J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 1 - 13*
https://doi.org/10.14317/jami.2022.001

THE LENGTH-BIASED POWERED INVERSE RAYLEIGH DISTRIBUTION WITH APPLICATIONS†

ABDELFATTAH MUSTAFA AND M.I. KHAN*

Abstract. This article introduces a new distribution called length-biased powered inverse Rayleigh distribution. Some of its statistical properties are derived. Maximum likelihood procedure is applied to report the point and interval estimations of all model parameters. The proposed distribution is also applied to two real data sets for illustrative purposes.

AMS Mathematics Subject Classification : 60E05, 62E10, 62F10.
Key words and phrases : Length-biased, power inverse Rayleigh distribution, maximum likelihood estimation.

1. Introduction

One of the most frequently used distribution in probability and reliability theory is the Rayleigh distribution. The Rayleigh distribution was basically applied in the fields of optics and acoustics, [25]. It is originated from two parameter Weibull distribution. It plays a pivotal role in analyzing lifetime data and modeling such as life testing studies, diagnostic imaging, theory of communication, physical sciences, oceanography, project effort loading modeling, applied statistics and clinical research.
The inverse Rayleigh distribution, (IRD), was introduced by Trayer [31]. It is an important model for lifetime analysis. The maximum likelihood of its scale parameter are studied by Voda [32]. In recent years, several extensions of probability distributions have been made to increase its applicability and flexibility.
The power transformation (PT) technique has been first introduced by Box and Cox [5]. The PT technique is develop new distributions from well-known distributions by adding a new parameter. A new parameter gives several desirable properties and offers a more flexible model. The PT technique has been used in many statistical aspects.

Received April 6, 2021. Revised May 30, 2021. Accepted May 31, 2021. *Corresponding author.

It can be shown that, the random variable (r.v.), $X = 1/T$ has an inverse Rayleigh distribution, if T has a Rayleigh distribution. Nashaat [21] introduced the powered inverse Rayleigh distribution (PIRD) through the powered transformation as follow

$$x = t^{\frac{1}{\alpha}}, \text{ where } \quad \alpha > 0, \quad t = x^{\alpha}, \; dt = \alpha x^{\alpha-1} dx.$$

Then probability density function (pdf) of two parameter PIRD will be

$$f(x;\alpha,\theta) = \frac{2\alpha}{\theta x^{2\alpha+1}} \exp\left\{-\frac{1}{\theta x^{2\alpha}}\right\}, \quad \alpha,\theta,x > 0, \tag{1}$$

where α, θ are shape and scale parameters. Clearly when $\alpha = 1$, we get one parameter IRD. The corresponding cumulative distribution function (cdf) of (1) is,

$$F(x;\alpha,\theta) = \exp\left\{-\frac{1}{\theta x^{2\alpha}}\right\}, \quad \alpha,\theta,x > 0, \tag{2}$$

The general formula of non-central moments is given by

$$\mu_r^{'} = E[X^r] = \left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}} \Gamma\left(1 - \frac{r}{2\alpha}\right), \quad r < 2\alpha. \tag{3}$$

The reset of the paper can be organized as follows. The length–biased powered inverse Rayleigh distribution (LBPIRD) is introduced in Section 2. The statistical properties of LBPIRD are addressed in Section 3. The parameter estimation of the model are studied in Section 4. Some applications and numerical results are presented in Sections 5. A brief conclusion is introduced in Section 6.

2. The Length-Biased Powered Inverse Rayleigh Distribution

The statistical insight of length-biased distributions was analyzed by Cox [6] in relation to renewal theory. To establish the relationships between the original distributions and their length-biased forms, numerous works are done, see [1, 2, 3, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 20, 22, 23, 24, 26, 27, 28, 29], and cited references therein.

