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AN EFFICIENT ALGORITHM FOR EVALUATION OF OSCILLATORY INTEGRALS HAVING CAUCHY AND JACOBI TYPE SINGULARITY KERNELS

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Abstract. Herein, an algorithm for efficient evaluation of oscillatory Fourier-integrals with Jacobi-Cauchy type singularities is suggested. This method is based on the use of the traditional Clenshaw-Curtis (CC) algorithms in which the given function is approximated by the truncated Chebyshev series, term by term, and the oscillatory factor is approximated by using Bessel function of the first kind. Subsequently, the modified moments are computed efficiently using the numerical steepest descent method or special functions. Furthermore, Algorithm and programming code in MATHEMATICA® 9.0 are provided for the implementation of the method for automatic computation on a computer. Finally, selected numerical examples are given in support of our theoretical analysis.

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1. Introduction

Herein, we are concerned with the mathematical calculation of Fourier-type integral of the form

$$I_{-1}^{1}\left(f;\mu\right)=\int_{-1}^{1}\frac{f\left(x\right)}{\left(1+x\right)^{\alpha}\left(1-x\right)^{\beta}\left(x-\mu\right)^{v}}e^{i\omega x}dx,\ |\mu|<1,\ i^2=-1 \qquad (1)$$

where $|\omega|$ is strictly greater than 1, $\alpha,\beta\in(-\infty,1)$, $v\in N$ and f is a sufficiently smooth function on the interval $[-1,1]$. For $v=0$, the integral (1) has been extensively investigated by several researchers, (see, for instance [1, 2, 3, 4, 5, 28, 29]). For $v=1$ and $\alpha=\beta=0$ the integrals become Cauchy Principal Value (PVC) integrals and various methods have been developed for their efficient

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computation of it [6, 7, 31, 32] and references in [30, p. 182]. Furthermore, the same CPV is well recognized as the Hilbert transform when the frequency, $\omega = 0$ and the sufficient condition for the existence of Hilbert transform, that $f(x)$ has to satisfy a Lipschitz and Hölder condition in a closed interval [-1, 1], is fulfilled. In this paper, we will consider the cases where $\alpha, \beta \neq 0$ and $v = 1$. For the case where $v = 0$ Gauss Jacobi method [8, p. 113] may be very efficient for handling the integral of type (1.1) when low frequency values are considered. However, when the frequency becomes large the method suffers several difficulties and may require a lot of time in computation and many function evaluations. The reason for this is that not only does the integral become highly oscillatory but also there exist singularities at the endpoints of the integration interval. The asymptotic behavior of such a case was studied in 1955, by A. Erdelyi [9], who performed integration by parts repeatedly, that is if $\beta \in [0, +\infty), \alpha \in (-\infty, 1)$ and $f(x)$ is N times continuously differentiable for $|x| \leq 1$, then

$$\int_{-1}^{1} \frac{f(x)\, e^{i\omega x}}{(1+x)^{\alpha}(1-x)^{\beta}} dx = D_N(\omega) - C_N(\omega) + \mathrm{O}\left(\omega^{-N-1}\right),\ as\ \omega \to \infty,$$

where

$$C_N(\omega) = \sum_{s=1}^{N} \tfrac{\Gamma(s+\alpha-1)}{\Gamma(s)} \sqrt{e^{\pi(s+\alpha-2)}}\omega^{-s-\alpha} e^{-i\omega} \tfrac{d^{s-1}}{dx^{s-1}} \left[(1-x)^{\beta} f(x)\right]_{x=-1},$$
$$D_N(\omega) = \sum_{s=1}^{N} \tfrac{\Gamma(s+\beta-1)}{\Gamma(s)} \sqrt{e^{\pi(s+\beta-2)}}\omega^{-s-\beta} e^{i\omega} \tfrac{d^{s-1}}{dx^{s-1}} \left[(1+x)^{\alpha} f(x)\right]_{x=1}.$$

Oscillatory integrals play a vital role in applied mathematics, computerized tomography, image processing, astronomy, electromagnetic, seismology, and quantum chemistry (for more see, [10, 11, 12, 13, 14, 15] and the references therein). There have been great papers with phenomenal methods for the efficient evaluation of oscillatory integrals. The earliest numerical method was formulated by Filon [16]. He achieved the successful approximation of $f(x)$ by a polynomial with a second degree to generate the oscillatory equivalent of Simpson's rule. Further, other numerical strategies of some weakly singular oscillatory integrals are Filo-Clenshaw-Curtis [17], Asymptotic methods [18]. However, all those methods exhibit some shortcomings when the frequency is very large.

In the case $\omega \geq 1$, $v = 1$ in (1) the integral presents more difficulties and the classical Gauss rules cannot directly be applied, due to the fact that not only does the integral becomes highly oscillatory but also becomes unbounded at $x = \mu$. Our main contribution in this paper is that we present a method that is efficient for quite low, moderate and very high value of frequency. In this approach, we combine the classical Clenshaw-Curtis (CC) methods and the Steepest descent method to produce a single efficient method. First, the given function is approximated by the truncated Chebyshev series, term by term, and the oscillatory factor is approximated by using Bessel function of the first kind. Subsequently, the modified moments are computed efficiently using the numerical steepest descent method or special functions in some cases.

