J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 205 - 213*
https://doi.org/10.14317/jami.2022.205

INTEGRAL REPRESENTATION OF SOME BASIC K-HYPERGEOMETRIC FUNCTIONS

ASAD ALI* AND MUHAMMAD ZAFAR IQBAL

Abstract. In this paper we give a simple and direct proof of an Euler integral representation for a special class of ${}_{q+1}F_{q,k}$ k-hypergeometric functions for $q \geq 2$. The values of certain ${}_3F_{2,k}$ and ${}_4F_{3,k}$ functions at $x = \frac{1}{k}$, some of which can be derived using other methods. We may conclude that for $k = 1$ the results are reduced to [3].

AMS Mathematics Subject Classification : 33C45, 42C05.
Key words and phrases : Hypergeometric function, k-hypergeometric function, k-gamma function, k-beta function and k-pochhammer symbol.

1. Introduction

The hypergeometric functions are important for obtaining various properties, such as, integral representation, generating functions, solution of Gauss differential equations [1, 2, 6]. We aim at deriving an integral representation for a family of the k-hypergeometric functions defined by Diaz and Pariguan [4]. They have introduced and proved some identities of k-gamma function, k-beta function and k-pochhammer symbol. They have also deduced an integral representation of k-gamma and k-beta function respectively given by

$$\Gamma_k(\alpha) = \int_o^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt,\ Re(\alpha) > 0,\ k > 0. \tag{1}$$

$$\beta_k(m,n) = \frac{1}{k}\int_o^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} dt,\ m > 0,\ n > 0. \tag{2}$$

In [7, 8] defined k-hypergeometric function

$${}_2F_{1,k}\left((a,k),(\beta,k);(c,k);x\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(\beta)_{n,k}x^n}{(c)_{n,k}n!}. \tag{3}$$

Received December 11, 2020. Revised March 7, 2021. Accepted April 7, 2021. *Corresponding author.

where

$$(\alpha)_{n,k} = \alpha(\alpha+k)(\alpha+2k)...(\alpha+(n-1)k) = \prod_{j=1}^{n}(\alpha+kj-k), \tag{4}$$

is called k-Pochhammer symbol and $\alpha \neq 0$, $(\alpha)_{0,k} = 1$, $k > 0$. The generalized k-hypergeometric series is a natural generalization to move from (3) the definition to a similar function with any number of numerator and denominator parameters. In [5] define an integral representation of the generalized k-hypergeometric function

$$_{q+1}F_{q,k}\left[\begin{matrix}(a,k),\left(\frac{b}{q},k\right),\left(\frac{b+k}{q},k\right),...,\left(\frac{b+(q-1)k}{q},k\right); \\ \left(\frac{c}{q},k\right),\left(\frac{c+k}{q},k\right),...,\left(\frac{c+(q-1)k}{q},k\right);\end{matrix}\ x\right]$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k\beta\Gamma_k(c-b)}\int_o^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{b_1-a_1}{k}-1}(1-kxt^q)^{\frac{-a}{k}}. \tag{5}$$

Generalize k-Pochhammer symbol defined as

$$(\alpha)_{mn,k} = m^{mn}\prod_{j=1}^{m}\left(\frac{a+kj-k}{m}\right)_{n,k}. \tag{6}$$

The Euler integral representation of the k-Gauss hypergeometric function, or $_2F_{1,k}$ function

$$\begin{aligned} &{}_2F_{1,k}\left((a,k),(\beta,k);(c,k);x\right) \\ &= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(1-kxt)^{\frac{-a}{k}}dt. \end{aligned} \tag{7}$$

Here, it is understood that $arg(t) = arg(1-t) = 0$ and $(1-kxt)^{\frac{-a}{k}}$ has its principle value.

$$_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\,; & \frac{1}{k} \\ (c-\alpha,k)\,; & \frac{1}{k}\end{matrix}\right] = \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)} \tag{8}$$

The case where one of the numerator parameters is a negative integer, thereby making the $_2F_{1,k}$ finite sum

$$_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta,k)\,; & \frac{1}{k} \\ (c-\alpha,k)\,; & \frac{1}{k}\end{matrix}\right] = \frac{(c-a)_{n,k}}{(c)_{n,k}} \tag{9}$$

