

## MULTIPLIERS FOR OPERATOR-VALUED BESSEL SEQUENCES AND GENERALIZED HILBERT-SCHMIDT CLASSES<sup>†</sup>

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**ABSTRACT.** In 1960, Schatten studied operators of the form  $\sum_{n=1}^{\infty} \lambda_n(x_n \otimes \overline{y_n})$ , where  $\{x_n\}_n$  and  $\{y_n\}_n$  are orthonormal sequences in a Hilbert space, and  $\{\lambda_n\}_n \in \ell^{\infty}(\mathbb{N})$ . Balazs generalized some of the results of Schatten in 2007. In this paper, we further generalize results of Balazs by studying the operators of the form  $\sum_{n=1}^{\infty} \lambda_n(A_n^*x_n \otimes \overline{B_n^*y_n})$ , where  $\{A_n\}_n$  and  $\{B_n\}_n$  are operator-valued Bessel sequences,  $\{x_n\}_n$  and  $\{y_n\}_n$  are sequences in the Hilbert space such that  $\{\|x_n\|\|y_n\|\}_n \in \ell^{\infty}(\mathbb{N})$ . We also generalize the class of Hilbert-Schmidt operators studied by Balazs.

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### 1. Introduction

In 1946, Gabor [9] introduced a method for reconstructing functions (signals) using a family of elementary functions. Later in 1952, Duffin and Schaeffer [8] presented a similar tool in the context of nonharmonic Fourier series and this is the starting point of frame theory. After some decades, Daubechies, Grossmann and Meyer [10] in 1986 announced formally the definition of frame in the abstract Hilbert spaces. After their work, the theory of frames began to be studied widely and deeply.

Frames can be viewed as redundant bases which are generalization of orthonormal bases. They provide robust, stable and usually non-unique representations of vectors in a Hilbert space. There have been a focus of study for more than three decades in applications where redundancy plays a vital and useful role. The redundancy and flexibility offered by frames has spurred their

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applications in a variety of areas throughout mathematics and engineering, such as operator theory [4], harmonic analysis [13], pseudo-differential operators [14], quantum computing [29], signal and image processing [3], wireless communication [20], and so on.

Let  $\mathcal{H}, \mathcal{H}_0$  be Hilbert spaces and  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  be the Banach space of all bounded linear operators from  $\mathcal{H}$  to  $\mathcal{H}_0$ . We write  $\mathcal{B}(\mathcal{H}, \mathcal{H})$  as  $\mathcal{B}(\mathcal{H})$ . The field of real scalars  $\mathbb{R}$  or complex scalars  $\mathbb{C}$  is denoted by  $\mathbb{K}$ . We use the following notation used in [24] for  $x, y \in \mathcal{H}$ , define

$$x \otimes \bar{y} : \mathcal{H} \ni h \mapsto \langle h, y \rangle x \in \mathcal{H}.$$

In Chapter 1 of [24], Schatten has made a detailed study of operators of the form

$$\sum_{n=1}^{\infty} \lambda_n (x_n \otimes \bar{y_n}), \quad (1)$$

where  $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ ,  $\{x_n\}_n$  and  $\{y_n\}_n$  are orthonormal sequences in a Hilbert space. It is shown that every compact operator can be written of the form (1) with all  $\lambda_n \geq 0$  and  $\lambda_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hilbert-Schmidt class operators  $\mathcal{S}(\mathcal{H})$  and trace class operators  $\mathcal{T}(\mathcal{H})$  are studied by Schatten and von Neumann in their joint paper [25]. It is shown that both  $\mathcal{S}(\mathcal{H})$  and  $\mathcal{T}(\mathcal{H})$  admit norms under which they are complete and they are two-sided star closed ideals in  $\mathcal{B}(\mathcal{H})$ . It is further proved that the norm on  $\mathcal{S}(\mathcal{H})$  comes from an inner product and hence  $\mathcal{S}(\mathcal{H})$  is a Hilbert space.

Balazs [18] generalized results for operators of the form given in (1) by allowing Bessel sequences in place of orthonormal sequences (we refer [16] for Bessel sequence and its properties). The first question while considering the operator in (1) is its existence. Given  $\{x_n\}_n$  and  $\{y_n\}_n$  are orthonormal, Schatten considered the following equality for  $n, m \in \mathbb{N}$  and  $h \in \mathcal{H}$ ,

$$\left\| \sum_{k=n}^m \lambda_k (x_k \otimes \bar{y_k}) h \right\|^2 = \sum_{k=n}^m \sum_{r=n}^m \lambda_k \bar{\lambda_r} \langle h, y_k \rangle \langle y_r, h \rangle \langle x_k, x_r \rangle = \sum_{k=n}^m |\lambda_k|^2 |\langle h, y_k \rangle|^2.$$

The existence of the operator given in (1) is guaranteed from Bessel's inequality. This idea may not work when we drop the condition of orthonormality. Balazs realized that the operator in (1) exists and it can be written as the composition of three bounded linear operators (Theorem 6.1 in [18]) acting on Hilbert spaces when  $\{x_n\}_n$  and  $\{y_n\}_n$  are Bessel sequences. Balazs and Stoeva studied invertibility of these operators in [5–7, 17, 27].

In Section 2 we generalize the work of Balazs by considering operator-valued Bessel sequence and we derive various properties of it and continuity of multipliers. The class of Hilbert-Schmidt operators which we define in Section 3 will depend upon a conjugate-linear isometry, an operator-valued orthonormal basis and a sequence in a Hilbert space. We show that generalized Hilbert-Schmidt class admits a semi-norm and it is a two-sided star closed ideal in  $\mathcal{B}(\mathcal{H})$ .

We now give some definitions and results which will be used in the sequel.

**Definition 1.1.** [8] A sequence  $\{x_n\}_n$  in  $\mathcal{H}$  is said to be a frame if there exist  $a, b > 0$  such that

$$a\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, x_n \rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

Constants  $a$  and  $b$  are called frame bounds. If  $a$  is allowed to take the value 0, then  $\{x_n\}_n$  is called as a Bessel sequence with bound  $b$ .

**Definition 1.2.** [28] A collection  $\{A_n\}_n$  in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  is said to be an operator-valued Bessel sequence (g-Bessel sequence) with bound  $b > 0$  if

$$\sum_{n=1}^{\infty} \|A_n h\|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

It can be seen easily that if  $\{A_n\}_n$  is an operator-valued Bessel sequence in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  with bound  $b$ , then  $\|A_n\| \leq \sqrt{b}, \forall n \in \mathbb{N}$ . In fact,  $\|A_n h\|^2 \leq \sum_{k=1}^{\infty} \|A_k h\|^2 \leq b\|h\|^2, \forall h \in \mathcal{H}, \forall n \in \mathbb{N}$ .

**Definition 1.3.** [28] A collection  $\{A_n\}_n$  in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  is said to be

- (i) an operator-valued Riesz basis (g-Riesz basis) if  $\{h \in \mathcal{H} : A_n h = 0, \forall n \in \mathbb{N}\} = \{0\}$  and there exist  $a, b > 0$  such that for every finite  $\mathbb{S} \subseteq \mathbb{N}$ ,

$$a \sum_{n \in \mathbb{S}} \|y_n\|^2 \leq \left\| \sum_{n \in \mathbb{S}} A_n^* y_n \right\|^2 \leq b \sum_{n \in \mathbb{S}} \|y_n\|^2, \quad \forall y_n \in \mathcal{H}_0.$$

- (ii) an operator-valued orthonormal basis (g-basis) if  $\langle A_n^* y, A_m^* z \rangle = \delta_{n,m} \langle y, z \rangle$ ,  $\forall n, m \in \mathbb{N}, \forall y, z \in \mathcal{H}_0$ , and  $\sum_{n=1}^{\infty} \|A_n h\|^2 = \|h\|^2, \forall h \in \mathcal{H}$ .

We refer the reader to [28] for examples and properties of operator-valued orthonormal bases and Riesz bases.

**Definition 1.4.** [28] A collection  $\{A_n\}_n$  in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  is said to be an operator-valued

- (i) orthogonal sequence if  $\langle A_n^* y, A_m^* z \rangle = \delta_{n,m} \langle y, z \rangle, \forall n, m \in \mathbb{N}, \forall y, z \in \mathcal{H}_0$ .
- (ii) orthonormal sequence if it is orthogonal and  $\sum_{n=1}^{\infty} \|A_n h\|^2 \leq \|h\|^2, \forall h \in \mathcal{H}$ .

We derive the following two theorems which we use in the paper.

**Theorem 1.5.** Let  $\{A_n\}_n$  and  $\{B_n\}_n$  be operator-valued orthonormal bases in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ . Then there exists a unique unitary  $U \in \mathcal{B}(\mathcal{H})$  such that  $A_n = B_n U$ , for all  $n \in \mathbb{N}$ .