The paper is arranged as follows: In the next section, we give the numerical formulation of the proposed method. In Section 3, we present the efficient computation of the modified moments. In Section 4, selected numerical examples are given to substantiate our theoretical analysis. Lastly, concluding remarks are presented in Section 5.

2. Main results

2.1. Numerical formulation of the method. In the sequel, we are concerned with the evaluation of the integral (1) for $v = 1$

$$I_{-1}^{1}(f;\mu) = \int_{-1}^{1} \frac{f(x)\, e^{i\omega x}}{(1+x)^{\alpha}(1-x)^{\beta}(x-\mu)} dx, \quad -1 < \mu < 1. \tag{2}$$

We start by writing the approximation of the smooth function $f(x)$ by the truncated Chebyshev series

$$f(x) = \sum_{l=0}^{N} a_{l,j} T_l(x), \tag{3}$$

where the double prime in the summation indicates that the first and the last terms have half weights and $T_l(x)$ is the l^{th}Chebyshev polynomial of the first kind. It is worth pointing out that the formula (3) would always give better approximation if $f(x)$ is smooth on $[-1,1]$, even for small values of N. To compute the coefficient $a_{l,j}$ we first assume that $x_j = \cos\frac{\pi j}{N}$, $0 \le j \le N$, then by orthogonality conditions [19], we have

$$\sum_{j=0}^{N}{}'' T_l(x_j)\, T_k(x_j) = \begin{cases} 0, & \text{if } l \neq k, \\ N, & \text{if } l = k = 0 \text{ or } N, \\ \frac{N}{2}, & \text{if } l = k \neq 0 \text{ or } N. \end{cases}$$

Multiplying on both sides of (3) by $T_l(x_j)$, yields

$$\begin{aligned} \sum_{j=0}^{N}{}'' f(x_j)\, T_l(x_j) &= \sum_{j=0}^{N}{}'' \sum_{l=0}^{N}{}'' a_{l,j} T_l(x_j)\, T_l(x_j) \\ &= \sum_{j=0}^{N}{}'' \sum_{l=0}^{N}{}'' a_{l,j} \cos\left(\frac{l\pi j}{N}\right) \cos\left(\frac{l\pi j}{N}\right) \\ &= \frac{N}{2} a_{l,j} \end{aligned}$$

Therefore, the coefficient $a_{l,j}$ can be computed using the formula

$$a_{l,j} = \frac{2}{N} \sum_{j=0}^{N}{}'' f\left(\cos\frac{\pi j}{N}\right) \cos\left(\frac{l\pi j}{N}\right), \tag{4}$$

or simply by writing

$$a_{l,j} = \frac{2}{N} \sum_{j=0}^{N} {}'' f(x_j)\, T_l(x_j)\,. \tag{5}$$

Throughout the paper $x_j = \cos \frac{\pi j}{N}$, $0 \le j \le N$, are the $(N+1)-$ Clenshaw-Curtis point set. If desired, one may also utilize the following formula:

$$a_{l,j}{}' = \frac{2}{N+1} \sum_{j=0}^{N} f\left(\cos \frac{\pi(2j+1)}{2(N+1)}\right) \cos\left(\frac{\pi l(2j+1)}{2(N+1)}\right). \tag{6}$$

In Table 6, we show the computational differences using both coefficients (5) and (6) for various values of the frequency ω and N fixed.

The method by Cooley and Turkey [20], can be employed to efficiently compute (4) and their algorithm costs only $\mathrm{O}(N \log N)$ operations by FFT (First Fourier Transform). Note that the CC quadrature methods which have been investigated in several papers [21, 22], interpolates $f(x)$ at CC points. Hence, substitute (3) into (2) we get

$$I_{N,l}(f;\mu) = \sum_{l=0}^{N} {}'' a_{l,j} M_{N,j}(\alpha,\beta,l,\mu,\omega)\,. \tag{7}$$

In the above expression $M_{N,j}(\alpha,\beta,l,\mu,\omega)$ is the modified moments which involves singularity types and it can be denoted as

$$M_{N,j}(\alpha,\beta,l,\mu,\omega) = \int_{-1}^{1} \frac{e^{i\omega x} T_l(x)}{(1+x)^{\alpha}(1-x)^{\beta}(x-\mu)} dx, \quad -1 < \mu < 1. \tag{8}$$