Let X be a r.v. of a PIRD with pdf $f(x)$, then $g(x) = \frac{xf(x)}{E(X)}$ is a pdf of LBPIRD(α, θ) is given by,

$$g(x) = \frac{2\alpha\theta^{\frac{1}{2\alpha}-1}}{\Gamma\left(1-\frac{1}{2\alpha}\right)} x^{-2\alpha} \exp\left\{-\frac{1}{\theta}x^{-2\alpha}\right\}, \quad x > 0, \tag{4}$$

where $\theta > 0$ and $\alpha > \frac{1}{2}$.
The cdf of LBPIRD can be obtained as follows

$$G(x) = \frac{1}{\Gamma\left(1-\frac{1}{2\alpha}\right)} \gamma\left(1 - \frac{1}{2\alpha}, \frac{1}{\theta}x^{-2\alpha}\right). \tag{5}$$

Where $\gamma(a,x)$ is the incomplete gamma function,

$$\gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt.$$

The reliability function (rf) is given by

$$R(x)=P(X>x)=1-\frac{1}{\Gamma\left(1-\frac{1}{2\alpha}\right)}\gamma\left(1-\frac{1}{2\alpha},\frac{1}{\theta}x^{-2\alpha}\right). \tag{6}$$

The hazard rate function (hrf) is

$$h(x)=\frac{g(x)}{R(x)}=\frac{2\alpha\theta^{\frac{1}{2\alpha}-1}x^{-2\alpha}\exp\left\{-\frac{1}{\theta}x^{-2\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)-\gamma\left(1-\frac{1}{2\alpha},\frac{1}{\theta}x^{-2\alpha}\right)}. \tag{7}$$

Figures 1 – 4 display the pdf, cdf, rf and hrf of LBPIRD for some values of α and θ, respectively.

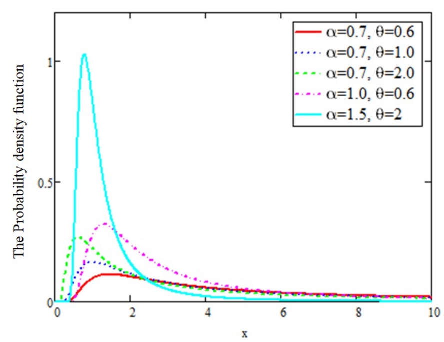

FIGURE 1. The pdf of the LBPIRD(α,θ).

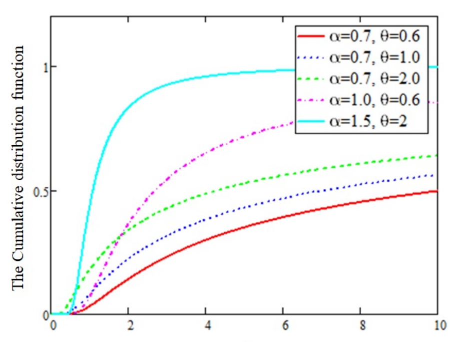

FIGURE 2. The cdf of the LBPIRD(α,θ).

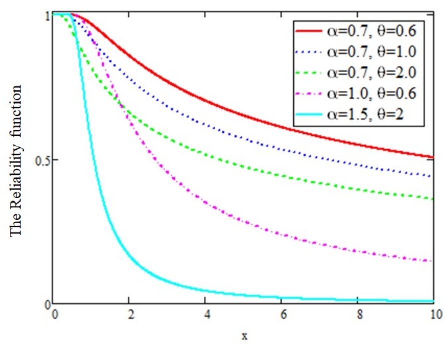

FIGURE 3. The rf of the LBPIRD(α, θ).

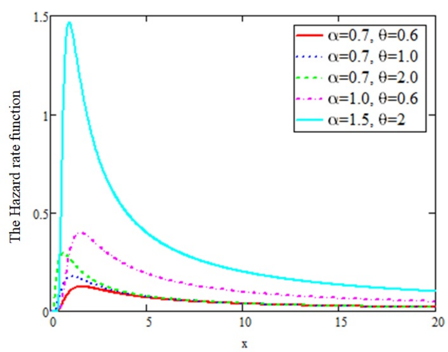

FIGURE 4. The hrf of the LBPIRD(α, θ).

We can conclude that: The shape of the LBPIRD is unimodal and positively skewed, see Figure 1. Also from Figure 4, it can be shown that the hazard rate is increasing on $(0, x_0)$, and then decreasing on (x_0, ∞).

3. Statistical Properties

Some important statistical properties of LBPIRD are obtained in this section. The moment generating function (MGF), and Rényi and Tsallis's entropies are also presented. The following theorem discussed the moments for the LBPIRD.