*Proof.* (Existence) Define  $U := \sum_{n=1}^{\infty} B_n^* A_n$ . This operator exists in the strong-operator topology, since for every  $n, m \in \mathbb{N}$  with  $n < m$  and  $h \in \mathcal{H}$ ,

$$\left\| \sum_{j=n}^m B_j^* A_j h \right\|^2 = \left\langle \sum_{j=n}^m B_j^* A_j h, \sum_{k=n}^m B_k^* A_k h \right\rangle$$

$$= \sum_{j=n}^m \left\langle A_j h, B_j \left( \sum_{k=n}^m B_k^* A_k h \right) \right\rangle = \sum_{j=1}^n \|A_j h\|^2.$$

Now  $B_n U = B_n (\sum_{m=1}^{\infty} B_m^* A_m) = A_n, \forall n \in \mathbb{N}$ . We now show that  $U$  is unitary. For,

$$UU^* = \left( \sum_{n=1}^{\infty} B_n^* A_n \right) \left( \sum_{m=1}^{\infty} A_m^* B_m \right) = \sum_{n=1}^{\infty} B_n^* \left( \sum_{m=1}^{\infty} A_n A_m^* B_m \right) = \sum_{n=1}^{\infty} B_n^* B_n = I_{\mathcal{H}}$$

and

$$U^* U = \left( \sum_{n=1}^{\infty} A_n^* B_n \right) \left( \sum_{m=1}^{\infty} B_m^* A_m \right) = \sum_{n=1}^{\infty} A_n^* \left( \sum_{m=1}^{\infty} B_n B_m^* A_m \right) = \sum_{n=1}^{\infty} A_n^* A_n = I_{\mathcal{H}}.$$

(Uniqueness) Let  $W \in \mathcal{B}(\mathcal{H})$  satisfy  $B_n U = B_n W = A_n, \forall n \in \mathbb{N}$ . Then  $U = I_{\mathcal{H}} U = \sum_{n=1}^{\infty} B_n^* (B_n U) = \sum_{n=1}^{\infty} B_n^* (B_n W) = W$ .  $\square$

**Theorem 1.6.** *Let  $\{F_n\}_n$  be an operator-valued orthonormal basis in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  and  $\{A_n\}_n$  be an operator-valued Riesz basis in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ . Then there exists a unique invertible  $T \in \mathcal{B}(\mathcal{H})$  such that  $A_n = F_n T$ , for all  $n \in \mathbb{N}$ .*

*Proof.* (Existence) From the definition of operator-valued Riesz basis, there exists an operator-valued orthonormal basis  $\{G_n\}_n$  in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  and invertible  $R : \mathcal{H} \rightarrow \mathcal{H}$  such that  $A_n = G_n R, \forall n \in \mathbb{N}$ . Define  $T := \sum_{n=1}^{\infty} F_n^* G_n R$ . Since  $\{F_n\}_n$  and  $\{G_n\}_n$  are orthonormal bases, similar to the proof of Theorem 1.5,  $T$  is well-defined. Now  $F_n T = G_n R = A_n, \forall n \in \mathbb{N}$ ,

$$\begin{aligned} T \left( R^{-1} \left( \sum_{k=1}^{\infty} G_k^* F_k \right) \right) &= \left( \sum_{n=1}^{\infty} F_n^* G_n R \right) \left( R^{-1} \left( \sum_{k=1}^{\infty} G_k^* F_k \right) \right) = \sum_{n=1}^{\infty} F_n^* \left( \sum_{k=1}^{\infty} G_n G_k^* F_k \right) \\ &= \sum_{n=1}^{\infty} F_n^* F_n = I_{\mathcal{H}} \end{aligned}$$

and

$$\begin{aligned} \left( R^{-1} \left( \sum_{k=1}^{\infty} G_k^* F_k \right) \right) T &= R^{-1} \left( \sum_{n=1}^{\infty} G_n^* F_n \right) \left( \sum_{k=1}^{\infty} F_k^* G_k R \right) \\ &= R^{-1} \left( \sum_{n=1}^{\infty} G_n^* \left( \sum_{k=1}^{\infty} F_n F_k^* G_k R \right) \right) \\ &= R^{-1} \left( \sum_{n=1}^{\infty} G_n^* G_n \right) R = I_{\mathcal{H}}. \end{aligned}$$

(Uniqueness) Let  $W \in \mathcal{B}(\mathcal{H})$  satisfy  $F_n T = F_n W = A_n, \forall n \in \mathbb{N}$ . Then  $T = I_{\mathcal{H}} T = \sum_{n=1}^{\infty} F_n^* (F_n T) = \sum_{n=1}^{\infty} F_n^* (F_n W) = W$ .  $\square$

**Theorem 1.7.** [24] (Polar decomposition) *Let  $A \in \mathcal{B}(\mathcal{H})$ . We denote  $[A] := (A^* A)^{1/2}$ . Then there exists a partial isometry  $W$  whose initial space is  $\overline{[A](\mathcal{H})}$  and the final space is  $\overline{A(\mathcal{H})}$ , satisfying the following conditions.*

- (i)  $A = W[A]$ .
- (ii)  $[A] = W^*A$ .
- (iii)  $A^* = W^*[A^*]$ .
- (iv)  $[A^*] = W[A]W^*$ .

The above decomposition of  $A$  is unique in the following sense: If  $A = W_1B_1$  where  $B_1 \geq 0$  and  $W_1$  is a partial isometry with initial space  $\overline{B_1(\mathcal{H})}$ , then  $B_1 = [A]$  and  $W_1 = W$ . Further, if  $A$  is a finite rank operator, then we can take  $W$  as unitary.

**Definition 1.8.** [25] Let  $\{e_n\}_n$  be an orthonormal basis for  $\mathcal{H}$ . The Hilbert-Schmidt class is defined as

$$\mathcal{S}(\mathcal{H}) := \left\{ A \in \mathcal{B}(\mathcal{H}) : \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty \right\}$$

with the norm of  $A \in \mathcal{S}(\mathcal{H})$  is  $\sigma(A) := \left( \sum_{n=1}^{\infty} \|Ae_n\|^2 \right)^{1/2}$ .

**Definition 1.9.** [25] Let  $\{e_n\}_n$  be an orthonormal basis for  $\mathcal{H}$ . The trace class is defined as

$$\mathcal{T}(\mathcal{H}) := \{AB : A, B \in \mathcal{S}(\mathcal{H})\}$$

with the trace of  $C \in \mathcal{T}(\mathcal{H})$  defined by  $\text{Tr}(C) := \sum_{n=1}^{\infty} \langle Ce_n, e_n \rangle$  and the norm of  $C \in \mathcal{T}(\mathcal{H})$  is given by  $\tau(C) := \text{Tr}([C])$ .

**Definition 1.10.** [1] Let  $\mathcal{X}$  and  $\mathcal{Y}$  be Banach spaces. An operator  $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$  is called nuclear if there exist sequences  $\{f_n\}_n$  in  $\mathcal{X}^*$  and  $\{y_n\}_n$  in  $\mathcal{Y}$  such that  $Tx = \sum_{n=1}^{\infty} f_n(x)y_n, \forall x \in \mathcal{X}$ . In this case, we define the nuclear-norm of  $T$  as

$$\|T\|_{\text{Nuc}} := \inf \left\{ \sum_{n=1}^{\infty} \|f_n\| \|y_n\| : T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \text{ is nuclear with } T(\cdot) = \sum_{n=1}^{\infty} f_n(\cdot)y_n \right\}.$$

## 2. Multipliers for operator-valued Bessel sequences

We first derive a result which allows to define the notion of multiplier for operator-valued Bessel sequences.

**Theorem 2.1.** Let  $\{A_n\}_n$  and  $\{B_n\}_n$  be operator-valued Bessel sequences in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  with bounds  $b, d$ , respectively. If  $\{\lambda_n\}_n \in \ell^{\infty}(\mathbb{N})$ , and  $\{x_n\}_n, \{y_n\}_n$  are sequences in  $\mathcal{H}_0$  such that  $\{\|x_n\| \|y_n\|\}_n \in \ell^{\infty}(\mathbb{N})$ , then the map

$$T : \mathcal{H} \ni h \mapsto \sum_{n=1}^{\infty} \lambda_n (A_n^* x_n \otimes \overline{B_n^* y_n}) h \in \mathcal{H}$$

is a well-defined bounded linear operator with norm at most

$$\sqrt{bd} \|\{\lambda_n\}_n\|_{\infty} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\|.$$

*Proof.* Let  $n, m \in \mathbb{N}$  with  $n \leq m$ . Then for each  $h \in \mathcal{H}$ ,

$$\begin{aligned}
& \left\| \sum_{k=n}^m \lambda_k (A_k^* x_k \otimes \overline{B_k^* y_k}) h \right\| = \sup_{g \in \mathcal{H}, \|g\| \leq 1} \left| \left\langle \sum_{k=n}^m \lambda_k (A_k^* x_k \otimes \overline{B_k^* y_k}) h, g \right\rangle \right| \\
&= \sup_{g \in \mathcal{H}, \|g\| \leq 1} \left| \sum_{k=n}^m \lambda_k \langle h, B_k^* y_k \rangle \langle A_k^* x_k, g \rangle \right| \leq \sup_{g \in \mathcal{H}, \|g\| \leq 1} \sum_{k=n}^m |\lambda_k \langle h, B_k^* y_k \rangle \langle A_k^* x_k, g \rangle| \\
&= \sup_{g \in \mathcal{H}, \|g\| \leq 1} \sum_{k=n}^m |\lambda_k \langle B_k h, y_k \rangle \langle x_k, A_k g \rangle| \leq \sup_{g \in \mathcal{H}, \|g\| \leq 1} \sum_{k=n}^m |\lambda_k| \|B_k h\| \|y_k\| \|x_k\| \|A_k g\| \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \sup_{g \in \mathcal{H}, \|g\| \leq 1} \sum_{k=n}^m \|B_k h\| \|A_k g\| \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \sup_{g \in \mathcal{H}, \|g\| \leq 1} \left( \sum_{k=n}^m \|B_k h\|^2 \right)^{\frac{1}{2}} \left( \sum_{k=n}^m \|A_k g\|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{b} \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \left( \sum_{k=n}^m \|B_k h\|^2 \right)^{\frac{1}{2}} \sup_{g \in \mathcal{H}, \|g\| \leq 1} \|g\| \\
&= \sqrt{b} \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \left( \sum_{k=n}^m \|B_k h\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and  $\sum_{k=1}^{\infty} \|B_k h\|^2$  converges with  $\sum_{k=1}^{\infty} \|B_k h\|^2 \leq d \|h\|^2$ . Hence  $T$  is well-defined linear. Above calculations also show that

$$\|T\| \leq \sqrt{ab} \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\|.$$

□

**Corollary 2.2.** *Let  $\{A_n\}_n$  be an operator-valued orthonormal sequence in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ ,  $\{B_n\}_n$  be an operator-valued Bessel sequence in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  with bound  $b$ . If  $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ ,  $\{x_n\}_n$ ,  $\{y_n\}_n$  are sequences in  $\mathcal{H}_0$  such that  $\{\|x_n\| \|y_n\|\}_n \in \ell^\infty(\mathbb{N})$ , then the map  $T : \mathcal{H} \ni h \mapsto \sum_{n=1}^{\infty} \lambda_n (A_n^* x_n \otimes \overline{B_n^* y_n}) h \in \mathcal{H}$  is a well-defined bounded linear operator with norm at most  $\sqrt{b} \|\{\lambda_n\}_n\|_\infty \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\|$ .*

