J. Appl. Math. & Informatics Vol. **40***(2022), No. 1 - 2, pp. 117 - 132*
https://doi.org/10.14317/jami.2022.117

REVERSE EDGE MAGIC LABELING OF CARTESIAN PRODUCT, UNIONS OF BRAIDS AND UNIONS OF TRIANGULAR BELTS

KOTTE AMARANADHA REDDY AND S. SHARIEF BASHA*

ABSTRACT. Reverse edge magic(REM) labeling of the graph $G = (V, E)$ is a bijection of vertices and edges to a set of numbers from the set, defined by $\lambda : V \cup E \to \{1, 2, 3, ..., |V| + |E|\}$ with the property that for every $xy \in E$, constant k is the weight of equals to a xy, that is $\lambda(xy) - [\lambda(x) + \lambda(x)] = k$ for some integer k. We given the construction of REM labeling for the Cartesian Product, Unions of Braids and Unions of Triangular Belts. The Kotzig array used in this paper is the $3 \times (2r + 1)$ kotzig array. we test the konow results about REM labelling that are related to the new results we found.

AMS Mathematics Subject Classification : 05C78.
Key words and phrases : Reverse edge magic labeling, Cartesian product, unions of braids, unions of triangular belts.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. Labeling of G is a bijection $f : V \cup E \to \{1, 2, 3, ..., |V| + |E|\}$. If $x, y \in V$ and if $e = xy \in E$, then the weight $w(e)$ of the edge e is given by $w(e) = f(e) - \{f(x) + f(y)\}$. The total labeling f is said to be reverse edge-magic (REM) labeling if the weight of each edge is a constant and this constant is called the magic constant of the REM labeling. REM labeling is called reverse super edge magic (RSEM) labeling if the vertices are labeled using the smallest $|V|$ integers. In [2], the result for REM labeling of a complete bipartite graph stated by Kotzig and Rosa. They used the terminology M-valuation, which is now known as EMT labeling and also stated the preservation an EMT labeling for the odd number of copies of certain graphs. They used the term edge-magic to describe a graph that has REM labeling. In [4], The method to expand the result in EMT labeling for some families

Received August 4, 2021. Revised October 24, 2021. Accepted December 7, 2021.
*Corresponding author.

of graphs is introduced by I. Singgih. In, [1] used the results for EMT labeling of 2-regular graphs for the method of generalized by S. Cichacz-Przenioslo et al. In this section, we will describe first the method that later applied to construct a REM labeling. This method preserves the REM (RSEM) properties as we extend the length of cycles, or multiplying the number of paths, by a factor of an odd number. In [3] Marr and Wallis give a definition of a Kotzig array as $d*m$ grid, each row being a permutation of $\{0,1,\ \ldots,\ m-1\}$ and each column having the same sum. The Kotzig array used in this paper is the $3*(2r+1)$ Kotzig array k that is given as an example after adding each the entry of the array by one:

$$k=\begin{bmatrix} 1 & 2\ \cdots\ r+1\ r+2\ \ldots\ 2r & 2r+1 \\ r+1 & r+2\ \ldots.\ 2r+1\ 1\ \ldots\ r-1 & r \\ 2r+1 & 2r-1\ \ldots\ 1\ 2r\ \ldots.\ 4 & 2 \end{bmatrix}$$

If we write the first two rows of k as a permutation cycle ,
we have:$\tau=(1,\ r+1,\ 2r+1,\ r,\ .\ .\ .,\ 3,\ r+3,\ 2,\ r+2)$

The difference between two consecutive elements in τ is equal to τ has taken modulo $(2r\ +\ 1)$. Note that $\tau$is a $(2r\ +\ 1)-$cycle. Since $(2r\ +\ 1)$ is an odd number for every non-negative integer r, then $gcd\,(2,\ 2r\ +\ 1)\ =\ 1$, and so we have τ^2 also a $(2r\ +\ 1)-$cycle. This fact plays an important role in preserving the properties of magic labeling of our REM and RSEM labeling as we extend the length of cycles. Let k' be the modified k, where we switched the first and second row of k' :

$$k=\begin{bmatrix} r+1 & r+2\ \ldots.\ 2r+1\ 1\ \ldots\ r-1 & r \\ 1 & 2\ \ldots.\ r+1\ r+2\ \ldots\ 2r & 2r+1 \\ 2r+1 & 2r-1\ \ldots.\ 1\ 2r\ \ldots.\ 4 & 2 \end{bmatrix}$$