3.1. Moments and moment generating function.

Theorem 3.1. *The r-th moments for LBPIR (α, θ) distribution is,*

$$\mu'_{r(LBPIRD)} = \frac{\Gamma\left(1 - \frac{r+1}{2\alpha}\right)}{\Gamma\left(1 - \frac{1}{2\alpha}\right)}, \quad 2\alpha > r + 1. \tag{8}$$

Proof. The rth moment is obtained by the following formula,

$$\begin{aligned}\mu'_{r(LBPIRD)} &= E[X^r] = \int_{-\infty}^{\infty} x^r g(x)dx = \int_{-\infty}^{\infty} x^r \frac{xf(x)}{E(X)}dx \\ &= \int_{-\infty}^{\infty} \frac{x^{r+1}f(x)}{E(X)}dx = \left[\frac{\mu'_{r+1}}{\mu'_1}\right]_{PRID}. \end{aligned} \tag{9}$$

Substituting from (3) into (9), then

$$\mu'_{r(LBPIRD)} = \frac{\Gamma\left(1-\frac{r+1}{2\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)}.$$

The proof is completed. □

From Theorem 3.1, we can compute the mean and variance for the LBPIR distribution, as follows.

$$E(X) = \mu'_{1(LBPIRD)} = \frac{\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)},$$

and

$$Var(X) = \mu'_{2(LBPIRD)} - [\mu'_{1(LBPIRD)}]^2 = \frac{\Gamma\left(1-\frac{3}{2\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)} - \left[\frac{\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{2\alpha}\right)}\right]^2.$$

Using (9), the skewness and kurtosis can be calculated by the following formulas.

$$sk = E\left[\left(\frac{X-E(X)}{\sigma}\right)^3\right] = \frac{\mu'_3 - 2\mu'_2\mu'_1 - \mu'^3_1}{\left(\mu'_2 - \mu'^2_1\right)^{3/2}},$$

and

$$ku = E\left[\left(\frac{X-E(X)}{\sigma}\right)^4\right] = \frac{\mu'_4 - 3\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1}{(\mu'_2 - \mu'^2_1)^2}.$$

The mode for the LBPIR distribution can be calculated by differentiating the natural log of pdf and equating to zero. From (4), the natural log for pdf is

$$\ln(g(x)) = \ln(2\alpha\theta^{\frac{1}{2\alpha}-1}) - \ln\Gamma\left(1-\frac{1}{2\alpha}\right) - 2\alpha\ln(x) - \frac{1}{\theta x^{2\alpha}}.$$

The first derivatives for $\ln g(x)$, with respect to x, is

$$\frac{\partial}{\partial x}\ln g(x) = -\frac{2\alpha}{x} + \frac{2\alpha}{\theta x^{2\alpha+1}} = -\frac{2\alpha}{\theta x^{2\alpha+1}}(\theta x^{2\alpha}-1) = 0,$$

then the critical value is

$$\theta x^{2\alpha} - 1 = 0 \Longrightarrow x = \left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}.$$

By using the second derivatives test, the second derivatives is

$$\frac{\partial^2}{\partial x^2}\ln g(x) = \frac{2\alpha}{x^2} - \frac{2\alpha(2\alpha+1)}{\theta x^{2\alpha+2}} = \frac{2\alpha}{\theta x^2}\left[1 - \frac{2\alpha+1}{\theta x^{2\alpha}}\right] = 0.$$

At $x = \left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}$, then

$$\left[\frac{\partial^2}{\partial x^2}\ln g(x)\right]_{x=\left(\frac{1}{\theta}\right)^{1/2\alpha}} = \frac{2\alpha}{x^2}\left[1-(2\alpha+1)\right] = \frac{2\alpha}{x^2}\left[-2\alpha\right] < 0.$$

Then the pdf for LBPIR distribution has the maximum value at $x = \left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}$. Therefore, the mode for LBPIR distribution is $\left(\frac{1}{\theta}\right)^{\frac{1}{2\alpha}}$.