*Proof.* Even though this is a Corollary of Theorem 2.1, we shall write a direct argument using orthonormality of  $A_n$ 's. Let  $n, m \in \mathbb{N}$  with  $n \leq m$  and  $h \in \mathcal{H}$ . Consider

$$\left\| \sum_{k=n}^m \lambda_k (A_k^* x_k \otimes \overline{B_k^* y_k}) h \right\|^2 = \left\langle \sum_{k=n}^m \lambda_k (A_k^* x_k \otimes \overline{B_k^* y_k}) h, \sum_{r=n}^m \lambda_r (A_r^* x_r \otimes \overline{B_r^* y_r}) h \right\rangle$$

$$\begin{aligned}
&= \left\langle \sum_{k=n}^m \lambda_k \langle h, B_k^* y_k \rangle A_k^* x_k, \sum_{r=n}^m \lambda_r \langle h, B_r^* y_r \rangle A_r^* x_r \right\rangle \\
&= \sum_{k=n}^m \lambda_k \langle h, B_k^* y_k \rangle \sum_{r=n}^m \overline{\lambda_r} \langle B_r^* y_r, h \rangle \langle A_k^* x_k, A_r^* x_r \rangle \\
&= \sum_{k=n}^m \lambda_k \langle h, B_k^* y_k \rangle \overline{\lambda_k} \langle B_k^* y_k, h \rangle \langle x_k, x_k \rangle = \sum_{k=n}^m |\lambda_k|^2 |\langle h, B_k^* y_k \rangle|^2 \|x_k\|^2 \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n|^2 \sum_{k=n}^m |\langle h, B_k^* y_k \rangle|^2 \|x_k\|^2 = \sup_{n \in \mathbb{N}} |\lambda_n|^2 \sum_{k=n}^m |\langle B_k h, y_k \rangle|^2 \|x_k\|^2 \\
&\leq \sup_{n \in \mathbb{N}} |\lambda_n|^2 \sum_{k=n}^m \|B_k h\|^2 \|y_k\|^2 \|x_k\|^2 \leq \sup_{n \in \mathbb{N}} |\lambda_n|^2 \sup_{n \in \mathbb{N}} \|x_n\|^2 \|y_n\|^2 \sum_{k=n}^m \|B_k h\|^2,
\end{aligned}$$

the last sum converges.  $\square$

**Corollary 2.3.** *Theorem 2.1 also holds good if the condition  $\{\|x_n\|\|y_n\|\}_n \in \ell^\infty(\mathbb{N})$  is replaced by the condition that both  $\{\|x_n\|\}_n$  and  $\{\|y_n\|\}_n$  are in  $\ell^\infty(\mathbb{N})$ .*

**Remark 2.1.** Corollary 2.3 can also be derived by using Theorem 6.1 in [18]. In fact, if  $\{\|x_n\|\}_n, \{\|y_n\|\}_n \in \ell^\infty(\mathbb{N})$ , then we observe that both  $\{A_n^* x_n\}_n, \{B_n^* y_n\}_n$  are Bessel sequences. For,

$$\begin{aligned}
&\sum_{n=1}^{\infty} |\langle h, A_n^* x_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle A_n h, x_n \rangle|^2 \\
&\leq \sup_{n \in \mathbb{N}} \|x_n\|^2 \sum_{n=1}^{\infty} \|A_n h\|^2 \leq a \sup_{n \in \mathbb{N}} \|x_n\|^2 \|h\|^2.
\end{aligned}$$

Similarly  $\sum_{n=1}^{\infty} |\langle h, B_n^* y_n \rangle|^2 \leq b \sup_{n \in \mathbb{N}} \|y_n\|^2 \|h\|^2, \forall h \in \mathcal{H}$ . Now (i) in Theorem 6.1 in [18] says that  $T$  is a well-defined bounded linear operator with

$$\|T\| \leq \sqrt{ab} \|\{\lambda_n\}_n\|_{\infty} \sup_{n \in \mathbb{N}} \|x_n\| \sup_{n \in \mathbb{N}} \|y_n\|.$$

A partial converse of Theorem 2.1 is given in Theorem 2.4 (which extends Theorem 1 of Chapter 1 in [24]).

**Theorem 2.4.** *Let  $\{A_n\}_n, \{B_n\}_n$  be operator-valued orthonormal sequences in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ ,  $\{x_n\}_n, \{y_n\}_n$  be sequences in  $\mathcal{H}_0$  such that  $\{\|x_n\|\|y_n\|\}_n \in \ell^\infty(\mathbb{N})$ ,  $\inf_{n \in \mathbb{N}} \|x_n\|\|y_n\| > 0$ , and let  $\{\lambda_n\}_n$  be a sequence of scalars. Then the family*

$$\{\lambda_n \langle h, B_n^* y_n \rangle A_n^* x_n\}_n$$

*is summable for every  $h \in \mathcal{H}$  if and only if  $\{\lambda_n\}_n$  is bounded. Whenever  $\{\lambda_n\}_n$  is bounded, the map  $\mathcal{H} \ni h \mapsto \sum_{n=1}^{\infty} \lambda_n (A_n^* x_n \otimes B_n^* y_n) h \in \mathcal{H}$  is a well-defined bounded linear operator with norm at most  $\|\{\lambda_n\}_n\|_{\infty} \sup_{n \in \mathbb{N}} \|x_n\|\|y_n\|$ .*

*Proof.* ( $\Leftarrow$ ) Follows from Theorem 2.1 (since an orthonormal operator-valued sequence is an operator-valued Bessel sequence). Note that in this case we can take  $a = b = 1$ .

( $\Rightarrow$ ) Let us suppose that  $\{\lambda_n\}_n$  is not bounded. Then we can extract a subsequence  $\{\lambda_{n_k}\}_{k=1}^{\infty}$  from  $\{\lambda_n\}_n$  such that  $|\lambda_{n_k}| \geq k, \forall k \in \mathbb{N}$ . For each  $m \in \mathbb{N}$ , define  $T_m : \mathcal{H} \ni h \mapsto \sum_{k=1}^m \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k} \in \mathcal{H}$ . Also define  $T : \mathcal{H} \ni h \mapsto \sum_{k=1}^{\infty} \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k} \in \mathcal{H}$ . Then for all  $r, s \in \mathbb{N}, r \leq s$ , since  $n_k \geq k, \forall k \in \mathbb{N}$ ,

$$\begin{aligned} & \left\| \sum_{k=r}^s \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k} \right\|^2 \\ &= \left\langle \sum_{k=r}^s \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k}, \sum_{j=r}^s \lambda_{n_j} \langle h, B_{n_j}^* y_{n_j} \rangle A_{n_j}^* x_{n_j} \right\rangle \\ &= \sum_{k=r}^s |\lambda_{n_k}|^2 |\langle h, B_{n_k}^* y_{n_k} \rangle|^2 \|x_{n_k}\|^2 \leq \sum_{k=r}^s |\lambda_k|^2 |\langle h, B_k^* y_k \rangle|^2 \|x_k\|^2 \\ &= \left\| \sum_{k=r}^s \lambda_k \langle h, B_k^* y_k \rangle A_k^* x_k \right\|^2 \end{aligned}$$

which is convergent (by assumption). Therefore  $T$  is a well-defined bounded linear operator. Further, for each fixed  $h \in \mathcal{H}$ ,

$$\begin{aligned} \|T_m h - Th\|^2 &= \left\| \sum_{k=1}^m \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k} - \sum_{k=1}^{\infty} \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k} \right\|^2 \\ &= \left\| \sum_{k=m+1}^{\infty} \lambda_{n_k} \langle h, B_{n_k}^* y_{n_k} \rangle A_{n_k}^* x_{n_k} \right\|^2 = \sum_{k=m+1}^{\infty} |\lambda_{n_k}|^2 |\langle h, B_{n_k}^* y_{n_k} \rangle|^2 \|x_{n_k}\|^2 \\ &\leq \sum_{k=m+1}^{\infty} |\lambda_k|^2 |\langle h, B_k^* y_k \rangle|^2 \|x_k\|^2 = \left\| \sum_{k=m+1}^{\infty} \lambda_k \langle h, B_k^* y_k \rangle A_k^* x_k \right\|^2 \rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Hence  $T_m \rightarrow T$  pointwise which says that  $\{T_m\}_{m=1}^{\infty}$  is bounded pointwise. By Uniform Boundedness Principle, there exists  $R > 0$  such that  $\sup_{m \in \mathbb{N}} \|T_m\| \leq R$ . Next, for each  $m \in \mathbb{N}$ , using orthonormality of  $B_n$ 's,

$$T_m \left( \frac{B_{n_m}^* y_{n_m}}{\|y_{n_m}\|} \right) = \sum_{k=1}^m \lambda_{n_k} \left\langle \frac{B_{n_m}^* y_{n_m}}{\|y_{n_m}\|}, B_{n_k}^* y_{n_k} \right\rangle A_{n_k}^* x_{n_k} = \lambda_{n_m} \|y_{n_m}\| A_{n_m}^* x_{n_m}$$

(the condition  $\inf_{n \in \mathbb{N}} \|x_n\| \|y_n\| > 0$  says that none of the  $y_n$ 's equals to zero). Previous equation along with the observation

$$\left\| \frac{B_{n_m}^* y_{n_m}}{\|y_{n_m}\|} \right\| \leq \frac{\|B_{n_m}^*\| \|y_{n_m}\|}{\|y_{n_m}\|} = \frac{1 \cdot \|y_{n_m}\|}{\|y_{n_m}\|} = 1$$

gives  $R \geq \sup_{m \in \mathbb{N}} \|T_m\| \geq \|T_m\| \geq |\lambda_{n_m}| \|y_{n_m}\| \|A_{n_m}^* x_{n_m}\| = |\lambda_{n_m}| \|y_{n_m}\| \|x_{n_m}\| \geq m \|y_{n_m}\| \|x_{n_m}\| \geq m \inf_{n \in \mathbb{N}} \|y_n\| \|x_n\|, \forall m \in \mathbb{N} \Rightarrow m \leq \frac{R}{\inf_{n \in \mathbb{N}} \|y_n\| \|x_n\|}, \forall m \in \mathbb{N}$

which is a contradiction.  $\square$

**Corollary 2.5.** *Theorem 2.4 holds good if the condition  $\inf_{n \in \mathbb{N}} \|x_n\| \|y_n\| > 0$  is replaced by one of the following conditions.*

- (i)  $\inf_{n \in \mathbb{N}} |\langle x_n, y_n \rangle| > 0$ .
- (ii)  $\inf_{n \in \mathbb{N}} \|x_n\| \inf_{n \in \mathbb{N}} \|y_n\| > 0$ .