We may proceed by approximating the oscillatory factor $e^{i\omega x}$ into Bessel function of order j as

$$e^{ix\omega} = 2 \sum_{j=0}^{\infty}{}' i^j J_j(\omega) T_j(x). \tag{9}$$

Then employ the following Chebyshev equality [27]

$$T_l(x) T_j(x) = \frac{1}{2}\left(T_{l+j}(x) + T_{|l-j|}(x)\right), \tag{10}$$

and

$$d_{l,j} = \frac{M_{l+j} + M_{|l-j|}}{2}. \tag{11}$$

Substituting the above equations (9), (10) and (11) into (7) we obtain a new approximation which can be written as

$$I_k(f,\mu) = 2 \sum_{l=0}^{N}{}'' \sum_{j=0}^{\infty}{}' i^j J_j(\omega) a_{l,j} M_k(\alpha,\beta,l,\mu,\omega), \tag{12}$$

where

$$M_k(\alpha,\beta,l,\mu,\omega) = \int_{-1}^{1} \frac{T_k(x)}{(1+x)^{\alpha}(1-x)^{\beta}(x-\mu)} dx. \tag{13}$$

For $\alpha = \beta = 1/2$ (13) can be computed directly from [23] as

$$\begin{aligned}&\int_{-1}^{1} \frac{T_k(x)\,dx}{(1+x)^{\alpha}(1-x)^{\beta}(x-\mu)} \\ &= \left(\frac{4^k\Gamma(1+K)^2}{\Gamma(1+2k)}\right)\pi\cot(-\pi\beta)(1+\mu)^{\alpha}(1-\mu)^{\beta}P_k^{(-\beta,-\alpha)}(\mu) \\ &-\frac{2^{-\alpha-\beta}\Gamma(-\beta)\Gamma(-\alpha+1+k)\,{}_2F_1\left(1+k,\beta+\alpha-k,\beta+1,\frac{1-\mu}{2}\right)}{\Gamma(-\beta-\alpha+1+k)},\end{aligned}$$

where $\Gamma(z) = \int_0^{\infty} y^{z-1}e^{-y}dy$ is the Gamma function, $P_k^{(a,b)}(z)$ is the Jacobi polynomials and ${}_2F_1(a,b;c;x)$ is the Hypergeometric function [24], that has the series expansion of

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k x^k}{k!(c)_k},$$

with $(.)_k$ being the Pochhammer's symbol.

3. Computation of the Modified Moments (8)

Herein, we demonstrate efficiently the computation of the modified moments (8) and its evaluation is given in the below theorem

Theorem 3.1. The modified moments (8) $M_{N,j}(\alpha,\beta,l,\mu,\omega)$, $-1 < \mu < 1$ can be transmuted as

$$M_{N,j}(\alpha,\beta,l,\mu,\omega) = G_1(\alpha,\beta,l,\mu,\omega) + G_2(\alpha,\beta,l,\mu,\omega) + \frac{i\pi T_l(\mu)e^{i\omega\mu}}{(1+\mu)^{\alpha}(1-\mu)^{\beta}}, \tag{14}$$

where

$$G_1(\alpha,\beta,l,\mu,\omega) = \omega^{(\alpha-1)}i^{(1-\alpha)}e^{-i\omega}\int_0^{\infty}\frac{T_l\left(-1+i\frac{t}{\omega}\right)e^{-t}}{t^{\alpha}\left(2-i\frac{t}{\omega}\right)^{\beta}\left(-1-\mu+i\frac{t}{\omega}\right)}dt, \tag{15}$$

and

$$G_2(\alpha,\beta,l,\mu,\omega) = \omega^{(\beta-1)}i^{(1-\beta)}e^{i\omega}\int_0^{\infty}\frac{T_l\left(1+i\frac{t}{\omega}\right)e^{-t}}{t^{\beta}\left(2+i\frac{t}{\omega}\right)^{\alpha}\left(1-\mu+i\frac{t}{\omega}\right)}dt. \tag{16}$$

Proof. Consider the region Γ in a closed complex plane bounded by $P = \cup_{k=1}^{8}P_k$, with

$$\begin{aligned}&P_1: y_1(x) = -1+re^{ix},\ 0\le x\le\tfrac{\pi}{2}, && P_5: y_5(x) = 1+re^{ix},\ \tfrac{\pi}{2}\le x\le\pi,\\ &P_2: y_2(x) = x,\ -1+r\le x\le\mu-r, && P_6: y_6(x) = 1+ix,\ r\le x\le R,\\ &P_3: y_3(x) = \mu+re^{ix},\ 0\le x\le\pi, && P_7: y_7(x) = x+iR,\ -1\le x\le 1,\\ &P_4: y_4(x) = x,\ \mu+r\le x\le 1-r, && P_8: y_8(x) = -1+ix,\ r\le x\le R.\end{aligned} \tag{17}$$

Assume that

$$\phi(z) = \frac{T_l(z)e^{i\omega z}}{(1+z)^{\alpha}(1-z)^{\beta}(z-\mu)},$$

since $\phi(z)$ is analytic at all points in the half-strip of the complex plane $-1 \leq \Re e(z) \leq 1, \Im(z) \geq 0$. Then by Cauchy-Goursat theorem we have

$$\oint_P \frac{T_l(z)e^{i\omega z}}{(1+z)^\alpha(1-z)^\beta(z-\mu)}dz = \sum_{k=1}^{8}\int_{P_k} \frac{T_l(x)e^{i\omega x}}{(1+x)^\alpha(1-x)^\beta(x-\mu)}dx = 0,$$

which can be re-written from the Fig.1 with ease as

$$\int_{-1+r}^{\mu-r}\phi(x)\,dx + \int_{\mu+r}^{1-r}\phi(x)\,dx = -\left(\oint_{P_1} + \oint_{P_3} + \oint_{P_5} + \sum_{m=6}^{8}\oint_{P_m}\right)\phi(z)\,dz, \tag{18}$$