Theorem 3.2. *The MGF for the LBPIR distribution is given by*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{\Gamma\left(1 - \frac{(r+1)}{2\alpha}\right)}{\Gamma\left(1 - \frac{1}{2\alpha}\right)} \frac{t^r}{r!}. \tag{10}$$

Proof. The MGF is obtained by the following formula,

$$M_X(t) = E[e^{Xt}] = \int_{-\infty}^{\infty} e^{xt} g(x)dx = \int_0^{\infty} e^{xt} \frac{xf(x)}{E(X)}dx.$$

By using the expansion for the exponential distribution $e^{xt} = \sum_{r=0}^{\infty} \frac{(xt)^r}{r!}$, then

$$M_X(t) = \sum_{r=0}^{\infty} \int_{-\infty}^{\infty} \frac{x^{r+1}f(x)}{E(X)} \frac{t^r}{r!}dx = \sum_{r=0}^{\infty} \left[\frac{\mu'_{r+1}}{\mu'_1}\right]_{PRID} \frac{t^r}{r!}. \tag{11}$$

Substituting from (3) into (11), then

$$M_X(t) = \sum_{r=0}^{\infty} \frac{\Gamma\left(1 - \frac{r+1}{2\alpha}\right)}{\Gamma\left(1 - \frac{1}{2\alpha}\right)} \frac{t^r}{r!}.$$

This completes the proof. □

The following table shows some statistical measures for LBPIR distribution.

From Table 1,

(1) the LBPIRD is positive skewness and has unimodal.
(2) The mean, mode, variance, skewness and kurtosis are decreasing when α is increasing.

Theorem 3.3. *The characteristic function for the LBPIRD can be obtained as follows*

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{\Gamma\left(1 - \frac{r+1}{2\alpha}\right)}{\Gamma\left(1 - \frac{1}{2\alpha}\right)} \frac{(it)^r}{r!}. \tag{12}$$

TABLE 1. The statistical measures for LBPIRD(α, θ), when $\theta = 0.5$.

α	Mean	Mode	Var(X)	Sk	Ku
3.0	1.200	1.122	0.131	3.685	51.656
3.5	1.154	1.104	0.078	2.917	26.259
4.0	1.125	1.091	0.052	2.516	18.519
4.5	1.104	1.080	0.037	2.266	14.873
5.0	1.089	1.072	0.028	2.095	12.777
5.5	1.078	1.065	0.022	1.970	11.425
6.0	1.069	1.059	0.017	1.874	10.483
6.5	1.062	1.055	0.014	1.799	9.792
7.0	1.056	1.051	0.012	1.737	9.263
7.5	1.052	1.047	0.010	1.687	8.846
8.0	1.048	1.044	0.0086	1.644	8.510

Proof. we have

$$\phi_X(t) = E[e^{iXt}] = \int_{-\infty}^{\infty} e^{ixt} g(x)dx = \int_0^{\infty} e^{ixt} \frac{xf(x)}{E(X)} dx.$$

By using the expansion for the exponential distribution $e^{ixt} = \sum_{r=0}^{\infty} \frac{(ixt)^r}{r!}$, then

$$\phi_X(t) = \sum_{r=0}^{\infty} \int_{-\infty}^{\infty} \frac{x^{r+1} f(x)}{E(X)} \frac{(it)^r}{r!} dx = \sum_{r=0}^{\infty} \left[\frac{\mu'_{r+1}}{\mu'_1} \right]_{PRID} \frac{(it)^r}{r!}. \tag{13}$$

Substituting from (3) into (13), we have

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{\Gamma\left(1 - \frac{r+1}{2\alpha}\right)}{\Gamma\left(1 - \frac{1}{2\alpha}\right)} \frac{(it)^r}{r!}.$$

Hence the Theorem 3.3 is proved. □

3.2. Measure of uncertainty. The Rényi and Tsallis entropy from LBPIR distribution are as follows, see [30].

$$R_r(X) = \frac{1}{1-r} \log \int_0^{\infty} f^r(x)dx = \log\left(\frac{2\alpha}{r}\theta^{\frac{1}{2\alpha}}\right) - \frac{2\alpha - 1}{2\alpha(1-r)} \log(r) + \log\left[\frac{\Gamma\left(r - \frac{1}{2\alpha}\right)}{\Gamma\left(1 - \frac{1}{2\alpha}\right)}\right]$$

and

$$T_r(X) = \frac{1}{1-r}\left[\int_0^{\infty} f^r(x)dx - 1\right] = \frac{1}{1-r}\left[\left(\frac{2\alpha\theta^{\frac{1}{2\alpha}}}{r}\right)^{r-1} \frac{\Gamma\left(r - \frac{1}{2\alpha}\right)}{r^{1-\frac{1}{2\alpha}}\Gamma\left(1 - \frac{1}{2\alpha}\right)} - 1\right],$$

where $r > 2\alpha, r \neq 1$.