It is clear that if we write the first two rows of a permutation cycle, we have τ^{-1}. All Cartesian products $C_k\square C_m$ that accept a group distance magic labelling by $Z_{k,m}$ are completely characterised by Dalibor Froncek in [10]. Toru Kojima was dividing three sections in [11] and they proved preliminary lemmas in Section 1 to get their results. Section 2 shows that the Cartesian product of paths and C_4- free super edge-magic graphs that satisfy certain conditions is C_4- supermagic. Section 3 demonstrates that the Cartesian product of paths, as well as some special classes of graphs like caterpillars, cycles, and the disjoint union of two copies of a caterpillar, are C_4 - super magic. Orientable Z_n- distance magic labelings of the Cartesian product of cycles were discovered by Bryan Freyberg and Melissa Keranen in [12]. Also proved even-ordered hypercubes are also orientable Z_n- distance magic. In [13], For any positive integer $n\geq 3$ and oddm ≥ 3, C. Palanivelu and N. Neela are given the existence of super (a,d)-edge antimagic total labelling of cartesian product of path and cycle $P_m\square C_m$.

In this article, we apply the method to construct a REM labeling for several other families of graphs and section 2, 3, 4 and 5 we estabilished the REM labelling of Cartesian Product $P_2\,\square\,C_n$, $P_2\,\square\,P_n$, Unions of Braids $mB\,(n)$ and Unions of Triangular Belts $mTB(\propto)$ by using new concept kotzig array to develop the multiplying the number of paths and extending the length of cycles. Also we develop the new results for each families of graphs are given by the following sections, we given some examples and describe how the method works.

Theorem 1.1. *The complete bipartite graph $K_{p,q}$ exists for all $p, q \geq 1$, form M-valuation.*

Theorem 1.2. *Say G is a 3-colorable edge-magic graph and H is the union of t disjoint copies of G, t odd. Then H is edge magic.*

Theorem 1.3. *Let G be a 2-regular graph that has a REM labeling γ. Let $G^{'}$ be a 2-regular graph obtained by extending the length of each component of G by an odd factor. Then there exists an REM labeling for $G^{'}$ that can be obtained by modifying the REM labeling of G.*

Proof. Let γ be a REM labeling for any 2-regular graph G. For every vertex and edge of G, let λ be the labeling obtained by decreasing the original label by 1, that is, let $\lambda(v) = \gamma(v) - 1$ and $\lambda(e) = \gamma(e) - 1$. For each cycle C_n in G, construct a $n \times 3$ table with entries as follows.

In the first column: For $i = 1, 2, \ldots, n$, the entry in the i^{th} row is the matrix

$$\Lambda = \begin{bmatrix} \lambda\left(v_i\right) \\ \lambda\left(v_{i+1}\right) \\ \lambda\left(e_{i+1}\right) \end{bmatrix}$$

In the second column: For $y = 1,\ 2,\ 3$ and $z = 1, 2, 3, \ldots, (2r+1)$ the entry in the i^{th} row is either k or $k^{'}$ depending on the value of i, namely $k = \left[k_{yz}\right]$, if $i \leq \left[\frac{n}{2}\right] + 1$, and $k^{'}{}_{yz}$, if $\left[\frac{n}{2}\right] + 1 < i \leq n$, where k_{yz} denotes the element on the y^{th} row and z^{th} column of k.

In the third column: for $i = 1, 2, \ldots., n$, the entry in the i^{th} row is the matrix

$$\Theta_i = \begin{cases} k_{yz} + (2r+1)\,\Lambda_{y1},\ if\ i \leq \left[\frac{n}{2}\right] + 1 \\ k'{}_{yz} + (2r+1)\,\Lambda_{y1},\ if\ \ \left[\frac{n}{2}\right] + 1 < i \leq n \end{cases}$$

If we multiply the permutation cycles of k and $k^{'}$ in the second column, we obtain $\tau^{\frac{n}{2}} + 1\tau^{n-\left(\left[\frac{n}{2}\right]-n+2\right)} = \tau^{2\left[\frac{n}{2}\right]-n+2}$ If n is odd we have $\tau^{(n-1)-n+2} = \tau$ and if n is even we have $\tau^{n-n+2} = \tau^2$.