In [2] prove the following theorem

Theorem 1.1. *If $Re(\alpha) > 0, Re(\beta) > 0$, $k > 0$, and if m is positive integer. Then inside the region of convergence of the resultant series are*

$$\frac{t^{1-\frac{\alpha+\beta}{k}}}{kB_k(\alpha,\beta)}\int_o^1 x^{\frac{\alpha}{k}-1}(t-x)^{\frac{\beta}{k}-1}{}_pF_{q,k}\left[\begin{matrix}(a_1,k),(a_2,k),...,(a_p,k)\,;\\(b_1,k),(b_2,k),\dots,(b_q,k)\,;\end{matrix}\;cx^m\right]\quad dx$$

$$= {}_{p+m}F_{q+m,k}\quad\times$$

$$\left[\begin{matrix}(a_1,k),...,(a_p,k);\left(\dfrac{\alpha}{m},k\right),...,\left(\dfrac{\alpha+(m-1)k}{m},k\right);\\(b_1,k),\dots,(b_q,k);\left(\dfrac{\alpha+\beta}{m},k\right),\left(\dfrac{\alpha+\beta+k}{m},k\right),...,\left(\dfrac{\alpha+\beta+(m-1)k}{m},k\right);\end{matrix}\;cx^m\right].\quad(10)$$

2. Main results

We begin with the proof of an Euler integral representation for the special class of k-hypergeometric functions ${}_3F_{2,k}\left((\alpha,k),\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right);\left(\frac{c}{2},k\right),\left(\frac{c+k}{2},k\right);x\right)$.

Theorem 2.1. *If $Re(c)>Re\beta>0$, then*

$$\begin{aligned}&{}_3F_{2,k}\left[\begin{matrix}(\alpha,k),\left(\dfrac{\beta}{2},k\right),\left(\dfrac{\beta+k}{2},k\right);\\\left(\dfrac{c}{2},k\right),\left(\dfrac{c+k}{2},k\right);\end{matrix}\;x\right]\\&=\frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(1-kxt^2)^{\frac{-\alpha}{k}}dt.\end{aligned}\qquad(11)$$

Proof. Suppose that $|x|<1$. Then the left hand side of (11) becomes

$$\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}\left(\frac{\beta}{2}\right)_{n,k}\left(\frac{\beta+k}{2}\right)_{n,k}}{\left(\frac{c}{2}\right)_{n,k}\left(\frac{c+k}{2}\right)_{n,k}}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}\,(\beta)_{2n,k}}{(c)_{2n,k}}\frac{x^n}{n!}.\qquad(12)$$

Since from (6)

$$(\alpha)_{2n,k}=2^{2n}\left(\frac{a}{2}\right)_{n,k}\left(\frac{a+k}{2}\right)_{n,k}.$$

$$\begin{aligned}\frac{(\beta)_{2n,k}}{(c)_{2n,k}}&=\frac{\Gamma_k(c)}{\Gamma_k(\beta)\Gamma_k(c-\beta)}\frac{\Gamma_k(\beta+2nk)\Gamma_k(c-\beta)}{\Gamma_k(c+2nk)}\\&=\frac{\Gamma_k(c)}{\Gamma_k(\beta)\Gamma_k(c-\beta)}\beta_k(\beta+2nk,c-\beta).\end{aligned}$$

Therefore, the right hand side of (12) becomes

$$\frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}x^n}{n!}\int_o^1 t^{\frac{\beta+2nk}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}dt,$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta+2nk}{k}-1}(1-t)^{\frac{c-\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}(xt^2)^n}{n!} dt,$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(1-kxt^2)^{\frac{-a}{k}} dt.$$

This proves the result for $|x| < 1$. □

Corollary 2.1. *If $Re(c) > Re(\beta) > 0$, then*

$$ {}_3F_{2,k}\left[\begin{matrix} (-nk,k), \left(\frac{\beta}{2},k\right), \left(\frac{\beta+k}{2},k\right); \\ \left(\frac{c}{2},k\right), \left(\frac{c+k}{2},k\right); \end{matrix} \; x\right]$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(1-kxt^2)^n dt. \tag{13}$$