3.3. Inequality measures. Bonferroni and Lorenz curves are defined as follows.

$$B(P) = \frac{1}{P\mu}\int_0^q xf(x)dx, \quad \text{and} \quad L(P) = \frac{1}{\mu}\int_0^q xf(x)dx,$$

where $\mu = E(X)$ and $q = F^{-1}(p)$, see Bonferroni [4].
Therefore, Bonferroni and Lorenz curves from LBPIRD are related as

$$B(P) = \frac{1}{P\theta^{\frac{1}{2\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)}\gamma\left(1-\frac{1}{2\alpha}, \frac{1}{\theta}q^{-2\alpha}\right),$$

and

$$L(P) = \frac{1}{\theta^{\frac{1}{2\alpha}}\Gamma\left(1-\frac{1}{\alpha}\right)}\gamma\left(1-\frac{1}{2\alpha}, \frac{1}{\theta}q^{-2\alpha}\right).$$

4. Parameters Estimation

The parameters estimation of the LBPIRD has been discussed. The maximum likelihood estimation (MLE) based on a complete sample is used.

4.1. Maximum likelihood estimators. Let $x_1, x_2, \cdots, x_n$ be a complete sample of size n from the LBPIRD(α, θ). The likelihood function is given by

$$L(\alpha, \theta|x) = \prod_{i=1}^{n} g(x_i; \alpha, \theta) = \prod_{i=1}^{n} \frac{2\alpha\theta^{\frac{1}{2\alpha}-1}}{\Gamma\left(1-\frac{1}{2\alpha}\right)} x^{-2\alpha} \exp\left\{-frac1\theta x^{-2\alpha}\right\}.$$

The log-likelihood function is

$$\ell = n\ln(2\alpha) + n\left(\frac{1}{2\alpha}-1\right)\ln(\theta) - n\ln\Gamma\left(1-\frac{1}{2\alpha}\right) - 2\alpha\sum_{i=1}^{n}\ln(x_i) - \frac{1}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}. \tag{14}$$

Differentiating the above equation with respect to α, θ and setting the derivatives equal to zero.

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \frac{n}{2\alpha^2}\ln(\theta) - \frac{n}{2\alpha^2}\psi\left(1-\frac{1}{2\alpha}\right) - 2\sum_{i=1}^{n}\ln(x_i) + \frac{2}{\theta}\sum_{i=1}^{n}x_i^{-2\alpha}\ln(x_i), \tag{15}$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta}\left(\frac{1}{2\alpha}-1\right) + \frac{1}{\theta^2}\sum_{i=1}^{n}x_i^{-2\alpha}, \tag{16}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$.

4.2. Asymptotic confidence bounds. Since, we could not derive an analytic solution to the MLEs, we could not derive explicit confidence intervals for the parameters. The normal approximation of the sampling distribution of the MLEs can be used to derive asymptotic intervals of the parameters.

It is known that the MLE of $\Theta = (\alpha, \theta)$, say $\hat{\Theta}$, approximately follows normal distribution with mean of Θ and variance covariance V, see [15]. That is, $\hat{\Theta} = (\hat{\alpha}, \hat{\theta}) \sim N_2(\Theta, V)$, where V is given

$$V = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \theta} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ell}{\partial \theta^2} \end{pmatrix}^{-1}, \tag{17}$$

where

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \frac{n}{\alpha^3}\ln(\theta) + \frac{n}{\alpha^3}\psi\left(1 - \frac{1}{2\alpha}\right) - \frac{n}{4\alpha^4}\psi'\left(1 - \frac{1}{2\alpha}\right) - \frac{4}{\theta}\sum_{i=1}^{n} x_i^{-2\alpha}[\ln(x_i)]^2, \tag{18}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = -\frac{n}{2\alpha^2 \theta} - \frac{2}{\theta^2}\sum_{i=1}^{n} x_i^{-2\alpha}\ln(x_i), \tag{19}$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{n}{\theta^2}\left(\frac{1}{2\alpha} - 1\right) - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i^{-2\alpha}. \tag{20}$$

A $100(1-\nu)\%$ confidence interval for $\Theta_j,\ j = 1, 2$, can be approximated by $\hat{\Theta}_j \pm z_{\nu/2} SE_j(\hat{\Theta}_j)$, where $z_{\nu/2}$ is the upper $100\frac{\nu}{2}th$ percentile of the standard normal distribution and $SE_j(\hat{\Theta}_j)$ is the square root of the jth element in the diagonal of V.