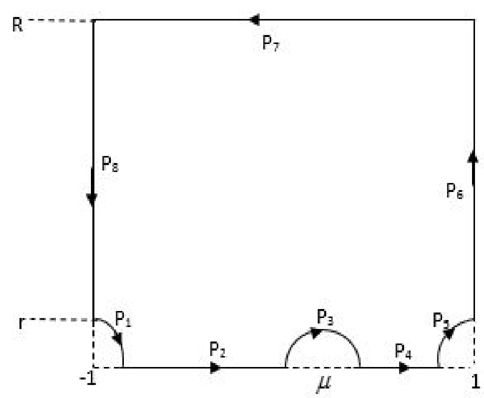

FIGURE 1. Integration paths for the integral (8)

where the direction of all contours taken counterclockwise as shown in Fig. 1. It is rather simple to show that the integrals over the quarter circle P_1 and P_5 results in zero as $r \to 0$. For instance, let $P_1 : z = -1 + re^{ix}, x \in \left[0, \frac{\pi}{2}\right]$, we have

$$\begin{aligned}
\left|\oint_{P_1} \frac{T_l(z)e^{i\omega z}}{(1+z)^\alpha(1-z)^\beta(z-\mu)}dz\right| &= \left|-\int_0^{\frac{\pi}{2}} \frac{T_l\left(-1+re^{ix}\right)e^{i\omega\left(-1+re^{ix}\right)}ire^{ix}}{(re^{ix})^\alpha(2-re^{ix})^\beta(-1+re^{ix}-\mu)}dx\right| \\
&\leq r^{1-\alpha}\left|\int_0^{\frac{\pi}{2}} \frac{T_l\left(-1+re^{ix}\right)e^{-i\omega}e^{i\omega re^{ix}}e^{ix-ix\alpha}}{(2-re^{ix})^\beta(-1+re^{ix}-\mu)}dx\right| \\
&\leq r^{1-\alpha}\left|\int_0^{\frac{\pi}{2}} F(r,x)\,dx\right| \to 0,\ as\ \ r \to 0.
\end{aligned} \tag{19}$$

In the above expressions, the function $F(y,\theta)$ is continuous on $y \leq r$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, which shows that the direct calculation leads to zero as $r \to 0$. Similar conclusion can be obtained on the integral over the quarter circle P_5 which gives $\oint_{P_5} \frac{T_l(z)e^{i\omega z}}{(1+z)^\alpha(1-z)^\beta(z-\mu)}dz \to 0$, as $r \to 0$. The parameterization on $P_7 : z =$

$x+iR,\, x\in[-1,1]$, gives us

$$\begin{aligned}\left|\oint_{P_7}\phi(z)\,dz\right| &= \left|-\int_{-1}^{1}\frac{T_l(x+iR)e^{i\omega(x+iR)}}{(1+x+iR)^{\alpha}(1-x+iR)^{\beta}(x+iR-\mu)}dx\right|\\ &\leq e^{-\omega R}\int_{-1}^{1}\frac{|T_l(x+iR)|\left|e^{i\omega x}\right|}{|1+x+iR|^{\alpha}|1-x+iR|^{\beta}|x+iR-\mu|}dx\\ &\to 0 \quad as\ R\to\infty.\end{aligned}\tag{20}$$

The integral over $P_6: z=1+ix,\, r\leq x\leq R$ yields

$$-\oint_{P_6}\phi(z)\,dz = e^{i\omega}(-i)^{1-\beta}\int_r^R\frac{T_l(1+ix)\,e^{-\omega x}}{(2+ix)^{\alpha}x^{\beta}(1+ix-\mu)}dx.\tag{21}$$

The integral along the path $P_8: z=-1+ix,\, x\in[r,R]$ gives

$$-\oint_{P_8}\phi(z)\,dz = e^{-i\omega}i^{1-\alpha}\int_r^R\frac{T_l(-1+ix)\,e^{-\omega x}}{x^{\alpha}(2-ix)^{\beta}(-1+ix-\mu)}dx.\tag{22}$$

Additionally, the integral over the half-circle with the following parameterization: $P_3: z=\mu+re^{ix},\, 0\leq x\leq\pi$ yields

$$\begin{aligned}-\oint_{P_3}\phi(z)\,dz &= \int_0^{\pi}\frac{T_l\left(\mu+re^{ix}\right)e^{i\omega\left(\mu+re^{ix}\right)}ire^{ix}}{(1+\mu+re^{ix})^{\alpha}(1-\mu-re^{ix})^{\beta}(re^{ix})}dx\\ &= \int_0^{\pi}\frac{T_l\left(\mu+re^{ix}\right)e^{i\omega re^{ix}}ie^{i\omega\mu}}{(1+\mu+re^{ix})^{\alpha}(1-\mu-re^{ix})^{\beta}}dx\\ &= ie^{i\omega\mu}\frac{T_l(\mu)}{(1+\mu)^{\alpha}(1-\mu)^{\beta}},\quad as\ r\to 0.\end{aligned}\tag{23}$$

Taking the limits of the above integrals results as $r\to 0, R\to\infty$ and change the variable where $x={}^{t}\!/_{\omega}$, then substitute them into (18) lead to the conclusion of the Theorem 3.1.