The cycle $C_{n(2r+1)}$ is obtained by tracking the numbers on Θ. Let θ^i_{yz} denote the elements of Θ_i in the y^{th} row and z^{th} column. In each Θ_i, the two numbers θ^i_{1z} and θ^i_{2z} will be the labels of teo adjacent vertices on $C_{n(2r+1)}$, and θ^i_{3z} will be the label of the edge they share. For each i, $[1 \leq i \leq n,]$ each pair of $\theta^{i+1}_{1z}$and θ^{i+2}_{1z} that are equal denotes the same vertex on $C_{n(2r+1)}$ and all pairs $\theta^i_{1z}$and θ^i_{1z} represent labels of adjacent vertices.

Recall that in the second column, τ is a permutation cycle of length $2r+1$. Both 1 and 2 are relatively prime to $2r+1$ for any integer r, so $\tau = \tau^1$ and τ^2 are also permutation cycles of length $2r+1$. Consequently, we can track the labeling of $C_n\,(2r+1)$ by connecting these vertices from the third column continuously until we get a full circle of longer length (not stopping until all numbers in the third column are used). Since $1 \leq z \leq 2r\ +\ 1$, the result from this process is the labeled extended cycle $C_n(2r+1)$. For path component of G we create the same table, but since there is no relation between the endpoints, when tracking adjacent vertices in Θ_i from $i = 1$ until $i = m$, we will not be able to go back to $i = 1$. Every time we track adjacent vertices from $i = 1$ until $i = m$, we will get one copy of P_m instead. Since we have $(2r+1)$ columns in each Θ_i we end up with $(2r+1)$ copies of P_m instead of $P_m(2r+1)$ Combining all extended components, we obtain REM labeling for $G^{'}$. □

2. Cartesian Product $P_2 \square C_n$

Theorem 2.1. *If n is odd then the graph $P_m \square C_n$ has REM labeling with magic constant $k = \left(m - \frac{1}{2}\right) n - \frac{1}{2}$.*

Theorem 2.2. *The generalized prism $P_m \square C_n$ has an RSEM labeling if n is odd and $m \geq 2$.*

Theorem 2.3. *The graph $P_m \square C_n$ does not have an REM labeling for $n \equiv 2 \pmod 4$.*

Theorem 2.4. *The friendship graph Fr_n has an RSEM labeling if and only of $n \in \{3, 4, 5, 7\}$.*

Here we give an example of how to provide alternative ways of constructing REM (RSEM) labelings of the Cartesian product $P_2 \square C_3$ by using theorem 1.3. Using this method we can use the known REM (RSEM) labeling of $P_m \square C_n$ to obtain an REM (RSEM) labeling of $P_m \square C_{n(2r+1)}$.

Example 1: $P_2 \square C_3 \rightarrow P_2 \square C_9$

Using above theorems, we have an RSEM labeling for $P_2 \square C_3$ with $k = 4$ as shown in Figure 1.

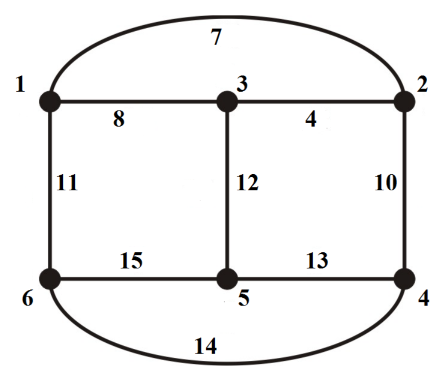

Figure 1: RSEM labeling for $P_2 \square C_3$

The tables are given below.