5. Numerical Results

In this section, two real data sets are analyzed to verify the flexibility and applicability of the LBPIR distribution. The certain analytical measures have been used to identify the best fit of competitive distribution. In this regard, The Akaike, Akaike corrected, Bayesian and Hannan–Quinn Information Criterion values are used to decide the most appropriate once.

Data set 1. The following data consists of 20 observations of patients receiving an analgesic, [12]: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

The MLEs, ℓ, AIC, AICC, BIC and HQIC for the four models, IRD, PIRD, LBIRD and LBPIRD are presented in Table 2.

From Table 2, LBPIRD has lower AIC, AICC, BIC and HQIC values than other competitor models. So, we come to an end that LBPIRD leads to better fit over the IRD, PIRD and LBIRD.

Table 2. The MLEs, ℓ, AIC, AICC, BIC and HQIC.

Model	$\hat{\alpha}$	$\hat{\theta}$	ℓ	AIC	AICC	BIC	HQIC
IRD	0.362	–	-692.933	1.388×10^3	1.388×10^3	$1.389\ \times10^3$	1.388×10^3
PIRD	0.045	0.949	-324.02	652.04	652.393	654.032	652.429
LBIRD	–	0.724	-27.931	57.862	57.973	58.858	652.429
LBPIRD	2.333	0.171	-15.458	34.916	35.269	36.908	35.305

The variance-covariance is

$$V = \begin{pmatrix} 0.082 & -5.214\times10^{-3} \\ -5.214\times10^{-3} & 1.259\times10^{-3} \end{pmatrix}$$

The 95% confidence interval for α and θ are (1.77312, 2.8933) and (0.10099, 0.24008). To show that the likelihood equations have a unique solution, the profile of the log-likelihood function of α and θ, are plotted in Figure 5.

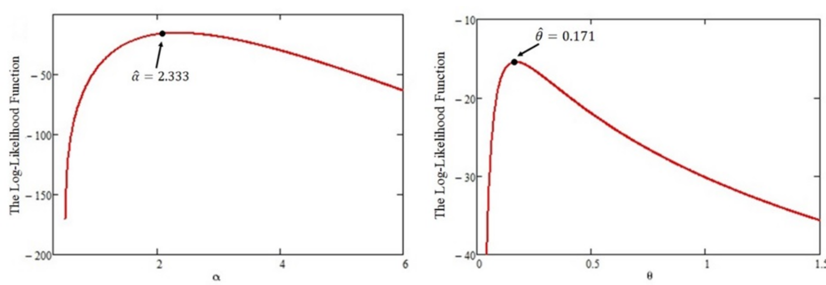

Figure 5. The profile of the log-likelihood function of α and θ.

Data set 2. The time to failure of turbocharger (103h) of one engine is, [33], 1.6, 8.4, 8.1, 7.9, 3.5, 2, 8.4, 4.8, 3.9, 2.6, 8.5, 5.4, 5, 4.5, 3, 6, 5.6, 5.1, 4.6, 6.5, 6.1, 5.8, 5.3, 7, 6.5, 6.3, 6, 7.3, 7.1, 6.7, 8.7, 7.7, 7.3, 7.3, 8.8, 8, 7.8, 7.7, 9, 8.3. Table 3, contains the MLEs, ℓ, AIC, AICC, BIC and HQIC for the four models, IRD, PIRD, LBIRD and LBPIRD.

Table 3. The MLEs, ℓ, AIC, AICC, BIC and HQIC.