□

3.1. Evaluation of the integral parts in (14) by Gauss-Laguerre rule. The semi-infinite integrals G_1 and G_2 in the theorem (14) can be accurately evaluated by the Generalized Gauss-Laguerre rule [25]. Moreover, they can be denoted in a simple form as

$$\begin{aligned}G_1(\alpha,\beta,l,\mu,\omega) &= \int_0^{\infty}H(t)\,t^{-\alpha}e^{-t}dt,\\ G_2(\alpha,\beta,l,\mu,\omega) &= \int_0^{\infty}Y(t)\,t^{-\beta}e^{-t}dt,\end{aligned}\tag{24}$$

where

$$H(t)=\omega^{(\alpha-1)}i^{(1-\alpha)}e^{-i\omega}\frac{T_l\left(-1+i\frac{t}{\omega}\right)}{\left(2-i\frac{t}{\omega}\right)^{\beta}\left(-1-\mu+i\frac{t}{\omega}\right)},$$

and

$$Y(t)=\omega^{(\beta-1)}i^{(1-\beta)}e^{i\omega}\frac{T_l\left(1+i\frac{t}{\omega}\right)}{\left(2+i\frac{t}{\omega}\right)^{\alpha}\left(1-\mu+i\frac{t}{\omega}\right)}.$$

Inasmuch as (24) agrees with the form $\int_0^{\infty}f(x)\,x^re^{-x}dx,\, r>-1$, then Generalized Gauss-Laguerre rule can be utilized to evaluate accurately both integrals G_1 and G_2. To do so, let $\left\{t_j^s, w_j^s\right\}_{j=1}^{N}$ be the nodes and weights of the N-point Generalized Gauss-Laguerre quadrature rule with respect to the weight function

$w_s = t^s e^{-t}$, where $s \in \{-\alpha, -\beta\}$ such that $-\alpha > -1$ and $-\beta > -1$. Therefore, G_1 and G_2 can be approximated by the sums

$$G_{1,N}(\alpha,\beta,l,\mu,\omega) = \sum_{j=1}^{N} w_{N,j}^{(-\alpha)} H\left(t_{N,j}^{-\alpha}\right) + E_1, \tag{25}$$

and

$$G_{2,N}(\alpha,\beta,l,\mu,\omega) = \sum_{j=1}^{N} w_{N,j}^{(-\beta)} Y\left(t_{N,j}^{-\beta}\right) + E_2, \tag{26}$$

where
$E_1 = \frac{C}{(2N)!} H^{(2N)}(\xi)$ and $E_2 = \frac{C}{(2N)!} Y^{(2N)}(\varsigma)$ for all $\xi, \varsigma \in \mathrm{C}$, and
$C = n!\Gamma(N+s+1)$ for $s \in \{-\alpha, -\beta\}$. Thus, the approximation of the modified moments (14) is given by the following formula

$$M_{N,j}(\alpha,\beta,l,\mu,\omega) = \sum_{j=1}^{N} w_{N,j}^{(-\alpha)} H\left(t_{N,j}^{-\alpha}\right) + \sum_{j=1}^{N} w_{N,j}^{(-\beta)} Y\left(t_{N,j}^{-\beta}\right) + \frac{i\pi T_l(\mu)\, e^{i\omega\tau}}{(1+\mu)^{\alpha}(1-\mu)^{\beta}}. \tag{27}$$

Theorem 3.2. *(*[26]*).* The errors approximate for the rule (27) can be estimated by

$$E_N[H;Y] = M_j(\alpha,\beta,l,\mu,\omega) - M_{N,j}(\alpha,\beta,l,\mu,\omega) = \mathrm{O}\left(\omega^{-2N-1+\max\{\alpha,\beta\}}\right),\ \omega \to \infty. \tag{28}$$

Theorem 3.3. Let $\alpha, \beta < 1$ and $F_N[f]$ be a polynomial approximate at the (N+1)-Clenshaw-Curtis points. Then the error estimate for $I_N[f;\mu]$ is

$$\left|I_{-1}^{1}[f;\mu] - I_{N,l}[f;\mu]\right| = \mathrm{O}(1),\ as\ \ \omega \to \infty. \tag{29}$$

