J. Appl. Math. & Informatics Vol. **40**(2022), No. 1 - 2, pp. 141 - 151
https://doi.org/10.14317/jami.2022.141

ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS IN A SECTOR†

IN HWA KIM AND NAK EUN CHO*

Abstract. The purpose of the present paper is to obtain some conditions for strongly starlikeness and univalence of normalized analytic functions in the open unit disk. Further, we prove an univalence and argument properties for certain integral operators.

AMS Mathematics Subject Classification : 30C45.
Key words and phrases : Univalent function, subordination, strongly starlike function, argument estimate.

1. Introduction

Let $\mathcal{A}$ denote the class of the functions f which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. A function f in $\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{U})$$

for some α $(0 \le \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order α in $\mathbb{U}$ and $\mathcal{S}^*(0) \equiv \mathcal{S}^*$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Further, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see [13, p. 36]).

A function $f \in \mathcal{A}$ is said to be strongly starlike of order α $(0 < \alpha \le 1)$ [5, 6] if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}). \tag{1}$$

Received August 20, 2021. Revised October 22, 2021. Accepted December 7, 2021.
*Corresponding author.
†This work was supported by a Research Grant of Pukyong National University(2021).

We denote by $\mathcal{S}[\alpha]$ the subclass of $\mathcal{A}$ consisting of strongly starlike functions of order α in $\mathbb{U}$ and $\mathcal{S}[0] \equiv \mathcal{S}^*$, which is the well-known class of starlike functions in $\mathbb{U}$. We note that the condition (1) can be written by

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

A function f in $\mathcal{A}$ is said to be in the class $\mathcal{S}^*(A, B)$ if it satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U})$$

for some A and B $(-1 \le B < A \le 1)$. The class $\mathcal{S}^*(A, B)$ $(-1 \le B < A \le 1)$ can be reduced to several well-known classes of starlike functions by selecting special values A and B. Note that $\mathcal{S}^*(1-2\alpha, -1) \equiv \mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$. Also the class $\mathcal{S}^*(A, B)$ $(-1 \le B < A \le 1)$ is called as the class of Janoski type starlike functions [4].

The strongly starlike functions and related interesting developments play a very important role in the study of pure and applied mathematical sciences and have been extensively studied by several authors (see, e.g., [1, 2, 3, 7, 8, 9, 10, 11, 14]. Additionally, the theory of integral operators and derivatives of an arbitrary real or complex order (see, for details, [16]; see also [12, 17]) has been applied not only in geometric function theory of complex analysis, but has also emerged as a potentially useful direction in the mathematical modeling and analysis of real-world problems in applied sciences (see, for example, [18]).

Motivated by the works mentioned above, in this paper, we will investigate some conditions for strongly starlikeness and univalence of functions belonging to $\mathcal{A}$. Also, we prove an univalence and argument properties of certain integral operators.

2. Main Results

In proving our results, we shall need the following lemma due to Numokawa[6].

Lemma 2.1. *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg\ p(z)| < \frac{\pi}{2}\alpha \ \textit{for}\ |z| < |z_0| \tag{2}$$

and

$$|\arg\ p(z_0)| = \frac{\pi}{2}\alpha \ (0 < \alpha \le 1) \tag{3}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k, \tag{4}$$

where

$$k \ge \frac{1}{2}\left(a + \frac{1}{a}\right) \ \textit{when}\ \arg\ p(z_0) = \frac{\pi}{2}\alpha, \tag{5}$$

$$k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right) \ \textit{when} \ \arg\ p(z_0) = -\frac{\pi}{2}\alpha, \tag{6}$$

and

$$\{p(z_0)\}^{\frac{1}{\alpha}} = \pm ia \ (a>0). \tag{7}$$

With the help of Lemma 2.1, we now derive the following theorem.