Model	$\hat{\alpha}$	$\hat{\theta}$	ℓ	AIC	AICC	BIC	HQIC
IRD	–	0.047	-8.37×10^3	1.674×10^4	1.674×10^4	1.674×10^4	1.674×10^4
PIRD	7.256×10^{-3}	0.975	-3.076×10^3	6.157×10^3	6.157×10^3	6.16×10^3	6.158×10^3
LBIRD	–	0.093	-109.046	220.092	220.144	221.78	6.158×10^3
LBPIRD	1.267	0.042	-104.809	213.617	213.78	216.995	214.839

It has been observed that from Table 3, the LBPIRD has smaller values of AIC, AICC, BIC and HQIC as compared to the IRD, PIRD and LBIRD. Hence, we

can conclude that our proposed model is the best.
The variance-covariance is

$$V = \begin{pmatrix} 5.661 \times 10^{-3} & -3.115 \times 10^{-4} \\ -3.115 \times 10^{-4} & 5.326 \times 10^{-5} \end{pmatrix}$$

The 95% confidence interval for α and θ are (1.11905, 1.41399) and (0.02751, 0.05612). Figure 6 shows that the likelihood equations have a unique solution of α and θ.

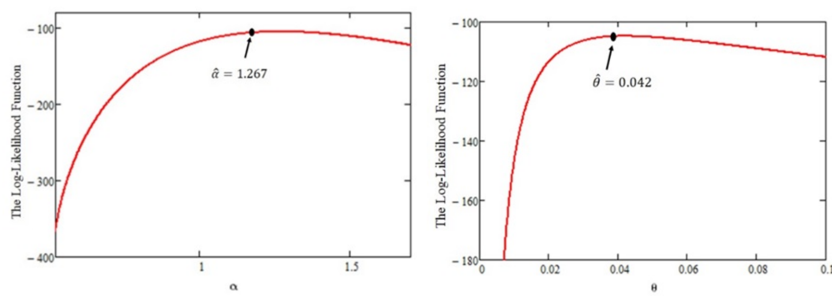

FIGURE 6. The profile of the log-likelihood function of α and θ.

6. Conclusion

The proposed distribution performs better as compared to the other model of Rayleigh distributions.

Acknowledgement : The researchers wish to extend their sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post-Publishing Program 1.

REFERENCES

1. A. Afaq, S.P. Ahmad and A. Ahmed, *Length–biased weighted Lomax distribution statistical properties and application*, Pakistan Journal of Statistics and Operation Research **12** (2016), 245-255.
2. M. Ajami and S.M.A. Jahanshahi, *Parameter estimation in weighted Rayleigh distribution*, Journal of Modern Applied Statistical Methods **16** (2017), 256-276.
3. A.K. Al-Khadim and A.N. Hussein, *New proposed length-biased weighted exponential and Rayleigh distribution with application*, Mathematical Theory and Modeling **4** (2014), 137-152.
4. C. Bonferroni, *Elementi di Statistica Generale*, Firenze: Seeber, 1930.
5. G.E.P. Box and D.R. Cox, *An analysis of transformations*, Journal of the Royal Statistical Society, Series B **26** (1964), 211-252.
6. D.R. Cox, *Renewal Theory*, Barnes & Noble, New York, 1962.
7. K.K. Das and T.D. Roy, *Applicability of length biased generalized Rayleigh distribution*, Advances in Applied Science Research **2** (2011), 320-327.