Proof. Let $F_N[f]$ be an interpolating polynomial of degree N approximation to f at Clenshaw-Curtis points and assume for simplification that $p(x) = f(x) - F_N(x)$ and $m(x) = \frac{1}{(1+x)^{\alpha}(1+x)^{\beta}}$. Then

$$\begin{aligned}
&\left|I_{-1}^{1}[f;\mu] - I_{N,l}[f;\mu]\right| \\
&= \left|\int_{-1}^{1} \tfrac{(f(x)-F_N(x))m(x)e^{i\omega x}dx}{(x-\mu)}\right| \\
&= \left|(p(x)-p(\mu))\int_{-1}^{1} \tfrac{m(x)e^{i\omega x}dx}{(x-\mu)} + p(\mu)\int_{-1}^{1} \tfrac{m(x)e^{i\omega x}dx}{(x-\mu)}\right| \\
&= \left|(p(x)-p(\mu))\int_{-1}^{1} \tfrac{m(x)e^{i\omega x}dx}{(x-\mu)} + p(\mu)\int_{-1}^{1} \tfrac{(m(x)-m(\mu))e^{i\omega x}dx}{(x-\mu)} + m(\mu)\,p(\mu)\int_{-1}^{1} \tfrac{e^{i\omega x}dx}{(x-\mu)}\right| \\
&\le \sup_{-1\le x\le 1}\left|\tfrac{p(x)-p(\mu)}{(x-\mu)}\right| \int_{-1}^{1} |m(x)|\,dx + p(\mu)\sup_{-1\le x\le 1}\left|\tfrac{m(x)-m(\mu)}{(x-\mu)}\right| + |m(\mu)|\,|p(\mu)|\left|\int_{-1}^{1} \tfrac{e^{i\omega x}}{(x-\mu)}dx\right| \\
&= \mathrm{O}(1),\ as\ \ \omega \to \infty.
\end{aligned}$$

□

3.4 Algorithm: Computation of (2)
Input values: $\alpha, \beta, \omega, N, l, \mu$;
Output:

(1) Compute the generalized Laguerre $L_N^s(x)$ roots $x_{N,j}^s$ for $s \in \{-\alpha, -\beta\}$;
(2) Compute the weights for α as $\omega_{N,j}^{(-\alpha)} = \frac{\Gamma(N+1-\alpha)x_{N,j}^{-\alpha}}{\Gamma(N+1)\left[(N+1)L_{N+1}^{-\alpha}\left(x_{N,j}^{-\alpha}\right)\right]^2}$, $j = 1, ..., N$;
(3) Compute the weights for β as $\omega_{N,j}^{(-\beta)} = \frac{\Gamma(N+1-\beta)x_{N,j}^{-\beta}}{\Gamma(N+1)\left[(N+1)L_{N+1}^{-\beta}\left(x_{N,j}^{-\beta}\right)\right]^2}$, $j = 1, ..., N$;
(4) Set $N = l$; compute $M_{N,j}(\alpha, \beta, l, \mu, \omega)$ for $j = 1, ..., N$ as shown in (27);
(5) If $j == 0\, or\, j == N$ compute $1/N$ otherwise $2/N$;
(6) Compute $a_{l,j}$ for $j = 0, ..., N$; If desired skip step (5) and compute $a'_{l,j}$ for $j = 0, ..., N$ as shown in (6);
(7) Approximate $I_{N,l}$ for $l = 0, ..., N$ using (7);
(8) Return $I_{N,l}(f, \mu)$;

3.2. Program: MATHEMATICA program for the Algorithm (3.4). In this sub-section, a programming code in Mathematica version 9.0 for the automatic computation of the algorithm (3.4) presented in this paper is provided. Moreover, in order to assess our algorithm, we will use the precision equal to 32 decimal digits. The program will also display the computation time in seconds.

```
FF = N[I*Pi*Exp[I*ohmeg*mu]*
    ChebyshevT[l, mu]/((1 + mu)^alf (1 - mu)^bt), dig];
While[Nn < 201;
  x1nodes = N[Solve[LaguerreL[Nn, -alf, y1] == 0], dig] ;
  x11nodes = y1 /. x1nodes;
  x1N[m_] := x11nodes[[m]];
  weightx1 =
   Gamma[Nn + 1 - alf]*
    x11nodes/(Gamma[
        Nn + 1]*((Nn + 1) LaguerreL[(Nn + 1), -alf, x11nodes])^2 );
  x2nodes = N[Solve[LaguerreL[Nn, -bt, y2] == 0], dig] ;
  x22nodes = y2 /. x2nodes;
  x2N[m_] := x22nodes[[m]];
  weightx2 =
   Gamma[Nn + 1 - bt]*
    x22nodes/(Gamma[
        Nn + 1]*((Nn + 1) LaguerreL[(Nn + 1), -bt, x22nodes])^2 );
  weights1[m_] := weightx1[[m]];
  weights2[m_] := weightx2[[m]];
  M[l_, ohmeg_] :=
   I^(-alf + 1)*Exp[-I*ohmeg]*ohmeg^(alf - 1)*
     Sum[weights1[m]*
       ChebyshevT[
         l, -1 + I*x1N[m]/ohmeg]/((2 - I*x1N[m]/ohmeg)^
```

```
          bt*(-1 + I*x1N[m]/ohmeg - mu)), {m, 1,
      Nn}] + (-I)^(-bt + 1)*Exp[I*ohmeg]*ohmeg^(bt - 1)*
    Sum[weights2[m]*
      ChebyshevT[l,
        1 + I*x2N[m]/ohmeg]/((2 + I*x2N[m]/ohmeg)^
          alf (1 + I*x2N[m]/ohmeg - mu)), {m, 1, Nn}] +
   N[I*Pi*Exp[I*ohmeg*mu]*
     ChebyshevT[l, mu]/((1 + mu)^alf (1 - mu)^bt), dig];
 K[j_, Nn_] := If[j == 0 \[Or] j == Nn, 1/Nn, 2/Nn];
 CK[Nn_, l_] :=
  N[Sum[K[j, Nn] f[Cos[Pi*j/Nn]]*Cos[l*Pi*j/Nn], {j, 0, Nn}], dig];
 Inl := N[
   CK[Nn, 0]*(M[0, ohmeg])/2 + CK[Nn, Nn]*(M[Nn, ohmeg])/2 +
    Sum[CK[Nn, l]*(M[l, ohmeg]), {l, 1, (Nn - 1)}], dig];];
Block[{$MaxExtraPrecision = 0}, Print[N[Inl, dig]]; A = N[Inl, dig];] //
     Timing
(***********************************************)
(*** INPUT VALUES ***)
dig = 32;(* Working Precision for internal Computation *)
alf = 1/10; bet = 1/2; mu = 1/2;
omg = 1000000;(* Frequency *)
f[t_] := t*Exp[t^2];(*Given Function*)
Nn = l = 24;(* Quadrature points *)
(** OUTPUT is displayed by Inl or A **)
```