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
5	2	3	1	17	18	16
10	1	3	2	31	33	32

Λ	κ or κ'			θ_i		
2	1	2	3	7	8	9
4	2	3	1	14	15	13
11	1	3	2	34	36	35

Λ	κ or κ'			θ_i		
2	1	2	3	4	5	6
4	2	3	1	11	12	10
11	1	3	2	28	30	29

Table 1: Tables for $P_2 \square C_3 \to P_2 \square C_9$ (vertical paths)

Λ	κ or κ'			θ_i			Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3	5	1	2	3	16	17	18
2	2	3	1	8	9	7	4	2	3	1	14	15	13
9	1	3	2	28	30	29	14	1	3	2	43	45	44
2	1	2	3	7	8	9	4	1	2	3	13	14	15
1	2	3	1	5	6	4	3	2	3	1	11	12	10
3	1	3	2	11	12	10	12	1	3	2	37	39	38
1	2	3	1	5	6	4	3	2	3	1	11	12	10
0	1	2	3	1	2	3	5	1	2	3	16	17	18
6	1	3	2	19	21	20	13	1	3	2	40	42	41

Table 2: Tables for $P_2 \square C_3 \to P_2 \square C_9$ (cycles)

From the tables we get an RSEM for $P_2 \square C_9$ with $k = 13$ as shown in Figure 2

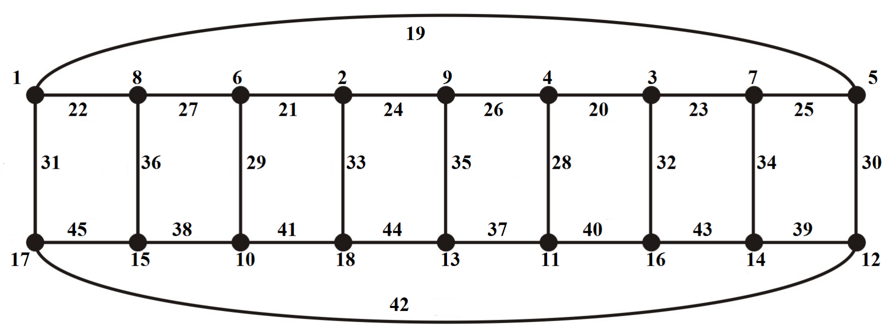

Figure 2: RSEM labeling for $P_2 \square C_9$

All results for REM labelings of $P_m \square C_n$ for odd values of n are already published. In this paper we proved only an alternative way to find such REM labelings of $P_m \square C_{n(2r+1)}$ from the known labelings of $P_m \square C_n$ for any positive r.

3. Cartesian Product $P_2 \square P_n$

Theorem 3.1. *If n is odd then the graph $P_2 \square P_n$ has an REM labeling with magic constant $k = n + 1$.*

Theorem 3.2. *If n is odd then the ladder $L_n \cong P_2 \square P_n$ has an RSEM labeling with magic constant $k = k + 1$.*

Unsurprisingly, when we apply theorem 1.3 to the graph $P_2 \square P_n$ it will multiply the number of the ladders instead of extending its length. We can obtain the RSEM labeling of $m\,(P_2 \square P_n)$ for any odd values of m as we explain in the following example.

Example 2: $P_2 \square P_5 \rightarrow m(P_2 \square P_5)$
RSEM labeling of $P_2 \square P_5$ with $k = 6$ is given as shown in Figure 3.

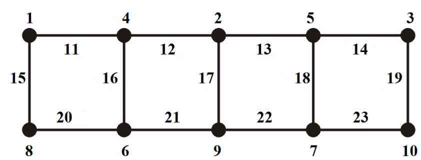

Figure 3: RSEM labeling for $P_2 \square P_5$

Tables for two P_5 are

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
10	1	3	2	31	33	32
3	1	2	3	10	11	12
1	2	3	1	5	6	4
11	1	3	2	34	36	35
1	1	2	3	4	5	6
4	2	3	1	14	15	13
12	1	3	2	37	39	38
4	2	3	1	14	15	13
2	1	2	3	7	8	9
13	1	3	2	40	42	41

Λ	κ or κ'			θ_i		
7	1	2	3	22	23	24
5	2	3	1	17	18	16
19	1	3	2	58	60	59
5	1	2	3	16	17	18
8	2	3	1	26	27	25
20	1	3	2	61	63	62
8	1	2	3	25	26	27
6	2	3	1	20	21	19
21	1	3	2	64	66	65
6	2	3	1	20	21	19
9	1	2	3	28	29	30
22	1	3	2	67	69	68