*Proof.* We apply Theorem 2.4 by noting

$$\begin{aligned} \inf_{n \in \mathbb{N}} \|x_n\| \|y_n\| &\geq \inf_{n \in \mathbb{N}} |\langle x_n, y_n \rangle| > 0, \\ \inf_{n \in \mathbb{N}} \|x_n\| \|y_n\| &\geq \inf_{n \in \mathbb{N}} \|x_n\| \inf_{n \in \mathbb{N}} \|y_n\| > 0. \end{aligned}$$

$\square$

Using Theorem 2.1 we define the notion of multiplier for operator-valued Bessel sequences as follows.

**Definition 2.6.** Let  $\{A_n\}_n$  and  $\{B_n\}_n$  be operator-valued Bessel sequences in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ . Let  $\{x_n\}_n$  and  $\{y_n\}_n$  be sequences in  $\mathcal{H}_0$  such that  $\{\|x_n\| \|y_n\|\}_n \in \ell^\infty(\mathbb{N})$ . For  $\{\lambda_n\}_n \in \ell^\infty(\mathbb{N})$ , the multiplier for  $\{A_n\}_n$  and  $\{B_n\}_n$  is defined as the operator

$$M_{\lambda, A, B, x, y} := \sum_{n=1}^{\infty} \lambda_n (A_n^* x_n \otimes \overline{B_n^* y_n}). \quad (2)$$

**Remark 2.2.** Let  $\mathcal{H}_0 = \mathbb{K}$  and  $\{e_n\}_n, \{f_n\}_n$  be Bessel (resp. orthonormal) sequences in  $\mathcal{H}$ . Define  $x_n := 1, y_n := 1, A_n : \mathcal{H} \ni h \mapsto \langle h, e_n \rangle \in \mathbb{K}, B_n : \mathcal{H} \ni h \mapsto \langle h, f_n \rangle \in \mathbb{K}, \forall n \in \mathbb{N}$ . Then  $\sum_{n=1}^{\infty} \lambda_n (A_n^* x_n \otimes \overline{B_n^* y_n}) = \sum_{n=1}^{\infty} \lambda_n (e_n \otimes \overline{f_n})$ . Thus Definition 2.6 reduces to the operator of the form  $\sum_{n=1}^{\infty} \lambda_n (e_n \otimes \overline{f_n})$ , considered by Balazs [18] (resp. Schatten and von Neumann [24]).

Following theorem collects various properties of  $M_{\lambda, A, B, x, y}$ .

**Theorem 2.7.** *Let  $\{A_n\}_n, \{B_n\}_n, \{C_n\}_n, \{D_n\}_n$  be operator-valued Bessel sequences in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  with bounds  $a, b, c, d$ , respectively,  $\{\lambda_n\}_n, \{\mu_n\}_n \in \ell^\infty(\mathbb{N})$ ,  $\alpha \in \mathbb{K}$  and let  $\{x_n\}_n, \{y_n\}_n, \{z_n\}_n, \{v_n\}_n, \{w_n\}_n$  be sequences in  $\mathcal{H}_0$  such that  $\{\|x_n\| \|y_n\|\}_n, \{\|y_n\| \|z_n\|\}_n, \{\|x_n\| \|z_n\|\}_n, \{\|z_n\| \|w_n\|\}_n \in \ell^\infty(\mathbb{N})$ . Then*

- (i)  $M_{\lambda, A, B, x, y}^* = M_{\bar{\lambda}, B, A, y, x}$ , where  $\bar{\lambda} := \{\overline{\lambda_n}\}_n$ . In particular, if  $\lambda$  is real valued, then  $M_{\lambda, A, B, x, y}$  is self-adjoint.

- (ii) If  $\{\lambda_n\|y_n\|\}_n \in \ell^\infty(\mathbb{N})$ ,  $\{A_n\}_n$  is orthonormal and  $\{B_n\}_n$  is orthogonal, then  $M_{\lambda,A,B,x,y}M_{\lambda,A,B,x,y}^* = M_{\mu,A,A,x,x}$ , where  $\mu := \{|\lambda_n|^2\|B_n^*y_n\|^2\}_n$ . In this case, if  $x_n \neq 0, \forall n \in \mathbb{N}$ , then  $(M_{\lambda,A,B,x,y}M_{\lambda,A,B,x,y}^*)^{1/2} = M_{\sqrt{\mu},A,A,x,x}$ , where  $\sqrt{\mu} := \{|\lambda_n|\frac{\|B_n^*y_n\|}{\|x_n\|}\}_n$ .
- (iii) If  $\{\lambda_n\|x_n\|\}_n \in \ell^\infty(\mathbb{N})$ ,  $\{A_n\}_n$  is orthogonal and  $\{B_n\}_n$  is orthonormal, then  $M_{\lambda,A,B,x,y}^*M_{\lambda,A,B,x,y} = M_{\gamma,B,B,y,y}$ , where  $\gamma := \{|\lambda_n|^2\|A_n^*x_n\|^2\}_n$ . In this case, if  $y_n \neq 0, \forall n \in \mathbb{N}$ , then  $(M_{\lambda,A,B,x,y}^*M_{\lambda,A,B,x,y})^{1/2} = M_{\sqrt{\gamma},B,B,y,y}$ , where  $\sqrt{\gamma} := \{|\lambda_n|\frac{\|A_n^*x_n\|}{\|y_n\|}\}_n$ .
- (iv) If  $\langle A_k^*x_k, B_n^*y_n \rangle = 0, \forall k, n \in \mathbb{N}$  with  $k \neq n$ , then for all  $k \in \mathbb{N}$ ,

$$M_{\lambda,A,B,x,y}^k = \sum_{n=1}^{\infty} \lambda_n^k \langle A_n^*x_n, B_n^*y_n \rangle^{k-1} (A_n^*x_n \otimes \overline{B_n^*y_n}).$$

In particular, if  $\{A_n\}_n$  is orthogonal, then

$$M_{\lambda,A,A,x,y}^k = \sum_{n=1}^{\infty} \lambda_n^k \langle A_n^*x_n, A_n^*y_n \rangle^{k-1} (A_n^*x_n \otimes \overline{A_n^*y_n}), \forall k \in \mathbb{N}.$$

- (v)  $M_{\alpha\lambda,A,B,x,y} = M_{\lambda,A,B,\alpha x,y} = M_{\lambda,A,\alpha B,x,y} = \alpha M_{\lambda,A,B,x,y}$ ,  
 $M_{\lambda,\alpha A,B,x,y} = M_{\lambda,A,B,x,\alpha y} = \bar{\alpha} M_{\lambda,A,B,x,y}$ .
- (vi)  $M_{\lambda+\mu,A,B,x,y} = M_{\lambda,A,B,x,y} + M_{\mu,A,B,x,y}$ .
- (vii)  $M_{\lambda,A+C,B,x,y} = M_{\lambda,A,B,x,y} + M_{\lambda,C,B,x,y}$ .
- (viii)  $M_{\lambda,A,B+C,x,y} = M_{\lambda,A,B,x,y} + M_{\lambda,A,C,x,y}$ .
- (ix)  $M_{\lambda,A,B,x+y,z} = M_{\lambda,A,B,x,z} + M_{\mu,A,B,y,z}$ .
- (x)  $M_{\lambda,A,B,x,y+z} = M_{\lambda,A,B,x,y} + M_{\mu,A,B,x,z}$ .
- (xi) If  $\{A_n\}_n$  is orthogonal, then

$$\|M_{\lambda\mu,A,B,x,y}\| \leq \min\{\sup_{n \in \mathbb{N}} |\lambda_n| \|M_{\mu,A,B,x,y}\|, \sup_{n \in \mathbb{N}} |\mu_n| \|M_{\lambda,A,B,x,y}\|\},$$

where  $\lambda\mu := \{\lambda_n\mu_n\}_n$ .

- (xii) (Symbolic calculus) If  $\langle A_k^*x_k, B_n^*y_n \rangle = 0, \forall k, n \in \mathbb{N}$  with  $k \neq n$ , then  $M_{\lambda,A,B,x,y}M_{\mu,A,B,x,y} = M_{\nu,A,B,x,y}$ , where  $\nu := \{\lambda_n\mu_n \langle A_n^*x_n, B_n^*y_n \rangle\}_n$ . Moreover, if  $A_n^*x_n = x, B_n^*y_n = y, \forall n \in \mathbb{N}$ , then  $M_{\lambda,A,B,x,y}M_{\mu,A,B,x,y} = \langle x, y \rangle M_{\lambda\mu,A,B,x,y}$ . In particular, if  $\langle x, y \rangle = 1$ , then  $M_{\lambda,A,B,x,y}M_{\mu,A,B,x,y} = M_{\lambda\mu,A,B,x,y}$ .
- (xiii) If  $\{A_n\}_n$  is orthogonal, then  $M_{\lambda,A,A,x,x}$  is normal.
- (xiv) If  $\{T_n\}_n$  in  $\mathcal{B}(\mathcal{H}_0)$  is such that  $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$ , then  $M_{\lambda,A,TB,x,y} = M_{\lambda,A,B,x,T^*y}$ , where  $T^*y := \{T_n^*y_n\}_n$ .
- (xv) If  $S \in \mathcal{B}(\mathcal{H})$ , then  $M_{\lambda,A,BS,x,y} = M_{\lambda,A,B,x,y}S$ .
- (xvi) If  $\{T_n\}_n$  in  $\mathcal{B}(\mathcal{H}_0)$  is such that  $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$ , then  $M_{\lambda,TA,B,x,y} = M_{\lambda,A,B,T^*x,y}$ , where  $T^*y := \{T_n^*x_n\}_n$ .
- (xvii) If  $S \in \mathcal{B}(\mathcal{H})$ , then  $M_{\lambda,AS,B,x,y} = S^*M_{\lambda,A,B,x,y}$ .
- (xviii) If  $\{T_n\}_n$  in  $\mathcal{B}(\mathcal{H}_0)$  is such that  $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$ , then  $M_{\lambda,A,B,x,Ty} = M_{\lambda,A,T^*B,x,y}$ , where  $T^*B := \{T_n^*B\}_n$ .