Executing the above program with the given input values, the program will display the following output values:
$-0.483358696297361025556467907026 45-2.70356368055024553392305885107 49i$
$\{0.687500, Null\}$
The values in braces are the computation time in seconds and for the above example, it is 0.687500 seconds. The relative error is $1.2880173\text{x}10^{-23}$. This proves that, the algorithm presented in this paper is accurate enough to give a 16-digit accurate approximation.

4. Numerical experiments

Herein, selected numerical examples are given to test the performance of the algorithm suggested for efficient computation of oscillatory integrals having Jacobi-Cauchy type singularities. All experiments were conducted using Mathematica version 9.0. In some examples computation time in seconds is provided. The approximation, CPU and relative errors achieved will depend on the values of $\alpha, \beta, \omega, \mu$ and N.
Example 1. We compute the integral

$$I_{-1}^{1}\left(f;\tau\right)=\int_{-1}^{1}\frac{xe^{x^{2}}e^{i\omega x}}{\left(1+x\right)^{1/10}\left(1-x\right)^{1/2}\left(x-\frac{1}{2}\right)}dx. \tag{30}$$

Achieved results are summarized in Table 1.

TABLE 1. Errors for the integral (30) with several values of N and ω.

	N				
ω	15	22	26	30	32
5	$2.6\text{x}10^{-8}$	$1.7\text{x}10^{-10}$	$1.0\text{x}10^{-10}$	$1.9\text{x}10^{-11}$	$8.6\text{x}10^{-12}$
10	$4.0\text{x}10^{-9}$	$3.6\text{x}10^{-13}$	$5.2\text{x}10^{-14}$	$4.7\text{x}10^{-15}$	$1.5\text{x}10^{-15}$
50	$2.2\text{x}10^{-10}$	$1.6\text{x}10^{-14}$	$1.2\text{x}10^{-17}$	$6.5\text{x}10^{-22}$	$7.2\text{x}10^{-23}$
10^2	$6.5\text{x}10^{-11}$	$1.4\text{x}10^{-14}$	$1.3\text{x}10^{-17}$	$1.3\text{x}10^{-22}$	$6.0\text{x}10^{-23}$
10^4	$5.1\text{x}10^{-14}$	$1.1\text{x}10^{-14}$	$9.7\text{x}10^{-18}$	$9.3\text{x}10^{-26}$	$4.5\text{x}10^{-23}$

Example 2. We consider the integral

$$I_{-1}^{1}\left(f;\tau\right)=\int_{-1}^{1}\frac{e^{i\omega x}\sin x}{\left(1+x\right)^{1/2}\left(1-x\right)^{1/4}\left(x-0.32\right)}dx. \tag{31}$$

Table 2, elaborate the obtained results.

TABLE 2. Errors for the integral (31) with various values of N,ω and $\mu=0,32$.