Table 3: Tables for $P_2 \square P_5 \rightarrow m(P_2 \square P_5)$ for P_5

Tables for five P_2 are

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
7	2	3	1	23	24	22
14	1	3	2	43	45	44

Λ	κ or κ'			θ_i		
2	1	2	3	10	11	12
5	2	3	1	17	18	16
15	1	3	2	46	48	47
Λ	κ or κ'			θ_i		
1	1	2	3	4	5	6
8	2	3	1	26	27	25
16	3	1	2	51	49	50
Λ	κ or κ'			θ_i		
4	1	2	3	13	14	15
6	2	3	1	20	21	19
19	1	3	2	52	54	53
Λ	κ or κ'			θ_i		
2	1	2	3	7	8	9
9	2	3	1	29	30	28
18	1	3	2	55	57	56

Table 4: Tables for $P_2\square P_5 \to m(P_2\square P_5)$ for P_2

From the tables we get RSEM for $P_2\square P_5$ with $k = 19$ as shown in Figure 4.

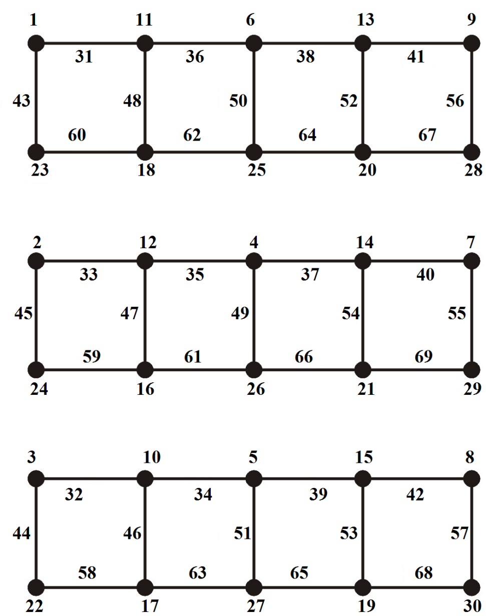

Figure 4: RSEM labeling for $3(P_2\square P_5)$

Hence we can summarize our new result in Theorem 3.3.

Theorem 3.3. *For any odd values of m and n, the graph $m(P_2 \square P_n)$ has an RSEM labeling with $k = m(n+1)+1$.*

Proof. The result follows from applying theorem 1.3 to result from Theorem 3.1 and 3.2 Performed on Friendship Graph Fr_n to RSEM labelings of friendship graphs, we obtain RSEM labeling for a new family of graph. To see how the method works we include the example below: □

Example 3: From Fr_3 using factor $(2r + 1) = 3$
RSEM labeling of Fr_3 with $k = 4$, given as shown in Figure 5.

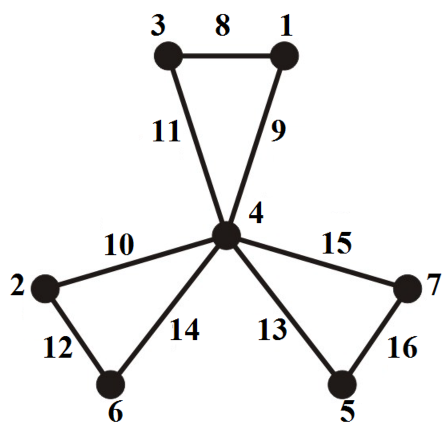

Figure 5: RSEM labeling for Fr_3

For the tables we treat each triangles in Fr_nas a cycle C_3 and make separate table for each cycle.

Λ	κ or κ'			θ_i		
3	1	2	3	10	11	12
0	2	3	1	2	3	1
8	1	3	2	25	27	26
0	1	2	3	1	2	3
2	2	3	1	8	9	7
7	1	3	2	22	24	23
2	2	3	1	8	9	7
3	1	2	3	10	11	12
10	1	3	2	31	33	32

Λ	κ or κ'			θ_i			Λ	κ or κ'			θ_i		
3	1	2	3	10	11	12	3	1	2	3	10	11	12
6	2	3	1	20	21	19	5	2	3	1	17	18	16
14	1	3	2	43	45	44	13	1	3	2	40	42	41
6	1	2	3	19	20	21	5	1	2	3	1 6	17	18
4	2	3	1	14	15	13	1	2	3	1	5	6	4
15	1	3	2	46	48	47	11	1	3	2	34	36	35
4	2	3	1	14	15	13	1	2	3	1	5	6	4
3	1	2	3	10	11	12	3	1	2	3	10	11	12
12	1	3	2	37	39	38	9	1	3	2	28	30	29

Table 5: Tables for theorem 1.3 performed on Fr_3 with factor $(2r + 1) = 3$

From the tables we get an RSEM for a new graph that shown in Figure 6.