- (xix) If  $\{T_n\}_n$  in  $\mathcal{B}(\mathcal{H}_0)$  is such that  $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$ , then  $M_{\lambda, A, B, Tx, y} = M_{\lambda, T^* A, B, x, y}$ .
- (xx) If  $\langle C_k^* z_k, B_n^* y_n \rangle = 0, \forall k, n \in \mathbb{N}$  with  $k \neq n$ , then  $M_{\lambda, A, B, x, y} M_{\mu, C, D, z, v} = M_{\lambda \mu \langle Cz, Dy \rangle, A, D, x, v}$ , where  $\lambda \mu \langle Cz, Dy \rangle := \{\lambda_n \mu_n \langle C_n^* z_n, D_n^* y_n \rangle\}$ . In particular, if  $\{B_n\}_n$  is orthogonal, then  $M_{\lambda, A, B, x, y} M_{\mu, B, D, z, v} = M_{\lambda \mu \langle Bz, Dy \rangle, A, D, x, v}$ .

**Theorem 2.8.** Let  $a$  and  $b$  be Bessel bounds for  $\{A_n\}_n$  and  $\{B_n\}_n$  respectively.

- (i) If  $\{\lambda_n\}_n \in c_0(\mathbb{N})$ , then  $M_{\lambda, A, B, x, y}$  is a compact operator.
- (ii) If  $\{\lambda_n\}_n \in \ell^1(\mathbb{N})$ , then  $\|M_{\lambda, A, B, x, y}\|_{\text{Nuc}} \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \|\{\lambda_n\}_n\|_1$ .
- (iii) If  $\{\lambda_n\}_n \in \ell^2(\mathbb{N})$  and  $\{A_n\}_n$  is orthogonal, then  $M_{\lambda, A, B, x, y}$  is a Hilbert-Schmidt operator with  $\sigma(M_{\lambda, A, B, x, y}) \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \|\{\lambda_n\}_n\|_2$ .

*Proof.* (i) For  $m \in \mathbb{N}$ , define  $M_{\lambda_m, A, B, x, y} := \sum_{n=1}^m \lambda_n (A_n^* x_n \otimes B_n^* y_n)$ .

Then  $M_{\lambda_m, A, B, x, y}(\mathcal{H}) \subseteq \text{span}\{A_n^* x_n\}_{n=1}^m$  and  $\|M_{\lambda_m, A, B, x, y} h - M_{\lambda, A, B, x, y} h\| = \|\sum_{n=m+1}^{\infty} \lambda_n (A_n^* x_n \otimes B_n^* y_n) h\| \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \|h\| \sup_{m+1 \leq n < \infty} |\lambda_n|$ ,  $\forall h \in \mathcal{H}$ . Hence  $\{M_{\lambda_m, A, B, x, y}\}_{m=1}^{\infty}$  is a sequence of finite rank operators and  $\|M_{\lambda_m, A, B, x, y} - M_{\lambda, A, B, x, y}\| \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \sup_{m+1 \leq n < \infty} |\lambda_n| \rightarrow 0$  as  $m \rightarrow \infty$ . Thus  $M_{\lambda_m, A, B, x, y}$  is compact.

- (ii) Define  $f_n : \mathcal{H} \ni h \mapsto \langle h, B_n^* y_n \rangle \in \mathbb{K}$  for each  $n \in \mathbb{N}$ . Then  $f_n \in \mathcal{H}^*$  and  $\|f_n\| = \|B_n^* y_n\|$  for all  $n \in \mathbb{N}$ . Then by calculation, we get that  $\|M_{\lambda, A, B, x, y}\|_{\text{Nuc}} \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \|\{\lambda_n\}_n\|_1$ .
- (iii) Let  $\{e_n\}_n$  be an orthonormal basis for  $\mathcal{H}$ . Then by calculation, we get that  $\sigma(M_{\lambda, A, B, x, y})^2 \leq ab \sup_{n \in \mathbb{N}} \|x_n\|^2 \|y_n\|^2 \|\{\lambda_n\}_n\|_2^2$ .

□

We now derive continuity of multiplier in the next proposition.

**Proposition 2.9.** Let  $\lambda_n^{(k)} = \{\lambda_n^{(k)}\}_n$ . If  $\{\lambda_n^{(k)}\}_{k=1}^{\infty}$  converges to  $\{\lambda_n\}_n$  as  $k \rightarrow \infty$  in

- (i)  $\ell^{\infty}(\mathbb{N})$ , then  $\{M_{\lambda^{(k)}, A, B, x, y}\}_{k=1}^{\infty}$  converges to  $M_{\lambda, A, B, x, y}$  as  $k \rightarrow \infty$  in the operator-norm.
- (ii)  $\ell^1(\mathbb{N})$ , then  $\{M_{\lambda^{(k)}, A, B, x, y}\}_{k=1}^{\infty}$  converges to  $M_{\lambda, A, B, x, y}$  as  $k \rightarrow \infty$  in the nuclear-norm.
- (iii)  $\ell^2(\mathbb{N})$  and  $\{A_n\}_n$  is orthogonal, then  $\{M_{\lambda^{(k)}, A, B, x, y}\}_{k=1}^{\infty}$  converges to  $M_{\lambda, A, B, x, y}$  as  $k \rightarrow \infty$  in the Hilbert-Schmidt norm.

*Proof.* Let  $a$  and  $b$  be Bessel bounds for  $\{A_n\}_n$  and  $\{B_n\}_n$  respectively.

- (i) For each  $h \in \mathcal{H}$ ,  $\|M_{\lambda^{(k)}, A, B, x, y} h - M_{\lambda, A, B, x, y} h\| = \|\sum_{n=1}^{\infty} (\lambda_n^{(k)} - \lambda_n)(A_n^* x_n \otimes B_n^* y_n) h\| \leq \sqrt{ab} \sup_{n \in \mathbb{N}} |\lambda_n^{(k)} - \lambda_n| \|h\| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\|$ . Therefore  $\|M_{\lambda^{(k)}, A, B, x, y} - M_{\lambda, A, B, x, y}\| \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \sup_{n \in \mathbb{N}} |\lambda_n^{(k)} - \lambda_n| \rightarrow 0$  as  $k \rightarrow \infty$ .
- (ii)  $\|M_{\lambda^{(k)}, A, B, x, y} - M_{\lambda, A, B, x, y}\|_{\text{Nuc}} = \|\sum_{n=1}^{\infty} (\lambda_n^{(k)} - \lambda_n)(A_n^* x_n \otimes B_n^* y_n)\|_{\text{Nuc}} \leq \sum_{n=1}^{\infty} |\lambda_n^{(k)} - \lambda_n| \|A_n^* x_n\| \|B_n^* y_n\| \leq \sqrt{ab} \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \sum_{n=1}^{\infty} |\lambda_n^{(k)} - \lambda_n| \rightarrow 0$  as  $k \rightarrow \infty$ .

- (iii) Starting with an orthonormal basis  $\{e_n\}_n$  for  $\mathcal{H}$ , we see that  $\sigma(M_{\lambda^{(k)}, A, B, x, y} - M_{\lambda, A, B, x, y})^2 = \sigma(\sum_{n=1}^{\infty} (\lambda_n^{(k)} - \lambda_n)(A_n^* x_n \otimes \overline{B_n^* y_n}))^2 = \sum_{r=1}^{\infty} \|\sum_{n=1}^{\infty} (\lambda_n^{(k)} - \lambda_n)(A_n^* x_n \otimes \overline{B_n^* y_n}) e_r\|^2 = \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} |\lambda_n^{(k)} - \lambda_n|^2 \|A_n^* x_n\|^2 |\langle e_r, B_n^* y_n \rangle|^2 = \sum_{n=1}^{\infty} |\lambda_n^{(k)} - \lambda_n|^2 \|A_n^* x_n\|^2 \|B_n^* y_n\|^2 \leq ab \sup_{n \in \mathbb{N}} \|x_n\|^2 \|y_n\|^2 \sum_{n=1}^{\infty} |\lambda_n^{(k)} - \lambda_n|^2 \rightarrow 0$  as  $k \rightarrow \infty$ .

□

Following proposition shows that the multiplier is bounded below in a special case.

**Proposition 2.10.** *Let  $\{A_n\}_n$  be an operator-valued orthonormal basis in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  and  $\{B_n\}_n$  be an operator-valued Riesz basis in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$ .*

- (i) *If  $x_n \neq 0 \neq y_n, \forall n \in \mathbb{N}$ , then the map  $S : \ell^{\infty}(\mathbb{N}) \ni \{\lambda_n\}_n \mapsto M_{\lambda, A, B, x, y} \in \mathcal{B}(\mathcal{H})$  is a well-defined injective bounded linear operator.*
- (ii) *There exists a unique invertible  $T \in \mathcal{B}(\mathcal{H})$  such that*

$$\sup_{g \in \mathcal{H}_0, g \neq 0} \sup_{n \in \mathbb{N}} \frac{|\lambda_n \langle g, y_n \rangle| \|x_n\|}{\|T^{-1} A_n^* g\|} \leq \|M_{\lambda, A, B, x, y}\| \leq \|T\| \sup_{n \in \mathbb{N}} |\lambda_n| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\|.$$

*In particular, if  $x_n \neq 0, \forall n \in \mathbb{N}$  (resp.  $y_n \neq 0, \forall n \in \mathbb{N}$ ), then*

$$\frac{\sup_{n \in \mathbb{N}} |\lambda_n \langle x_n, y_n \rangle|}{\|T^{-1}\|} \leq \|M_{\lambda, A, B, x, y}\| \quad \left( \text{resp. } \sup_{n \in \mathbb{N}} \frac{|\lambda_n \|x_n\| \|y_n\|}{\|T^{-1}\|} \leq \|M_{\lambda, A, B, x, y}\| \right).$$

*Proof.* Let  $T \in \mathcal{B}(\mathcal{H})$  be the unique invertible operator such that  $B_n = A_n T, \forall n \in \mathbb{N}$ , given by Theorem 1.6.