	N			
ω	15	20	30	40
5	$2.0\text{x}10^{-9}$	$9.1\text{x}10^{-11}$	$4.8\text{x}10^{-13}$	$5.9\text{x}10^{-15}$
10	$8.9\text{x}10^{-13}$	$1.1\text{x}10^{-14}$	$6.2\text{x}10^{-15}$	$1.1\text{x}10^{-20}$
50	$3.5\text{x}10^{-20}$	$7.5\text{x}10^{-21}$	$7.5\text{x}10^{-21}$	$7.1\text{x}10^{-21}$
10^2	$2.5\text{x}10^{-20}$	$2.8\text{x}10^{-24}$	$2.8\text{x}10^{-24}$	$2.8\text{x}10^{-24}$

Example 3. We compute the integral

$$I_{-1}^{1}\left(f;\tau\right)=\int_{-1}^{1}\frac{\left(3x^{3}-2x+5\right)e^{i\omega x}}{\left(1+x\right)^{1/100}\left(1-x\right)^{1/300}\left(x-3\right)\left(x-0.4\right)}dx. \tag{32}$$

Errors and time in seconds achieved are summarized in Table 3.

TABLE 3. Relative errors and computation time in seconds for the integral (32) with $N=22$.

ω	Relative Errors	Execution time
10	$1.680\text{x}10^{-14}$	0.984
10^2	$6.076\text{x}10^{-17}$	0.968
10^3	$6.157\text{x}10^{-17}$	1.062
10^4	$3.162\text{x}10^{-17}$	1.034
10^5	$6.165\text{x}10^{-17}$	1.032
10^6	$6.162\text{x}10^{-17}$	1.015

Example 4. We consider the Cauchy principal value integral

$$I_{-1}^{1}(f;\tau)=\int_{-1}^{1}\frac{(x+1)\ln(x+5)\,e^{i\omega x}}{(1+x)^{99/100}(1-x)^{1/6}(x^2+1)(x-0.79)}dx. \qquad (33)$$

Accuracy achieved are shown in Table 4.

Table 4. Relative errors and computation time in seconds for the integral (33) with $N=38$.

ω	Relative Errors	Execution time
10	1.827x10^{-11}	2.718
10^2	2.138x10^{-16}	2.593
10^3	2.351x10^{-17}	1.765
10^4	2.562x10^{-17}	2.390
10^5	2.911x10^{-17}	2.750
10^6	2.908x10^{-17}	4.406

Example 5. In the bellow example we consider the following Cauchy integral

$$I_{-1}^{1}(f;\tau)=\int_{-1}^{1}\frac{f_j(x)\,e^{i\omega x}}{(1+x)^{\alpha}(1-x)^{\beta}(x-\mu)}dx,\quad j=1,2,3, \qquad (34)$$

with various values of α,β,μ and N,ω are fixed. Obtained results are summarized in Table 5.

Table 5. Errors for the integral (34) with various values of $\alpha,\beta,\mu$and $N=38,\omega=10^3$.

$f_j(x)$	α	β	μ	Relative Errors
$\frac{xe^x}{1+x^2}$	1/4	1/3	0.54	$3.49x10^{-15}$
	1/100	1/100	0.5	$3.51x10^{-15}$
	1/10	1/2	0.5	$2.03x10^{-15}$
$\frac{\cos x}{10+x^2}$	1/4	1/3	0.90	$3.43x10^{-20}$
	1/100	1/100	0.90	$3.28x10^{-20}$
	1/10	1/2	0.90	$4.43x10^{-20}$
$\frac{\tan x}{75+x^2}$	1/4	1/3	0.45	$4.71x10^{-17}$
	1/100	1/100	0.45	$2.77x10^{-17}$
	1/10	1/2	0.45	$4.58x10^{-17}$

Example 6. Consider the integral

$$I_{-1}^{1}(f;\tau)=\int_{-1}^{1}\frac{e^{i\omega x}}{(x^2+10)(1+x)^{1/5}(1-x)^{2/3}(x-0.26)}dx. \qquad (35)$$

Moreover, we compare the integral (35) using two coefficients given in (5) and (6) with various values of ω and N fixed. Results are shown in Table 6.

TABLE 6. Errors for the integral (35) using coefficients (5) and (6) with various values of ω and N fixed.

	Relative Errors	
μ	$a_{l,j}$	$a_{l,j}'$
10	$2.77\text{x}10^{-15}$	$2.73\text{x}10^{-15}$
10^2	$2.57\text{x}10^{-18}$	$3.39\text{x}10^{-17}$
10^3	$2.59\text{x}10^{-18}$	$2.89\text{x}10^{-17}$
10^4	$2.48\text{x}10^{-18}$	$3.03\text{x}10^{-17}$
10^5	$2.37\text{x}10^{-18}$	$3.14\text{x}10^{-17}$
10^6	$2.41\text{x}10^{-18}$	$3.10\text{x}10^{-17}$

5. Conclusion

In this paper, we have efficiently computed oscillatory integrals having Jacobi-Cauchy type singularities kernel using the method proposed. These types of integrals can be found in many physical and engineering problems. The method employed can be regarded as the application of the traditional Clenshaw-Curtis algorithms to the oscillatory Cauchy-type singular integrals. The proposed method exhibited the expected high accuracy and efficiency for quite small, moderate, and large frequency values. Moreover, it can be shown that the efficiency of the proposed method depends on the frequency ω and N. The given tables illustrate the validity of the proposed method. Lastly, it has been established that the proposed method can be also applied in the computation of hypersingular integrals.

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