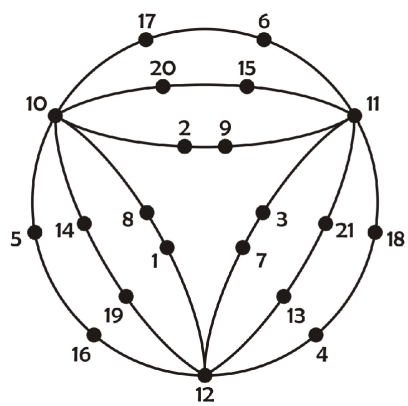

Figure 6: RSEM labeling for new graph from Fr_3

Due to limited space in the graph, the edge labels are not included in the figure. They can be found in the tables if required. For convenience let us denote this resulting graph by $C_t(n,m)$, where t is the number of vertices in each cycle, m is the number of common vertices where the distance between common vertices is always 2 and n is the number of triangles in the original friendship graph, which will become the number of layers of cycles (from inner to outer cycles) in the resulting graph. This way the graph in Figure 6 is denoted as $C_9(3,3)$ It has 3 layers of cycles (inner, middle and outer cycle), that do not share any common edges.

Theorem 3.4. *The friendship graph Fr_n has an RSEM labeling if and only of $n \in \{3,4,5,7\}$.*

Theorem 3.5. *The graph $C_3m(n,m)$ has an RSEM labeling when m is odd and $n \in \{3,4,5,7\}$.*

Proof. Observe that by applying Method 4 to a friendship graph using factor $m = 2r + 1$, every triangle in Fr_n will become a cycle of length $3m$, so the number of triangles (n) will become the number of layers in the new graph. Hence from RSEM labelings of all feasible values of n for friendship graph Fr_n that are stated in Theorem 3.4, we get the result above. □

$P_{2n}\,(+)\,N_m \rightarrow\ C_{(2n+1)(2r+1)}\,[+]\,N_m$

In performing the method to the graph $P_{2n}\,(+)\,N_m$, decompose the graph into a cycle with vertices $\{v_1, v_2, \ldots, v_{2n}, y_1\}$ and the paths $\{v_1, y_1, v_{2n}\}$, $i = 2, 3, \ldots, m$.

Example 4: $P_2\,(+)\,N_2 \rightarrow\ C_{15}\,[+]\,5N_1$

RSEM labeling of $P_2\,(+)\,N_2$ was given as shown in Figure 7.

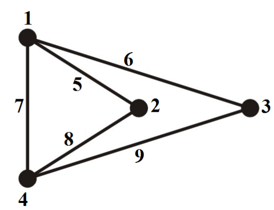

Figure 7: RSEM labeling for $P_2\,(+)\,N_2$

The tables are

Λ	κ or κ'					θ_i				
0	1	2	3	4	5	1	2	3	4	5
1	3	4	5	1	2	8	9	10	6	7
4	1	3	5	2	4	21	23	22	22	24
1	1	2	3	4	5	6	7	8	9	10
3	3	4	5	1	2	18	19	20	16	17
7	1	3	5	2	4	36	38	40	37	39
3	3	4	5	1	2	18	19	20	16	17
0	1	2	3	4	5	1	2	3	4	5
6	1	3	5	2	4	31	33	35	32	34

Table 6: Table for $P_2\,(+)\,N_2 \rightarrow\ C_{15}\,[+]\,5N_1$ (cycle)

Λ	κ or κ'					θ_i				
0	1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	13	14	15	11	12
5	1	3	5	2	4	26	28	30	27	29
2	1	2	3	4	5	11	12	13	14	15
3	3	4	5	1	2	18	19	20	16	17
8	1	3	5	2	4	41	43	45	42	44

Table 7: Table for $P_2\,(+)\,N_2 \rightarrow\ C_{15}\,[+]\,5N_1$ (path)

From the tables we get an RSEM for $C_{15}\,[+]\,5N_1$ with $k = 12$ as shown below.