Proof. Subtituting $\alpha = -nk$ in Theorem 2.1, the yeilds is (13). □

Theorem 2.2. *If $Re(c) > Re(\beta) > 0$ and $Re(c-\beta-\alpha) > 0$, then*

$$ {}_3F_{2,k}\left[\begin{matrix} (\alpha,k), \left(\frac{\beta}{2},k\right), \left(\frac{\beta+k}{2},k\right); \\ \left(\frac{c}{2},k\right), \left(\frac{c+k}{2},k\right); \end{matrix} \; \frac{1}{k}\right] \tag{14}$$

$$= \frac{\Gamma_k(c)\Gamma_k(c-\beta-a)}{\Gamma_k(c-\beta)\Gamma_k(c-a)} {}_2F_{1,k}\left[\begin{matrix} (\alpha,k),(\beta,k); \\ (c-\alpha,k); \end{matrix} \; -\frac{1}{k}\right] .$$

Proof. Suppose that $|x| = \frac{1}{k}$ in (13). Then the left hand side of (14) becomes

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(1-t)^{\frac{-a}{k}}(1+t)^{\frac{-a}{k}} dt,$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1}(1+t)^{\frac{-a}{k}} dt,$$

Since

$$(1+t)^{\frac{-a}{k}} = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}\left(-\frac{t}{k}\right)^n}{n!}.$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}t^n}{n!} dt,$$

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta+nk}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!} dt,$$

By using (2)

$$
{}_3F_{2,k}\left[\begin{matrix}(\alpha,k)\,,\left(\dfrac{\beta}{2},k\right),\left(\dfrac{\beta+k}{2},k\right); \\ \left(\dfrac{c}{2},k\right),\left(\dfrac{c+k}{2},k\right);\end{matrix}\ \dfrac{1}{k}\right]
$$

$$
= \frac{\Gamma_k(c)}{\Gamma_k(\beta)\Gamma_k(c-\beta)}\frac{\Gamma_k(\beta+nk)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a+nk)}\sum_{n=0}^{\infty}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!},
$$

$$
= \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\frac{\Gamma_k(\beta+nk)\Gamma_k(c-a)}{\Gamma_k(\beta)\Gamma_k(c-a+nk)}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!},
$$

$$
= \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}(\beta)_{n,k}}{(c-a)_{n,k}}\frac{(-\frac{1}{k})^n}{n!},
$$

$$
= \frac{\Gamma_k(c)\Gamma_k(c-\beta-a)}{\Gamma_k(c-\beta)\Gamma_k(c-a)}{}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\,; \\ (c-\alpha,k)\,;\end{matrix}\ -\frac{1}{k}\right] .
$$

□

Corollary 2.2. *If* $Re(c) > Re(\beta) > 0$*, then*

$$
{}_3F_{2,k}\left[\begin{matrix}(-nk,k)\,,\left(\dfrac{\beta}{2},k\right),\left(\dfrac{\beta+k}{2},k\right); \\ \left(\dfrac{c}{2},k\right),\left(\dfrac{c+k}{2},k\right);\end{matrix}\ \dfrac{1}{k}\right]
= \frac{(c-\beta)_{n,k}}{(c)_{n,k}}{}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta,k)\,; \\ (c+nk,k)\,;\end{matrix}\ -\frac{1}{k}\right] . \tag{15}
$$

Generalising (13) in the obvious way yields the following result.

It follows from (5), with $q = 3$ that for $Re(c) > Re(\beta) > 0$, then

$$
{}_4F_{3,k}\left[\begin{matrix}(\alpha,k),\left(\dfrac{\beta}{3},k\right),\left(\dfrac{\beta+k}{3},k\right),\left(\dfrac{\beta+2k}{3},k\right); \\ \left(\dfrac{c}{3},k\right),\left(\dfrac{c+k}{3},k\right),\left(\dfrac{c+2k}{3},k\right);\end{matrix}\ x\right]
$$

$$
= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\int_o^1 t^{\frac{a_1}{k}-1}(1-t)^{\frac{\beta_1-a_1}{k}-1}(1-kxt^3)^{\frac{-a}{k}}. \tag{16}
$$

and this leads to the following result.