Theorem 2.2. *Let p be analytic in $\mathbb{U}$ with $p(0)=1$ and $p(z) \neq 0$ in $\mathbb{U}$. If*

$$\left|\operatorname{Im}\left(\frac{\beta}{p(z)} - \gamma p(z) - \frac{zp'(z)}{p(z)}\right)\right| < 2\sqrt{\beta\gamma}\sin\frac{\pi}{2}\alpha + \alpha \ (0<\alpha\leq 1,\ \beta,\gamma>0,\ z\in\mathbb{U}), \tag{8}$$

then

$$|\arg\ p(z)| < \frac{\pi}{2}\alpha \quad (z\in\mathbb{U}).$$

Proof. If there exists a point $z_0 \in \mathbb{U}$ such that the conditions (2) and (3) are satisfied, then (by Lemma 2.1) we obtain (4) under the restrictions (5), (6), and (7). At first, we suppose that $p(z_0) = (ia)^\alpha$ $(a>0)$. Then we obtain

$$\begin{aligned}\operatorname{Im}\left(\frac{\beta}{p(z_0)} - \gamma p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) &= \operatorname{Im}\left(\beta(ia)^{-\alpha} - \gamma(ia)^\alpha - i\alpha k\right)\\ &= -\frac{\beta}{a^\alpha}\sin\frac{\pi}{2}\alpha - a^\alpha\gamma\sin\frac{\pi}{2}\alpha - \alpha k\\ &\leq -\sin\frac{\pi}{2}\alpha\cdot g(t) - \alpha,\end{aligned}$$

where

$$g(t) = \frac{\beta}{t} + \gamma t \quad (t = b^\alpha > 0).$$

Since $g(t) \geq g(\sqrt{\beta/\gamma})$, we have

$$\begin{aligned}\operatorname{Im}\left(\frac{\beta}{p(z_0)} - \gamma p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) &\leq -\sin\frac{\pi}{2}\alpha\cdot g(t) - \alpha\\ &= -\left(2\sqrt{\beta\gamma}\sin\frac{\pi}{2}\alpha + \alpha\right).\end{aligned}$$

This evidently contradicts the assumption (8). Next, we suppose that $p(z_0) = (-ia)^\alpha$ $(a>0)$. Applying the same method as the above, we have

$$\operatorname{Im}\left(\frac{\beta}{p(z_0)} - \gamma p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \geq 2\sqrt{\beta\gamma}\sin\frac{\pi}{2}\alpha + \alpha,$$

which contradicts the assumption (8). Therefore we complete the proof of Theorem 2.1. □

Taking $p(z) = \frac{f(z)}{zf'(z)}$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $f \in \mathcal{A}$ with $f(z)f'(z) \neq 0$ in $\mathbb{U}\backslash\{0\}$. If*

$$\left|\operatorname{Im}\left((\beta-1)\frac{zf'(z)}{f(z)} - \gamma\frac{f(z)}{zf'(z)} + 1 + \frac{zf''(z)}{f'(z)}\right)\right| < 2\sqrt{\beta\gamma}\sin\frac{\pi}{2}\alpha + \alpha \quad (z\in\mathbb{U}),$$

then f is a strongly starlike of order α.

Putting $\beta = 1$ and $\gamma = \alpha = \frac{1}{2}$ in corollary 2.3, we have the following corollary.

Corollary 2.4. *Let $f \in \mathcal{A}$. If*

$$\left|\mathrm{Im}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{1}{2}\frac{f(z)}{zf'(z)}\right)\right| < \frac{3}{2} \quad (z \in \mathbb{U}),$$

then f is a strongly starlike of order $\frac{1}{2}$.

Theorem 2.5. *Let $f, g \in \mathcal{A}$ with $f'(z) \neq 0$ in $\mathbb{U}\backslash\{0\}$ and let*

$$\rho = \sup_{|z|\leq 1}\left|\frac{g''(z)}{g'(z)}\right| \leq \alpha \quad (0 < \alpha \leq 1).$$