- (i) By using Theorem 2.1,

$$\|S\{\lambda_n\}_n\| = \|M_{\lambda, A, B, x, y}\| \leq \|T\| \sup_{n \in \mathbb{N}} \|x_n\| \|y_n\| \|\{\lambda_n\}_n\|_{\infty}, \forall \{\lambda_n\}_n \in \ell^{\infty}(\mathbb{N})$$

which implies  $S$  is bounded. From (v) and (vi) in Theorem 2.7 we see that  $S$  is linear. Now suppose  $S\{\lambda_n\}_n = 0$  for some  $\{\lambda_n\}_n \in \ell^{\infty}(\mathbb{N})$ . Then  $\sum_{n=1}^{\infty} \lambda_n \langle Th, A_n^* y_n \rangle A_n^* x_n = \sum_{n=1}^{\infty} \lambda_n \langle h, B_n^* y_n \rangle A_n^* x_n = \sum_{n=1}^{\infty} \lambda_n (A_n^* x_n \otimes \overline{B_n^* y_n}) h = M_{\lambda, A, B, x, y} h = (S\{\lambda_n\}_n) h = 0, \forall h \in \mathcal{H}$ . By taking  $h = T^{-1} A_k^* y_k, k \in \mathbb{N}$  we get  $\lambda_k \|y_k\|^2 x_k = A_k (\lambda_k \|y_k\|^2 A_k^* x_k) = A_k 0 = 0, \forall k \in \mathbb{N}$ . Hence  $S$  is injective.

- (ii) Let  $n \in \mathbb{N}$  be fixed and  $g \in \mathcal{H}_0$  be nonzero. We note that  $A_n^* g \neq 0$ . Else  $0 = A_n A_n^* g = g$ , which is forbidden. Upper bound for the operator norm is clear and lower bound is shown below.

$$\begin{aligned} \|M_{\lambda, A, B, x, y}\| &= \sup_{h \in \mathcal{H}, \|h\| \leq 1} \|M_{\lambda, A, B, x, y} h\| = \sup_{h \in \mathcal{H}, \|h\| \leq 1} \left\| \sum_{k=1}^{\infty} \lambda_k \langle Th, A_k^* y_k \rangle A_k^* x_k \right\| \\ &\geq \left\| \sum_{k=1}^{\infty} \lambda_k \langle T \frac{T^{-1} A_n^* g}{\|T^{-1} A_n^* g\|}, A_k^* y_k \rangle A_k^* x_k \right\| = \frac{|\lambda_n \langle g, y_n \rangle| \|x_n\|}{\|T^{-1} A_n^* g\|}. \end{aligned}$$

If  $x_n \neq 0, \forall n \in \mathbb{N}$  (resp.  $y_n \neq 0, \forall n \in \mathbb{N}$ ), then we take  $g = x_n$  (resp.  $g = y_n$ ) to get

$$\begin{aligned} \frac{|\lambda_n \langle x_n, y_n \rangle| \|x_n\|}{\|T^{-1} A_n^* x_n\|} &\geq \frac{|\lambda_n \langle x_n, y_n \rangle| \|x_n\|}{\|T^{-1}\| \|A_n^* x_n\|} = \frac{|\lambda_n \langle x_n, y_n \rangle| \|x_n\|}{\|T^{-1}\| \|x_n\|} = \frac{|\lambda_n \langle x_n, y_n \rangle|}{\|T^{-1}\|} \\ \left( \text{resp. } \frac{|\lambda_n \langle y_n, y_n \rangle| \|x_n\|}{\|T^{-1} A_n^* y_n\|} \geq \frac{|\lambda_n| \|y_n\|^2 \|x_n\|}{\|T^{-1}\| \|A_n^* y_n\|} = \frac{|\lambda_n| \|y_n\| \|x_n\|}{\|T^{-1}\|} \right). \end{aligned}$$

□

### 3. Generalized Hilbert-Schmidt class

Let  $\{x_n\}_n$  be a frame for  $\mathcal{H}$ . Operators  $A \in \mathcal{B}(\mathcal{H})$  satisfying  $\sum_{n=1}^{\infty} \|Ax_n\|^2 < \infty$  are studied in [15, 19]. It is shown in Lemma 1.2 of [15] that if  $\{x_n\}_n$  and  $\{y_n\}_n$  are Parseval frames, then  $\sum_{n=1}^{\infty} \|Ax_n\|^2 < \infty$  if and only if  $\sum_{n=1}^{\infty} \|Ay_n\|^2 < \infty$  and in this case,  $\sigma(A)^2 = \sum_{n=1}^{\infty} \|Ax_n\|^2 = \sum_{n=1}^{\infty} \|Ay_n\|^2 = \sum_{n=1}^{\infty} \|A^* x_n\|^2$ .

In the situation of operator-valued sequences, we set up the following definition.

**Definition 3.1.** Let  $\theta : \mathcal{H}_0 \rightarrow \mathcal{H}_0$  be a conjugate-linear isometry. Let  $\{F_n\}_n$  be an operator-valued orthonormal basis in  $\mathcal{B}(\mathcal{H}, \mathcal{H}_0)$  and  $\{x_n\}_n$  be a sequence in  $\mathcal{H}_0$ . Define (the generalized Hilbert-Schmidt class)

$$\begin{aligned} \mathcal{S}_{\theta, F, x}(\mathcal{H}) := \Big\{ A \in \mathcal{B}(\mathcal{H}) : &\sum_{n=1}^{\infty} \|AF_n^* x_n\|^2 < \infty, \\ &\theta(F_m V^* A^* U^* F_n^* x_n) = F_n U A V F_m^* x_m, \\ &\theta(F_m V^* A U^* F_n^* x_n) = F_n U A^* V F_m^* x_m, \forall n, m \in \mathbb{N}, \forall U, V \in \mathcal{B}(\mathcal{H}) \Big\}. \end{aligned}$$

If  $A \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ , then we define  $\sigma_{\theta, F, x}(A) := (\sum_{n=1}^{\infty} \|AF_n^* x_n\|^2)^{\frac{1}{2}}$ .

**Remark 3.1.** Definition 3.1 reduces to the definition of class of Hilbert-Schmidt operators (as well as Hilbert-Schmidt norm), Definition 1.8, given by Schatten and von Neumann, whenever  $\mathcal{H}_0 = \mathbb{K}$ , map  $\theta$  is conjugation,  $x_n = 1, F_n : \mathcal{H} \ni h \mapsto \langle h, e_n \rangle \in \mathbb{K}, \forall n \in \mathbb{N}$ , where  $\{e_n\}_n$  is an orthonormal basis for  $\mathcal{H}$ . Then  $\{F_n\}_n$  is an operator-valued orthonormal basis in  $\mathcal{B}(\mathcal{H}, \mathbb{K})$ . Observe now that the conditions  $\overline{F_m V^* A^* U^* F_n^* x_n} = F_n U A V F_m^* x_m$ ,  $\overline{F_m V^* A U^* F_n^* x_n} = F_n U A^* V F_m^* x_m, \forall n, m \in \mathbb{N}$  hold for all  $A, U, V \in \mathcal{B}(\mathcal{H})$ . Indeed, for  $A, U, V \in \mathcal{B}(\mathcal{H})$ ,  $\overline{F_m V^* A^* U^* F_n^* x_n} = \overline{F_m V^* A^* U^* F_n^* 1} = \overline{F_m V^* A^* U^* e_n} = \langle V^* A^* U^* e_n, e_m \rangle = \langle e_n, U A V e_m \rangle = \langle U A V e_m, e_n \rangle = F_n U A V e_m = F_n U A V F_m^* x_m, \forall n, m \in \mathbb{N}$ . Similarly,  $\overline{F_m V^* A U^* F_n^* x_n} = F_n U A^* V F_m^* x_m, \forall n, m \in \mathbb{N}, \forall U, V \in \mathcal{B}(\mathcal{H})$ .

**Theorem 3.2.** Let  $A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ ,  $\alpha \in \mathbb{K}$ ,  $T \in \mathcal{B}(\mathcal{H})$ . Then

- (i)  $\alpha A \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$  and  $\sigma_{\theta, F, x}(\alpha A) = |\alpha| \sigma_{\theta, F, x}(A)$ .
- (ii)  $A + B \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$  and  $\sigma_{\theta, F, x}(A + B) \leq \sigma_{\theta, F, x}(A) + \sigma_{\theta, F, x}(B)$ .
- (iii)  $\mathcal{S}_{\theta, F, x}(\mathcal{H})$  is a subspace of  $\mathcal{B}(\mathcal{H})$ .

- (iv)  $A^* \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$  and  $\sigma_{\theta,F,x}(A^*) = \sigma_{\theta,F,x}(A)$ .
- (v)  $TA \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$  and  $\sigma_{\theta,F,x}(TA) \leq \|T\|\sigma_{\theta,F,x}(A)$ .
- (vi)  $AT \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$  and  $\sigma_{\theta,F,x}(AT) \leq \|T\|\sigma_{\theta,F,x}(A)$ .
- (vii) If there exist  $a > 0$  and  $2 \leq p < \infty$  such that

$$a\|h\| \leq \left( \sum_{n=1}^{\infty} |\langle h, F_n^* x_n \rangle|^p \right)^{\frac{1}{p}}, \quad \forall h \in \mathcal{H},$$

then  $\|A\| \leq \sigma_{\theta,F,x}(A)/a, \forall A \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$ ,  $\sigma_{\theta,F,x}(\cdot)$  is a norm on  $\mathcal{S}_{\theta,F,x}(\mathcal{H})$  and  $\mathcal{S}_{\theta,F,x}(\mathcal{H})$  is complete in this norm. In particular, if  $\{F_n^* x_n\}_n$  is a frame for  $\mathcal{H}$ , then  $\mathcal{S}_{\theta,F,x}(\mathcal{H})$  is complete.