Theorem 2.3. *If* $Re(c) > Re(\beta) > 0$*, and* $Re(c-\beta-a) > 0$ *then*

$$
\begin{aligned}
&{}_4F_{3,k}\left[\begin{matrix}(\alpha,k),\left(\frac{\beta}{3},k\right),\left(\frac{\beta+k}{3},k\right),\left(\frac{\beta+2k}{3},k\right); \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right);\end{matrix}\ \frac{1}{k}\right] \\
&= \frac{\Gamma_k(c)\Gamma_k(c-\beta-a)}{\Gamma_k(c-\beta)\Gamma_k(c-a)} \\
&\times\ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k); \\ (c-\alpha,k);\end{matrix}\ -\frac{1}{k}\right]{}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta+nk,k); \\ (c-a+nk,k);\end{matrix}\ -\frac{1}{k}\right] \quad . \qquad (17)
\end{aligned}
$$

Proof. Suppose that $q = 3$ and $|x| = \frac{1}{k}$ in (5), then we have

$$
\begin{aligned}
&{}_4F_{3,k}\left[\begin{matrix}(\alpha,k),\left(\frac{\beta}{3},k\right),\left(\frac{\beta+k}{3},k\right),\left(\frac{\beta+2k}{3},k\right); \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right);\end{matrix}\ \frac{1}{k}\right] \\
&= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(1-t^3)^{\frac{-a}{k}}dt, \\
&= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1}(1+t+t^2)^{\frac{-a}{k}}dt, \\
&= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1}\sum_{n=0}^{\infty}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}t^n(1+t)^n}{n!}dt, \\
&= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!}\int_o^1 t^{\frac{\beta+nk}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1}(1+t)^n dt, \\
&= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!}\int_o^1 t^{\frac{\beta+nk}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1}\sum_{r=0}^{n}\binom{n}{r}t^r dt, \\
&= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!}\binom{n}{r}\int_o^1 t^{\frac{\beta+(r+n)k}{k}-1}(1-t)^{\frac{c-a-\beta}{k}-1}dt, \\
&= \frac{\Gamma_k(c)}{\Gamma_k(\beta)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!}\binom{n}{r}\frac{\Gamma_k(\beta+(r+n)k)\Gamma_k(c-\beta-a)}{\Gamma_k(\beta)\Gamma_k(c-a+(r+n)k)}, \\
&= \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!}\binom{n}{r}\frac{(\beta)_{r+n,k}}{(c-a)_{r+n,k}}, \\
&= \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!}\binom{n}{r}\frac{(\beta)_{n,k}(\beta+nk)_{r,k}}{(c-a)_{n,k}(c-a+nk)_{r,k}},
\end{aligned}
$$

$$= \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}(\beta)_{n,k}}{(c-a)_{n,k}n!} \sum_{r=0}^{n} \binom{n}{r} \frac{(\beta+nk)_{r,k}}{(c-a+nk)_{r,k}},$$

$$= \frac{\Gamma_k(c)\Gamma_k(c-a-\beta)}{\Gamma_k(c-a)\Gamma_k(c-\beta)} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}(\beta)_{n,k}}{(c-a)_{n,k}n!} \sum_{r=0}^{n} \frac{\left(-\frac{1}{k}\right)^r(-nk)_{r,k}(\beta+nk)_{r,k}}{(c-a+nk)_{r,k}r!},$$

$$= \frac{\Gamma_k(c)\Gamma_k(c-\beta-a)}{\Gamma_k(c-\beta)\Gamma_k(c-a)}$$

$$\times {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\,; & -\frac{1}{k}\\ (c-\alpha,k)\,; & \end{matrix}\right] \; {}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta+nk,k)\,; & -\frac{1}{k}\\ (c-\alpha+nk,k)\,; & \end{matrix}\right] \quad .$$

□

Theorem 2.4. *If $Re(c) > Re(\beta) > 0$, then*

$${}_3F_{2,k}\left[\begin{matrix}(\alpha,k)\,,\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right); & \frac{1}{2k}\\ \left(\frac{c}{2},k\right),\left(\frac{c+k}{2},k\right); & \end{matrix}\right] = 2^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}(c-\beta)_{n,k}}{(c)_{n,k}n!} \times$$

$${}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta,k)\,; & -\frac{1}{k}\\ (c+nk,k)\,; & \end{matrix}\right] \quad . \tag{18}$$

Proof. Suppose that $|x| = \frac{1}{2k}$ in (13) . Then the left hand side of (18) becomes

$$= \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}\left(1-\frac{t^2}{2}\right)^{\frac{-a}{k}} dt,$$

$$= 2^{\frac{a}{k}} \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}-1}(2-t^2)^{\frac{-a}{k}} dt,$$