If

$$\left|\arg\left(\alpha\frac{f'(z)}{g'(z)} + \frac{zf''(z)}{g'(z)}\right)\right| < \frac{\pi}{2}\delta(\rho, \alpha) \quad (z \in \mathbb{U}),$$

where

$$\delta(\rho, \alpha) = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha - \rho}{\alpha + \rho}\right), \tag{9}$$

then

$$\left|\arg\frac{f'(z)}{g'(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Proof. Let us put

$$p(z) = \frac{f'(z)}{g'(z)}. \tag{10}$$

Then we see that p is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. From (10), we have

$$\alpha\frac{f'(z)}{g'(z)} + \frac{zf''(z)}{g'(z)} = zp'(z) + p(z)\left(\alpha + \frac{zg''(z)}{g'(z)}\right).$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that the conditions (2) and (3) hold. Then (by Lemma 2.1) we get (4) with conditions (5), (6) and (7). For $p(z_0) = (ia)^\alpha$, we obtain

$$\begin{aligned}\arg\left(\alpha\frac{f'(z)}{g'(z)} + \frac{z_0 f''(z_0)}{g'(z_0)}\right) &= \arg p(z_0) + \arg\left(\alpha + \frac{z_0 g''(z_0)}{g'(z_0)} + \frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha k + \mathrm{Im}\frac{z_0 g''(z_0)}{g'(z_0)}}{\alpha + \mathrm{Re}\frac{z_0 g''(z_0)}{g'(z_0)}}\right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha - \rho}{\alpha + \rho}\right) \\ &= \frac{\pi}{2}\delta(\rho, \alpha),\end{aligned}$$

where $\delta(\rho,\alpha)$ is given by (9). This is a contradiction to the assumption of Theorem 2.5. Similarly, for the case $p(z_0) = (-ia)^\alpha$ $(a > 0)$, we have

$$\begin{aligned}\arg\left(\alpha\frac{f'(z)}{g'(z)} + \frac{z_0 f''(z_0)}{g'(z_0)}\right) &\le -\frac{\pi}{2}\alpha - \tan^{-1}\left(\frac{\alpha-\rho}{\alpha+\rho}\right)\\ &= -\frac{\pi}{2}\delta(\rho,\alpha),\end{aligned}$$

where $\delta(\rho,\alpha)$ is given by (9), which contradicts the assumption of our Theorem 2.5. Therefore we have the result. □

Letting $g(z) = z$ in Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ in $\mathbb{U}\backslash\{0\}$. If*

$$|\arg(\alpha f'(z) + zf''(z))| < \frac{\pi}{2}\left(\alpha + \frac{1}{2}\right) \quad (0 < \alpha \le 1,\ z \in \mathbb{U}),$$

then

$$|\arg f'(z)| < \frac{\pi}{2}\alpha \quad (\ z \in \mathbb{U}).$$

Taking $\alpha = 1$ in Corollary 2.6, we have the following result.

Corollary 2.7. *Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ in $\mathbb{U}\backslash\{0\}$. If*

$$|\arg(\alpha f'(z) + zf''(z))| < \frac{3}{4}\pi \quad (\ z \in \mathbb{U}),$$

then

$$\mathrm{Re}\ f'(z) > 0 \quad (\ z \in \mathbb{U}),$$

that is, f is univalent.

By using a similar method of the proof in Theorem 2.5, we have the following result.

Theorem 2.8. *Let $f, g \in \mathcal{A}$ and let*

$$\rho = \sup_{|z|\le 1}\left|\frac{g''(z)}{g'(z)}\right| \le \alpha \quad (0 < \alpha \le 1).$$

If

$$\left|\arg\frac{f'(z)}{g'(z)}\right| < \frac{\pi}{2}\eta(\rho,\alpha) \quad (z \in \mathbb{U}),$$

where

$$\eta(\rho,\alpha) = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha-\rho}{1+\rho}\right),$$

then

$$\left|\arg\frac{F'(z)}{g'(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where F is the integral operator given by

$$F(z) = \int_0^z \frac{f(t)}{t}dt. \tag{11}$$

Proof. Let us put

$$p(z) = \frac{F'(z)}{g'(z)}.$$

Then p is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. By a simple calculation, we have

$$\frac{f'(z)}{g'(z)} = zp'(z) + p(z)\left(1 + \frac{zg''(z)}{g'(z)}\right).$$

The remaining part of the proof is a similar to that of Theorem 2.5 and we omit it. □

Putting $\alpha = 1$ in Theorem 2.8, we have the following corollary.