- (viii) If  $A^*A \leq B^*B$ , then  $\sigma_{\theta,F,x}(A) \leq \sigma_{\theta,F,x}(B)$ .
- (ix) If  $C \in \mathcal{B}(\mathcal{H})$ , then  $C \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$  if and only if  $[C] \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$ . In this case,  $\sigma_{\theta,F,x}(C) = \sigma_{\theta,F,x}([C])$ .
- (x) If  $\|A\| < 1$ , then  $\sigma_{\theta,F,x}(A^n) \rightarrow 0$  as  $n \rightarrow \infty$ .
- (xi) If  $F_n^* x_n$  is an eigenvector for  $A$  with eigenvalue  $\lambda_n$  for each  $n \in \mathbb{N}$ , then  $\sigma_{\theta,F,x}(A)^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 \|x_n\|^2$ .

*Proof.* (i)  $\sigma_{\theta,F,x}(\alpha A)^2 = \sum_{n=1}^{\infty} \|\alpha A F_n^* x_n\|^2 = |\alpha|^2 \sum_{n=1}^{\infty} \|A F_n^* x_n\|^2$   
 $= \alpha \theta(F_m V^* A^* U^* F_n^* x_n) = \alpha F_n U A V F_m^* x_m = F_n U (\alpha A) V F_m^* x_m$ .

Similarly  $\theta(F_m V^*(\alpha A) U^* F_n^* x_n) = F_n U (\alpha A)^* V F_m^* x_m$ .

(ii)  $\sigma_{\theta,F,x}(A+B) = (\sum_{n=1}^{\infty} \|A F_n^* x_n + B F_n^* x_n\|^2)^{1/2} \leq (\sum_{n=1}^{\infty} \|A F_n^* x_n\|^2)^{1/2} + (\sum_{n=1}^{\infty} \|B F_n^* x_n\|^2)^{1/2} = \sigma_{\theta,F,x}(A) + \sigma_{\theta,F,x}(B)$ ,  $\theta(F_m V^*(A+B)^* U^* F_n^* x_n) = \theta(F_m V^* A^* U^* F_n^* x_n) + \theta(F_m V^* B^* U^* F_n^* x_n) = F_n U A V F_m^* x_m + F_n U B V F_m^* x_m = F_n U (A+B) V F_m^* x_m$ , and  $\theta(F_m V^*(A+B) U^* F_n^* x_n) = F_n U (A+B)^* V F_m^* x_m$ ,  $\forall n, m \in \mathbb{N}, \forall U, V \in \mathcal{B}(\mathcal{H})$ .

(iii) comes from (i) and (ii).

(iv) For every  $k \in \mathbb{N}$ ,

$$\begin{aligned} \sum_{n=1}^k \|A^* F_n^* x_n\|^2 &= \sum_{n=1}^k \sum_{m=1}^{\infty} \|F_m A^* F_n^* x_n\|^2 = \sum_{n=1}^k \sum_{m=1}^{\infty} \|\theta(F_m A^* F_n^* x_n)\|^2 \\ &= \sum_{n=1}^k \sum_{m=1}^{\infty} \|F_n A F_m^* x_m\|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^k \|F_n A F_m^* x_m\|^2 \\ &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|F_n (A F_m^* x_m)\|^2 = \sum_{m=1}^{\infty} \|A F_m^* x_m\|^2. \end{aligned}$$

Therefore  $\sum_{n=1}^{\infty} \|A^* F_n^* x_n\|^2 < \infty$ . A similar procedure gives  $\sigma_{\theta,F,x}(A^*)^2 = \sum_{n=1}^{\infty} \|A^* F_n^* x_n\|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|F_n (A F_m^* x_m)\|^2 = \sum_{m=1}^{\infty} \|A F_m^* x_m\|^2 = \sigma_{\theta,F,x}(A)^2$ .

(v)  $\sigma_{\theta,F,x}(TA)^2 = \sum_{n=1}^{\infty} \|T A F_n^* x_n\|^2 \leq \|T\|^2 \sum_{n=1}^{\infty} \|A F_n^* x_n\|^2 = \|T\|^2 \sigma_{\theta,F,x}(A)^2$ ,  $\theta(F_m V^*(TA)^* U^* F_n^* x_n) = \theta(F_m V^* A^* (T^* U^*) F_n^* x_n) = F_n (T^* U^*)^* A V F_m^* x_m = F_n U (TA) V F_m^* x_m$  and  $\theta(F_m V^*(TA) U^* F_n^* x_n) = F_n U (TA)^* V F_m^* x_m, \forall n, m \in \mathbb{N}, \forall U, V \in \mathcal{B}(\mathcal{H})$ .

- (vi) It is enough to show  $(AT)^* \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$  (using (iv)). For,  $\sigma_{\theta,F,x}((AT)^*)^2 = \sum_{n=1}^{\infty} \|T^* A^* F_n^* x_n\|^2 \leq \|T^*\|^2 \sum_{n=1}^{\infty} \|A^* F_n^* x_n\|^2 = \|T\|^2 \sigma_{\theta,F,x}(A)^2$ . Further,  $\theta(F_m V^*(AT)^* U^* F_n^* x_n) = F_n U(AT) V F_m^* x_m$  and  $\theta(F_m V^*(AT) U^* F_n^* x_n) = F_n U(AT)^* V F_m^* x_m, \forall n, m \in \mathbb{N}, \forall U, V \in \mathcal{B}(\mathcal{H})$ .
- (vii) For all  $h \in \mathcal{H}$ ,

$$\begin{aligned} a \|A^* h\| &\leq \left( \sum_{n=1}^{\infty} |\langle A^* h, F_n^* x_n \rangle|^p \right)^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} |\langle h, AF_n^* x_n \rangle|^p \right)^{\frac{1}{p}} \\ &\leq \|h\| \left( \sum_{n=1}^{\infty} \|AF_n^* x_n\|^p \right)^{\frac{1}{p}} \leq \|h\| \left( \sum_{n=1}^{\infty} \|AF_n^* x_n\|^2 \right)^{\frac{1}{2}} = \|h\| \sigma_{\theta,F,x}(A). \end{aligned}$$

This gives  $a \|A^*\| = a \|A\| \leq \sigma_{\theta,F,x}(A)$  which tells  $\sigma_{\theta,F,x}(A) = 0 \Rightarrow A = 0$ . Let  $\{A_n\}_{n=1}^{\infty}$  be a Cauchy sequence in  $\mathcal{S}_{\theta,F,x}(\mathcal{H})$ . Then  $\|A_n - A_m\| \leq \sigma_{\theta,F,x}(A_n - A_m)/a, \forall n, m \in \mathbb{N}$  tells that  $\{A_n\}_{n=1}^{\infty}$  is a Cauchy sequence in  $\mathcal{B}(\mathcal{H})$  with respect to the operator-norm. Let  $A := \lim_{n \rightarrow \infty} A_n$ , in the operator-norm. Choose  $N \in \mathbb{N}$  such that  $\sum_{k=1}^{\infty} \|(A_n - A_m)F_k^* x_k\|^2 = \sigma_{\theta,F,x}(A_n - A_m) < 1, \forall n, m \geq N$ . This gives that for each  $r \in \mathbb{N}$ ,  $\sum_{k=1}^r \|(A_n - A_m)F_k^* x_k\|^2 < 1, \forall n, m \geq N \Rightarrow$  for each  $r \in \mathbb{N}$ ,  $\sum_{k=1}^r \|(A_n - A)F_k^* x_k\|^2 = \lim_{m \rightarrow \infty} \sum_{k=1}^r \|(A_n - A_m)F_k^* x_k\|^2 \leq 1, \forall n \geq N \Rightarrow \sum_{k=1}^{\infty} \|(A_n - A)F_k^* x_k\|^2 \leq 1, \forall n \geq N$ .

Now for  $n, m \in \mathbb{N}$  and  $U, V \in \mathcal{B}(\mathcal{H})$ ,

$$\begin{aligned} &\theta(F_m V^*(A - A_N)^* U^* F_n^* x_n) \\ &= \theta(F_m V^* \lim_{k \rightarrow \infty} (A_k^* - A_N^*) U^* F_n^* x_n) \\ &= \lim_{k \rightarrow \infty} \theta(F_m V^* (A_k^* - A_N^*) U^* F_n^* x_n) \\ &= \lim_{k \rightarrow \infty} (\theta(F_m V^* A_k^* U^* F_n^* x_n) - \theta(F_m V^* A_N^* U^* F_n^* x_n)) \\ &= \lim_{k \rightarrow \infty} (F_n U A_k V F_m^* x_m - F_n U A_N V F_m^* x_m) \\ &= F_n U \lim_{k \rightarrow \infty} (A_k - A_N) V F_m^* x_m = F_n U (A - A_N) V F_m^* x_m. \end{aligned}$$

Similarly  $\theta(F_m V^*(A - A_N) U^* F_n^* x_n) = F_n U (A - A_N)^* V F_m^* x_m$ . Therefore  $A - A_N \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$ . Hence  $A = (A - A_N) + A_N \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$ .

$$\begin{aligned} (\text{viii)} \quad &\sigma_{\theta,F,x}(A)^2 = \sum_{n=1}^{\infty} \|AF_n^* x_n\|^2 = \sum_{n=1}^{\infty} \langle A^* AF_n^* x_n, F_n^* x_n \rangle \\ &\leq \sum_{n=1}^{\infty} \langle B^* BF_n^* x_n, F_n^* x_n \rangle = \sum_{n=1}^{\infty} \|BF_n^* x_n\|^2 = \sigma_{\theta,F,x}(B)^2. \end{aligned}$$

