_k \not\Rightarrow |C,0|_s$.

Proof of Theorem 2.1. We first note from a result of Bennett [1] that a factorable matrix T defines a bounded linear operator $L_T : \ell_k \to \ell_s$ such that $L_T(x) = T(x)$ for all $x \in \ell_k$ iff

$$\left(\sum_{v=0}^{m} |a_v|^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} |b_n|^{s}\right)^{1/s} = O(1), \tag{2.3}$$

where $k^*$is the conjugate of indices k. Now, let σ_n^0 and T_n be Cesàro $(C,0)$ and Riesz means (R,r_n) of the series Σx_n and $\Sigma\lambda_n x_n$, respectively. Then, by (2.1),

$$\sigma_n^0 = \sum_{v=0}^{n} x_v$$

$$V_n = \frac{1}{R_n}\sum_{v=0}^{n} r_v \sum_{r=0}^{v} \lambda_r x_r$$

and so $\Delta V_0 = \lambda_0 x_0$,

$$\Delta V_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1}\lambda_v x_v, \text{ for } n \geq 1.$$

Now, say $T_n' = n^{1/s^*}\Delta V_n$ and $\sigma_n^{0\prime} = n^{1/k^*} x_n$ for $n \geq 1$. Then, it can be written that

$$\begin{aligned} T_n' &= \frac{n^{1/s^*} r_n}{R_n R_{n-1}} \sum_{v=1}^{n} \frac{R_{v-1}\lambda_v}{v^{1/k^*}} \sigma_v^{0\prime} \\ &= \sum_{v=1}^{\infty} t_{nv}\sigma_v^{0\prime} \end{aligned}$$

where

$$t_{nv} = \begin{cases} \frac{n^{1/s^*} r_n R_{v-1}\lambda_v}{R_n R_{n-1} v^{1/k^*}}, & 1 \leq v \leq n, \\ 0, & v > n. \end{cases}$$

This means that the consequence of the theorem holds iff $(T_n') \in \ell_s$ for all $\left(\sigma_n^{0\prime}\right) \in \ell_k$, or, $T = (t_{nv}) : \ell_k \to \ell_s$ is a bounded operator. Thus, by applying (2.3) to the matrix T, we obtain (2.1).

Proof of Theorem 2.2. Let V_n and σ_n^0 be means of Riesz (R,r_n) and Cesàro $(C,0)$ of the series Σx_n and $\Sigma\lambda_n x_n$, respectively. Then, as above, $\Delta\sigma_n^0 = \lambda_n x_n$, and also $\Delta V_0 = x_0$,

$$\Delta V_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} x_v, \text{for } n \geq 1 \tag{2.4}$$

By inversion of (2.4), it can be stated that, for $n \geq 1$,

$$x_n = \frac{1}{R_{n-1}}\left(\frac{R_{n-1}R_n}{r_n}\Delta V_n - \frac{R_{n-1}R_{n-2}}{r_{n-1}}\Delta V_{n-1}\right)$$

Say $T_n' = n^{1/k^*}\Delta V_n$ and $\sigma_n^{0\prime} = n^{1/s^*}\lambda_n x_n$ for $n \geq 1$. Then, it can be written that

$$\begin{aligned}\sigma_n^{0\prime} &= \frac{n^{1/s^*}\lambda_n}{R_{n-1}}\left(\frac{R_{n-1}R_n}{n^{1/k^*}r_n}T_n' - \frac{R_{n-1}R_{n-2}}{(n-1)^{1/k^*}r_{n-1}}T_{n-1}'\right)\\ &= \sum_{v=1}^{\infty} d_{nv}\sigma_v^{0\prime}\end{aligned}$$

where

$$D_{nv} = \begin{cases} \frac{n^{1/s^*}\lambda_n}{R_{n-1}}\left(-\frac{R_{n-1}R_{n-2}}{(n-1)^{1/k^*}r_{n-1}}\right), & v = n-1\\ \frac{n^{1/s^*}\lambda_n}{R_{n-1}}\left(\frac{R_{n-1}R_n}{n^{1/k^*}r_n}\right), & v = n\\ 0, & v > n.\end{cases}$$

The reminder is similarly proved.

References

1. G. Bennett, *Some elemantery inequalities*, Quart. J. Math. Oxford **38** (1987), 401-425.
2. H. Bor, *A new result on the high indices theorem*, Analysis **29** (2009), 403-405.
3. H. Bor and B. Thorpe, *On some absolute summability methods*, Analysis **7** (1987), 145-152.
4. L.S. Bosanquet and G. Das, *Absolute summability factors for Nörlund means*, Proc. London Math. Soc. **38** (1979) 1-52.
5. T.M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113-141.
6. G.C. Hazar and M.A. Sarıgöl, *On factor relations between weighted and Nörlund means*, Tamkang J. Math. **50** (2019), 61-69.
7. S.M. Mazhar, *On the absolute summability factors of infinite series*, Tohoku Math. J. **23** (1971), 433-451.
8. M.R. Mehdi, *Summability factors for generalized absolute summability I*, Proc. London Math. Soc. **10** (1960), 180-199.
9. R.N. Mohapatra, *On absolute Riesz summability factors*, J. Indian Math. Soc. **32** (1968), 113-129.
10. C. Orhan and M.A. Sarıgöl, *On absolute weighted mean summability*, Rocky Mount. J. Math. **23** (1993), 1091-1097.
11. M.A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci. **42** (2015), 28-35.
12. M.A. Sarıgöl, *Characterization of summability methods with high indices*, Math. Slovaca **63** (2013), 1-6.
13. M.A. Sarıgöl and H. Bor, *Characterization of absolute summability factors*, J. Math. Anal. Appl. **195** (1995), 537-545.
14. M.A. Sarıgöl, *On two absolute Riesz summability factors of infinite series*, Proc. Amer. Math. Soc. **118** (1993), 485-488.

Mehmet Ali Sarıgöl received his Ph.D. degree from Department of Mathematics, Ankara University in 1985, and has been studing in Pamukkale University in Denizli since 1994. His research interests include summability, sequence spaces, mathematical analysis, functional Analysis, real analysis, Fourier analysis. He has more than 80 research papers published in the reputed international mathematics journals, and also serves as referee in many mathematical journals.

Department of Mathematics, Faculty of Art and Sicence, Pamukkale university, Denizli, 20160, Turkey.
e-mail: msarigol@pau.edu.tr