Corollary 2.9. *Let $f, g \in \mathcal{A}$ and let*

$$\rho = \sup_{|z|\leq 1}\left|\frac{g''(z)}{g'(z)}\right| \leq 1.$$

If

$$\left|\arg\frac{f'(z)}{g'(z)}\right| < \frac{\pi}{2}\eta(\rho) \quad (z \in \mathbb{U}),$$

where

$$\eta(\rho) = 1 + \frac{2}{\pi}\tan^{-1}\left(\frac{1-\rho}{1+\rho}\right),$$

then

$$\mathrm{Re}\ \frac{F'(z)}{g'(z)} > 0 \quad (z \in \mathbb{U}),$$

where F is the integral operator given by (11).

Letting $g(z) = z$ in corollary 2.9, we have the following result.

Corollary 2.10. *Let $f \in \mathcal{A}$. If*

$$|\arg f'(z)| < \frac{3}{4}\pi \quad (z \in \mathbb{U}),$$

then the integral operator given by (11) *is univalent.*

Theorem 2.11. *Let $g \in \mathcal{A}$ with*

$$\rho = \sup_{|z|\leq 1}\left|\frac{zg'(z)}{g(z)}\right| \leq \alpha \quad (0 < \alpha \leq 1). \tag{12}$$

If $f \in S^*[\beta(\alpha)]$, where

$$\beta(\alpha) = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha-\rho}{\rho}\right), \tag{13}$$

then

$$\left|\arg\frac{zF'(z)}{F(z)}\right| < \pi\alpha + \tan^{-1}\left(\frac{\alpha-\rho}{\rho}\right) \quad (z \in \mathbb{U}),$$

where

$$F(z) = \int_0^z \frac{g(t)}{f(t)} f'(t)dt. \tag{14}$$

Proof. From the assumption (23), we note that $g(z) \neq 0$ in $\mathbb{U}$. Since

$$F'(z) = \frac{g(z)f'(z)}{f(z)},$$

where F is integral operator given by (14) and $f \in S^*[\beta(\alpha)]$, it follows that $F'(z) \neq 0$ for $z \in \mathbb{U}$. Hence

$$p(z) = \frac{f'(z)F(z)}{f(z)F'(z)} \tag{15}$$

is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. A simple calculation of (15). shows that

$$zp'(z) + \frac{zg'(z)}{g(z)}p(z) = \frac{zf'(z)}{f(z)}. \tag{16}$$

If there exists a point $z_0 \in \mathbb{U}$ satisfying the conditions (2) and (3), then by Lemma 2.1, we obtain (4) under the restrictions (5), (6), and (7). At first, for $p(z_0) = (ia)^\alpha$ $(a > 0)$, from (16) we have

$$\begin{aligned}
\arg \frac{zf'(z_0)}{f(z_0)} &= \arg \left(z_0 p'(z_0) + \frac{z_0 g'(z_0)}{g(z_0)} p(z_0) \right) \\
&= \arg p(z_0) + \arg \left(\frac{z_0 g'(z_0)}{g(z_0)} + \frac{z_0 p'(z_0)}{p(z_0)} \right) \\
&= \frac{\pi}{2}\alpha + \tan^{-1} \left(\frac{\operatorname{Im} \frac{z_0 g'(z_0)}{g(z_0)} + \alpha k}{\operatorname{Re} \frac{z_0 g'(z_0)}{g(z_0)}} \right) \\
&\geq \frac{\pi}{2}\alpha + \tan^{-1} \left(\frac{\alpha - \rho}{\rho} \right) \\
&= \frac{\pi}{2}\beta(\alpha),
\end{aligned}$$

where $\beta(\alpha)$ is given by (13). This contradicts the assumption, $f \in S^*[\beta(\alpha)]$. Next,we suppose that $p(z_0) = (-ia)^\alpha (a > 0)$. Applying the same method as the above, we have

$$\arg \frac{z_0 f'(z_0)}{f(z_0)} \leq -\frac{\pi}{2}\alpha - \tan^{-1} \left(\frac{\alpha - \rho}{\rho} \right),$$

which also contradicts the assumption. Hence we have

$$\begin{aligned}
\frac{\pi}{2}\alpha &> |\arg p(z)| \\
&= \left| \arg \frac{f'(z)F(z)}{f(z)F'(z)} \right| \\
&\geq \left| \arg \frac{zF'(z)}{F(z)} \right| - \left| \arg \frac{zf'(z)}{f(z)} \right|.
\end{aligned} \tag{17}$$

From (17), we obtain

$$\left|\arg \frac{zF'(z)}{F(z)}\right| \leq \frac{\pi}{2}\alpha + \left|\arg \frac{zf'(z)}{f(z)}\right| < \pi\alpha + \tan^{-1}\left(\frac{\alpha-\rho}{\rho}\right).$$

Therefore we completes the proof of Theorem 2.11. □

Finally, we prove the following results by using Lemma 2.1.