Since

$$\left(1+(1-t^2)\right)^{\frac{-a}{k}} = \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}\left(-\frac{1}{k}\right)^n(1-t^2)^n}{n!}$$

$$= 2^{\frac{a}{k}} \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}+n-1} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}(1+t)^n}{n!} dt,$$

$$= 2^{\frac{a}{k}} \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!} \int_o^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{c-\beta}{k}+n-1} \sum_{r=0}^{n} \binom{n}{r} t^r dt,$$

$$= 2^{\frac{a}{k}} \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!} \sum_{r=0}^{n} \binom{n}{r} \int_o^1 t^{\frac{\beta+rk}{k}-1}(1-t)^{\frac{c-\beta}{k}+n-1} dt,$$

From [2]

$$\beta_k(\beta+rk, c-\beta+nk) = \frac{\Gamma_k(\beta+rk)\Gamma_k(c-\beta+nk)}{\Gamma_k(c+(r+n)k)}.$$

$$= 2^{\frac{a}{k}} \frac{\Gamma_k(c)}{k\Gamma_k(\beta)\Gamma_k(c-\beta)} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma_k(\beta+rk)\Gamma_k(c-\beta+nk)}{\Gamma_k(c+(r+n)k)}$$

$$= 2^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{(\beta)_{r,k}(c-\beta)_{n,k}}{(c)_{r+n,k}}$$

$$(c)_{r+n,k} = (c)_{n,k}(c+n)_{r,k}$$

$$= 2^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(\frac{1}{k})^n(-a)_{n,k}(c-\beta)_{n,k}}{(c)_{n,k}n!} \sum_{r=0}^{n} \binom{n}{r} \frac{(\beta)_{r,k}}{(c+n)_{r,k}}$$

Since from (9)

$$\sum_{r=0}^{n} \binom{n}{r} \frac{(\beta)_{r,k}}{(c+n)_{r,k}} = {}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta,k)\,; \\ (c+nk,k)\,;\end{matrix}\ -\frac{1}{k}\right] .$$

$$= 2^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(\frac{1}{k})^n(-a)_{n,k}(c-\beta)_{n,k}}{(c)_{n,k}n!} {}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta,k)\,; \\ (c+nk,k)\,;\end{matrix}\ -\frac{1}{k}\right] .$$

□

Corollary 2.3. *If $Re(c) > Re(\beta) > 0$, then*

$${}_3F_{2,k}\left[\begin{matrix}(\alpha,k)\,,\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right); \\ \left(\frac{c}{2},k\right),\left(\frac{c+k}{2},k\right);\end{matrix}\ \frac{1}{2k}\right]$$

$$= 2^{\frac{a}{k}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{k})^n(\alpha)_{n,k}}{n!} {}_3F_{2,k}\left[\begin{matrix}(-nk,k)\,,\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right); \\ \left(\frac{c}{2},k\right),\left(\frac{c+k}{2},k\right);\end{matrix}\ \frac{1}{2k}\right] . \tag{19}$$

Proof. From Corollary 2.2, we can see that

$${}_3F_{2,k}\left[\begin{matrix}(-nk,k)\,,\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right); \\ \left(\frac{c}{2},k\right),\left(\frac{c+k}{2},k\right);\end{matrix}\ \frac{1}{k}\right]$$

$$= \frac{(c-\beta)_{n,k}}{(c)_{n,k}} {}_2F_{1,k}\left[\begin{matrix}(-nk,k),(\beta,k)\,; \\ (c+nk,k)\,;\end{matrix}\ -\frac{1}{k}\right] .$$

which along with Theorem 2.4 gives the result. □

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Asad Ali received M.Phil degree from University of Agriculture Faisalabad Pakistan. His research area is Special Functions, Orthogonal Polynomials and Fractional calculus.

Department of Mathematics and Statistics, University of agriculture Faisalabad(38000), Pakistan.
e-mail: `mrasadali5100@gmail.com`

Muhammad Zafar Iqbal received Ph.D. degree from National College of Business Administration & Economics Lahore Pakistan. Currently he is working as chairman of department of mathematics and statistics at University of Agriculture Faisalabad Pakistan. His research area is probability theory, Analysis, Mathematical statistics and Special functions.

Department of Mathematics and Statistics, University of agriculture Faisalabad(38000), Pakistan.
e-mail: `mzts2004@hotmail.com`