- (ix) Let  $C = W[C]$  be the polar decomposition of  $C$  as in Theorem 1.7.  
 $(\Rightarrow)$  From (ii) in Theorem 1.7 we have  $[C] = W^* C$ . Now (v) tells that  $[C] = W^* C \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$ .  
 $(\Leftarrow)$  Polar decomposition of  $C$  and (v) give  $C \in \mathcal{S}_{\theta,F,x}(\mathcal{H})$ . Further, using (v),  $\sigma_{\theta,F,x}(C) = \sigma_{\theta,F,x}(W[C]) \leq \|W\| \sigma_{\theta,F,x}([C]) = \sigma_{\theta,F,x}([C]) = \sigma_{\theta,F,x}(W^* C) \leq \|W^*\| \sigma_{\theta,F,x}(C) = \sigma_{\theta,F,x}(C)$ .

$$(\text{x}) \quad 0 \leq \sigma_{\theta,F,x}(A^n) \leq \|A^{n-1}\| \sigma_{\theta,F,x}(A) \leq \|A\|^{n-1} \sigma_{\theta,F,x}(A) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$(\text{xi}) \quad \sigma_{\theta,F,x}(A)^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 \|F_n^* x_n\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2 \|x_n\|^2.$$

□

Theorem 3.2 says that  $\mathcal{S}_{\theta,F,x}(\mathcal{H})$  is a two-sided ideal in  $\mathcal{B}(\mathcal{H})$ . Further, if assumption in (vii) holds, then it is a two-sided closed ideal.

**Lemma 3.3.** *If  $A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ , then the series  $\sum_{n=1}^{\infty} |\langle AF_n^* x_n, BF_n^* x_n \rangle|$  converges. Further, if  $\mathcal{H}$  and  $\mathcal{H}_0$  are complex Hilbert spaces, then*

$$\begin{aligned} 4 \sum_{n=1}^{\infty} \langle AF_n^* x_n, BF_n^* x_n \rangle &= \sigma_{\theta, F, x}(A+B)^2 - \sigma_{\theta, F, x}(A-B)^2 \\ &\quad + i\sigma_{\theta, F, x}(A+iB)^2 - i\sigma_{\theta, F, x}(A-iB)^2 \end{aligned}$$

and if  $\mathcal{H}$  and  $\mathcal{H}_0$  are real Hilbert spaces, then

$$4 \sum_{n=1}^{\infty} \langle AF_n^* x_n, BF_n^* x_n \rangle = \sigma_{\theta, F, x}(A+B)^2 - \sigma_{\theta, F, x}(A-B)^2.$$

*Proof.* Let  $\mathcal{H}$  and  $\mathcal{H}_0$  be over  $\mathbb{C}$ . For all  $m \in \mathbb{N}$ , we have  $\sum_{n=1}^m |\langle AF_n^* x_n, BF_n^* x_n \rangle| \leq (\sum_{n=1}^m \|AF_n^* x_n\|^2)^{1/2} (\sum_{n=1}^m \|BF_n^* x_n\|^2)^{1/2} \leq \sigma_{\theta, F, x}(A)\sigma_{\theta, F, x}(B)$ . We next use the polarization identity,

$$\begin{aligned} 4 \sum_{n=1}^{\infty} \langle AF_n^* x_n, BF_n^* x_n \rangle &= \sum_{n=1}^{\infty} (\|AF_n^* x_n + BF_n^* x_n\|^2 - \|AF_n^* x_n - BF_n^* x_n\|^2 \\ &\quad + i\|AF_n^* x_n + iBF_n^* x_n\|^2 - i\|AF_n^* x_n - iBF_n^* x_n\|^2) \\ &= \sum_{n=1}^{\infty} \|(A+B)F_n^* x_n\|^2 - \sum_{n=1}^{\infty} \|(A-B)F_n^* x_n\|^2 \\ &\quad + i \sum_{n=1}^{\infty} \|(A+iB)F_n^* x_n\|^2 - i \sum_{n=1}^{\infty} \|(A-iB)F_n^* x_n\|^2 \\ &= \sigma_{\theta, F, x}(A+B)^2 - \sigma_{\theta, F, x}(A-B)^2 \\ &\quad + i\sigma_{\theta, F, x}(A+iB)^2 - i\sigma_{\theta, F, x}(A-iB)^2. \end{aligned}$$

The argument is similar if  $\mathcal{H}$  and  $\mathcal{H}_0$  are over  $\mathbb{R}$ .  $\square$

**Definition 3.4.** Given  $A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ , we define

$$\langle A, B \rangle := \sum_{n=1}^{\infty} \langle AF_n^* x_n, BF_n^* x_n \rangle. \quad (3)$$

Lemma 3.3 says that the series in Equation (3) is well-defined. We observe that  $\sigma_{\theta, F, x}(A)^2 = \langle A, A \rangle, \forall A \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ .

**Proposition 3.5.** (i)  $\langle A, A \rangle \geq 0, \forall A \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ .  
(ii) If there exist  $a > 0$  and  $2 \leq p < \infty$  such that

$$a\|h\| \leq \left( \sum_{n=1}^{\infty} |\langle h, F_n^* x_n \rangle|^p \right)^{\frac{1}{p}}, \quad \forall h \in \mathcal{H},$$

then  $\langle A, A \rangle = 0$  implies that  $A = 0$ .

(iii)  $\langle A+B, C \rangle = \langle A, C \rangle + \langle B, C \rangle$ ,  $\langle \alpha A, B \rangle = \alpha \langle A, B \rangle, \forall A, B, C \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ ,  $\forall \alpha \in \mathbb{K}$ .

- (iv)  $\overline{\langle A, B \rangle} = \langle B, A \rangle, \forall A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H}).$
- (v)  $\langle A^*, B^* \rangle = \overline{\langle A, B \rangle}, \forall A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H}).$
- (vi) If  $A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$  and  $T \in \mathcal{B}(\mathcal{H})$ , then  $\langle TA, B \rangle = \langle A, T^*B \rangle$  and  $\langle AT, B \rangle = \langle A, BT^* \rangle.$
- (vii)  $|\langle A, B \rangle| \leq \sigma_{\theta, F, x}(A)\sigma_{\theta, F, x}(B), \forall A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H}).$

*Proof.* (i)  $\langle A, A \rangle = \sum_{n=1}^{\infty} \|AF_n^*x_n\|^2 \geq 0.$

(ii) From (vii) in Theorem 3.2,  $0 = \langle A, A \rangle = \sigma_{\theta, F, x}(A) \geq a\|A\|.$

(iii)  $\langle A+B, C \rangle = \sum_{n=1}^{\infty} \langle AF_n^*x_n, CF_n^*x_n \rangle + \sum_{n=1}^{\infty} \langle BF_n^*x_n, CF_n^*x_n \rangle = \langle A, C \rangle + \langle B, C \rangle, \langle \alpha A, B \rangle = \alpha \sum_{n=1}^{\infty} \langle AF_n^*x_n, BF_n^*x_n \rangle = \alpha \langle A, B \rangle.$

(iv)  $\overline{\langle A, B \rangle} = \sum_{n=1}^{\infty} \langle BF_n^*x_n, AF_n^*x_n \rangle = \langle B, A \rangle.$

(v) We consider the case  $\mathbb{K} = \mathbb{C}$ , the case  $\mathbb{K} = \mathbb{R}$  is similar. For  $A, B \in \mathcal{S}_{\theta, F, x}(\mathcal{H})$ , by applying the Lemma 3.3, we get

$$\begin{aligned}
& 4\langle A^*, B^* \rangle \\
&= 4 \sum_{n=1}^{\infty} \langle A^*F_n^*x_n, B^*F_n^*x_n \rangle \\
&= \sigma_{\theta, F, x}(A^* + B^*)^2 - \sigma_{\theta, F, x}(A^* - B^*)^2 + i\sigma_{\theta, F, x}(A^* + iB^*)^2 \\
&\quad - i\sigma_{\theta, F, x}(A^* - iB^*)^2 \\
&= \sigma_{\theta, F, x}((A + B)^*)^2 - \sigma_{\theta, F, x}((A - B)^*)^2 + i\sigma_{\theta, F, x}((A - iB)^*)^2 \\
&\quad - i\sigma_{\theta, F, x}((A + iB)^*)^2 \\
&= \sigma_{\theta, F, x}(A + B)^2 - \sigma_{\theta, F, x}(A - B)^2 + i\sigma_{\theta, F, x}(A - iB)^2 - i\sigma_{\theta, F, x}(A + iB)^2 \\
&= \overline{\sigma_{\theta, F, x}(A + B)^2 - \sigma_{\theta, F, x}(A - B)^2 + i\sigma_{\theta, F, x}(A + iB)^2 - i\sigma_{\theta, F, x}(A - iB)^2} \\
&= 4 \sum_{n=1}^{\infty} \langle AF_n^*x_n, BF_n^*x_n \rangle = 4\overline{\langle A, B \rangle}.
\end{aligned}$$

(vi)  $\langle TA, B \rangle = \sum_{n=1}^{\infty} \langle TAF_n^*x_n, BF_n^*x_n \rangle = \sum_{n=1}^{\infty} \langle AF_n^*x_n, T^*BF_n^*x_n \rangle = \langle A, T^*B \rangle,$   
 $\langle AT, B \rangle = \langle B^*, (AT)^* \rangle = \overline{\langle A^*, (BT^*)^* \rangle} = \langle A, BT^* \rangle.$

(vii) We have

$$\begin{aligned}
|\langle A, B \rangle| &\leq \sum_{n=1}^{\infty} \|AF_n^*x_n\| \|BF_n^*x_n\| \\
&\leq \left( \sum_{n=1}^{\infty} \|AF_n^*x_n\|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \|BF_n^*x_n\|^2 \right)^{1/2} \\
&= \sigma_{\theta, F, x}(A)\sigma_{\theta, F, x}(B).
\end{aligned}$$

□

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