Theorem 2.12. *Let* $0 < \alpha \leq 1$ *and let* $f, g \in \mathcal{A}$ *with the condition*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1,\ z \in \mathbb{U}). \tag{18}$$

If

$$\left|\arg \frac{g'(z)f(z)}{g(z)f'(z)}\right| < \frac{\pi}{2}\beta(\alpha) \quad (z \in \mathbb{U}),$$

where

$$\beta(\alpha) = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha \sin \frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + \alpha \cos\frac{\pi}{2}(1-t(A,B))}\right), & (B \neq -1) \\ \alpha, & (B = -1), \end{cases} \tag{19}$$

where

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB}\right), \tag{20}$$

then

$$\left|\arg \frac{g'(z)F(z)}{g(z)F'(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where F *is the integral operator given by*

$$F(z) = \int_0^z \frac{f(t)}{g(t)} g'(t)dt. \tag{21}$$

Proof. Let

$$h(z) = \frac{zf'(z)}{f(z)}.$$

Then, form (18), we observe [15] that

$$\left|h(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (z \in \mathbb{U}, B \neq -1). \tag{22}$$

and

$$\text{Re } h(z) > \frac{1-A}{2} \quad (z \in \mathbb{U}, B = -1). \tag{23}$$

From (22) and (23), we have

$$h(z) = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$\begin{cases} \frac{1-A}{1-B} < \rho < \frac{1+A}{1+B} \\ -A(A,B) < \phi < t(A,B) \quad for\ B \neq -1, \end{cases}$$

when $t(A,B)$ is given by (20), and

$$\begin{cases} \frac{1-A}{2} < \rho < \infty \\ -1 < \phi < 1 \qquad for\ B = -1. \end{cases}$$

From (18) and the definition of F, it follows that $F'(z) \neq 0$ for $z \in \mathbb{U}$, and hence

$$p(z) = \frac{g'(z)F(z)}{g(z)F'(z)}$$

is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. A straightforward computation shows that

$$\frac{g'(z)f(z)}{g(z)f'(z)} = p(z) + \frac{f(z)}{zf'(z)} zp'(z).$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ satisfying the conditions of Lemma 2.1. At first, we assume that $p(z_0) = (ia)^{\alpha} (b > 0)$. For the case $B \neq -1$, we obtain

$$\begin{aligned} \arg\left(\frac{g'(z_0)f(z_0)}{g(z_0)f'(z_0)}\right) &= \arg\left(p(z_0) + \frac{z_0 p'(z_0)}{h(z_0)}\right) \\ &= \arg p(z_0) + \arg\left(1 + \frac{1}{h(z_0)} \cdot \frac{zp'(z_0)}{p(z_0)}\right) \\ &= \frac{\pi}{2}\alpha + \arg(1 + (\rho e^{i\frac{\pi}{2}\phi})^{-1} i\alpha k) \\ &= \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha k \sin\frac{\pi}{2}(1-\phi)}{\rho + \alpha k \cos\frac{\pi}{2}(1-\phi)}\right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha k \sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + \alpha k \cos\frac{\pi}{2}(1-t(A,B))}\right) \\ &= \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha k \sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + \alpha k \cos\frac{\pi}{2}(1-t(A,B))}\right)\right), \end{aligned}$$

where $t(A,B)$ and $\beta(\alpha)$ are given by (20) and (19), respectively. Similarly, for the case $B = -1$, we have

$$\begin{aligned} \arg\left(\frac{g'(z)f(z)}{g(z)f'(z)}\right) &\leq -\frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha k \sin\frac{\pi}{2}(1-t(A,B))}{\frac{1+A}{1+B} + \alpha k \cos\frac{\pi}{2}(1-t(A,B))}\right)\right), \\ &= -\frac{\pi}{2}\beta(\alpha), \end{aligned}$$

where $t(A,B)$ and $\beta(\alpha)$ are given by (20) and (19), respectively and for the case $B = -1$, we have

$$\arg\left(\frac{g'(z)f(z)}{g(z)f'(z)}\right) \leq -\frac{\pi}{2}\alpha,$$

which contradict the assumption. this completes the proof of our theorem. □

Taking $A = 1, B = -1$ and $g(z) = z$ in theorem 2.12, we have the following corollary.