8. K.K. Das and T.D. Roy, *On some length-biased weighted Weibull distribution*, Advances in Applied Science Research **2** (2011), 465-475.
9. K. Fatima and S.P. Ahmad, *Weighted inverse Rayleigh distribution*, International Journal of Statistics and Systems **12** (2017), 119-137.
10. R.C. Gupta and J.P. Keating, *Relations for reliability measures under length biased sampling*, Scandanavian Journal of Statistics **13** (1985), 49-56.
11. P.L. Gupta and R.C. Tripathi, *Effect of length-biased sampling on the modeling error*, Communications Statistics-Theory and Methods **19** (1990), 1483-1491.
12. A.J. Gross and V.A. Clark, *Survival Distributions: Reliability Applications in the Biometrical Sciences*, John Wiley, New York, 1975.
13. J. Kersey and B.O. Oluyede, *Theoretical properties of the length-biased inverse Weibull distribution*, Mathematical Sciences Publishers **5** (2012), 379-392.
14. R. Khattree, *Characterization of inverse-Gaussian and gamma distributions through their length-biased distributions*, IEEE Transactions on Reliability **38** (1989), 610-611.
15. J.F. Lawless, *Statistical Models and Methods for Lifetime Data*, 2nd Edition, Wiley, Canada, 2003.
16. K.A. Mir, A. Ahmed and J.A. Reshi, *Structural properties of length biased Beta distribution of first kind*, American Journal of Engineering Research **2** (2013), 01-06.
17. K. Modi, *Length-biased weighted Maxwell distribution*, Pakistan Journal of Statistics and Operation Research **11** (2015), 465-472.
18. S. Mudasir and S.P. Ahmad, *Structural properties of length liased Nakagami distribution*, International Journal of Modern Mathematical Sciences **13** (2015), 217-227.
19. S. Mudasir and S.P. Ahmad, *Characterization and estimation of the length biased Nakagami distribution*, Pakistan Journal of Statistics and Operation Research **14** (2018), 697-715.
20. N. Nanuwong and W. Bodhisuwan, *Length biased beta Pareto distribution and its structural properties with application*, Journal of Mathematics and Statistics **10** (2014), 49-57.
21. J.M.A. Nashaat, *Estimation of two parameter powered inverse Rayleigh distribution*, Pakistan Journal of Statistics **36** (2020), 117-133.
22. B.O. Oluyede, *On inequalities and selection of experiments for length-biased distributions*, Probability in the Engineering and Informational Sciences **13** (1999), 169-185.
23. Z. Praveen and M. Ahmad, *Some properties of size - biased weighted Weibull distribution*, International Journal of Advanced and Applied Sciences **5** (2018), 92-98.
24. M.V. Ratnaparkhi and U.V. Naik-Nimbalkar, *The length biased lognormal distribution and its application in the analysis of data from oil field exploration studies*, Journal of Modern Applied Statistical Methods **11** (2012), 225-260.
25. J.W.S. Rayleigh, *On the resultant of a large number of vibrations of the same pitch and of arbitrary phase*, Philosophical Magazine, 5th Series **10** (1880), 73-78.
26. A. Saghir, S. Tazeem and I. Ahmad, *The length-biased weighted exponentiated inverted Weibull distribution*, Cogent Mathematics **3** (2016), DOI: 10.1080/23311835.2016.1267299.
27. A. Saghir, A. Khadim and Z. Lin, *The Maxwell -length-biased distribution: Properties and estimation*, Journal of Statistical Theory and Practice **11** (2017), 26-40.
28. P. Seenoi, T. Supapakorn and W. Bodhisuwan, *The length-biased exponentiated inverted Weibull distribution*, International Journal of Pure and Applied Mathematics **92** (2014), 191-206.
29. S.A. Shaban and N.A. Boudrissa, *The Weibull length-biased distribution: Properties and estimation*, Interstat, (2007), http://interstat.statjournals.net/YEAR/2007/articles/0701002.pdf
30. M.S. Tabass, G.R.M. Borzadaran and M. Amini, *Rényi entropy in continuous case is not the limit of discrete case*, Mathematical Sciences and Applications E-Notes **4** (2016), 113-117.
31. V.N. Trayer, *Proceedings of the Academy of Science Belarus*, USSR, 1964.

32. V.G.H. Voda, *On the inverse Rayleigh distributed random variable*, Rep. Statistics Application and Research, JUSE **19** (1972), 13-21.
33. K. Xu, M. Xie, L.C. Tang, and S.L. Ho, *Application of neural networks in forecasting engine system reliability*, Applied Soft Computing **2** (2003), 255-268.

Abdelfattah Mustafa received M.Sc. from Mansoura University and Ph.D. at Mansoura University. Since 1997 he has been at Mansoura University. His research interests include reliability engineering and estimation theory.

Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
E-mail: amelsayed@mans.edu.eg

M.I. Khan received M.Sc. and Ph.D. from Aligarh Muslim University, India. Currently, he is working as an assistant professor at Islamic University of Madinah, Kingdom od Saudi Arabia, since 2014. His research has been focused in the area of mathematical statistics and ordered random variables.

Mathematics Department, Faculty of Science, Islamic University of Madinah, Kingdom of Saudi Arabia.
e-mail: izhar.stats@gmail.com