Corollary 2.13. *Let* $f \in \mathcal{A}$ *and* $0 < \alpha \leq 1$. *If*

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

then

$$\left|\arg \frac{zF'(z)}{F(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where F *is integral operator given by* (11).

References

1. M.F. Ali and A. Vasudevarao, *Logarithmic coefficients of some close-to-convex functions*, Bull. Aust. Math. Soc. **95** (2017), 228–237.
2. D.A. Brannan and W.E. Kirwan, *On some classes of bounded univalent functions*, J. Lond. Math. Soc. **1** (1969), 431–443.
3. A. Gangadharan and V. Ravichandran, *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl. **211** (1997), 303–313.
4. W. Janowski, *Some extremal problems for certain families of analytic functions*, I, Ann. Polon. Math. **28** (1973), 297-326.
5. P.T. Mocanu, *Alpha-convex integral operator and strongly starlike function*, Studia Univ. Babes-Bolyai Math. **34** (1989), 18-24.
6. M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A **69** (1993), 234-237.
7. M. Nunokawa and D.K. Thomas, *On convex and starlike functions in a sector*, J. Aust. Math. Soc. Ser. A **60** (1996), 363–368.
8. M. Nunokawa, S. Owa, H. Saitoh, A. Ikeda, and N. Koike, *Some results for strongly starlike functions*, J. Math. Anal. Appl. **212** (1997), 98–106.
9. M. Nunokawa, J. Sokól and K. Trabka-Wieclaw, *On the order of strongly starlikeness in some classes of starlike functions*, Acta Math. Hung. **145** (2015), 142–149.
10. M. Obradorić and S. Owa, *Some sufficient conditions for strongly starlikeness*, Int. J. Math. Math. Sci. **24** (2000), 643–647.
11. S. Owa, H.M. Srivastava, T. Hayami and K. Kuroki, *A new general idea for starlike and convex functions*, Tamkang J. Math. **47** (2016), 445–454.
12. J.H. Park, H.M. Srivastava, and N.E. Cho, *Univalence and convexity conditions for certain integral operatorsassociated with the Lommel function of the first kind*, AIMS Mathematics **6** (2021), 11380-11402.
13. Ch. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, 1975.
14. H. Shiraishi, S. Owa and H.M. Srivastava, *Sufficient conditions for strongly Carathéodory functions*, Comput. Math. Appl. **62** (2011), 2978–2987.
15. H. Silverman and E.M. Silvia, *Sudclasses of starlike functions subordinate to convex functions*, Can. J. Math. **37** (1985), 48-61.
16. H.M. Srivastava and S. Owa, *Univalent Functions, Fractional Calculus, and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989.
17. H.M. Srivastava, *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A: Sci. **44** (2020), 327-344.

18. H.M. Srivastava, R. Jan, A. Jan, W. Deebai and M. Shutaywi, *Fractional-calculus analysis of the transmission dynamics of the dengue infection*, Chaos **31** (2021), 1-18.

In Hwa Kim received her M.S. and Ph.D. from Pukyong National University, as well as her Ph.D. from Middle Tennessee State University. Her research interests are Complex Analysis and Economics.

Department of Economics and International Business, Sam Houston State University, Huntsville, TX 77341, USA.
e-mail: ihyahootn@gmail.com;ixk013@shsu.edu

Nak Eun Cho received M.S. and Ph.D. from Kyungpook National University. He is currently a professor at Pukyong National University since 1985. His research interests are Complex Analysis, Geometric Function Theory and Special Functions.

Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 608-737, Korea.
e-mail: necho@pknu.ac.kr