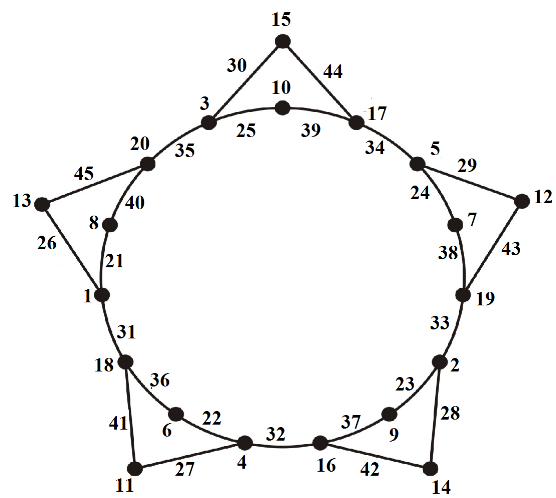

Figure 8: RSEM labeling for $C_{15}\,[+]\,5N_1$

4. Unions of Braids $mB\,(n)$

In this section we apply Method 4 to the result about braid graph $B(n)$ mentioned in this Section.

Example 7: $B\,(3) \to 3B\,(3)$

RSEM labeling of $B\,(3)$ with $k = 3$ was given as shown in Figure 9

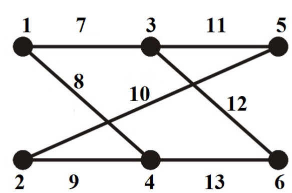

Figure 9. RSEM labeling for $B\,(3)$

Treat braid $B\,(3)$ as set of paths, consisting one central path with vertices label $(1, 3, 5, 2, 4, 6)$ and $2P_2$ with vertices label $(1, 4)$ and $(3, 6)$. For the second column of our method table, use all κ (not using κ' at all). Hence the tables for the method are

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
6	1	3	2	19	21	20
2	1	2	3	7	8	9
4	2	3	1	14	15	13
10	1	3	2	31	33	32
4	1	2	3	13	14	15
1	2	3	1	5	6	4
9	1	3	2	28	30	29
1	1	2	3	4	5	6
3	2	3	1	11	12	10
8	1	3	2	25	27	26
3	1	2	3	10	11	12
5	2	3	1	17	18	16
12	1	3	2	37	39	38

0	1	2	3	1	2	3
3	2	3	1	11	12	10
7	1	3	2	22	24	23

2	1	2	3	7	8	9
5	2	3	1	17	18	16
11	1	3	2	34	36	35

Table 8: Tables for $B\,(3) \to 3B\,(3)$

From the tables we get an RSEM for $3B\,(3)$ with $k = 10$ as shown in Figure 10.

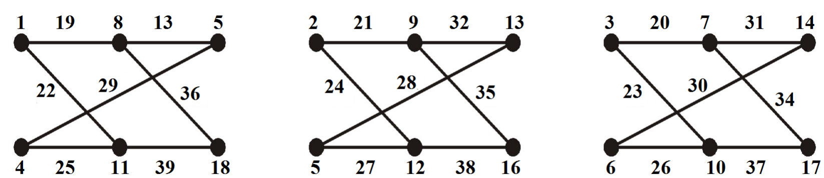

Figure 10. RSEM labeling for $3B\,(3)$

Applying Theorem 1.3 to braid graphs in general, we have the following theorem for unions of braids.

Theorem 4.1. *The braid graph $B(n)$ has an RSEM labeling for all $n \geq 3$.*

Theorem 4.2. *The union of braids $mB(n)$ has an RSEM labeling when m is odd.*

Proof. The result follows from applying theorem 1.3 to the graph in Theorem 4.2.

□

5. Unions of Triangular Belts $mTB(\propto)$

Next we apply Method 4 to the result about triangular belt $TB(\propto)$.
Example 8: $TB\ (\downarrow^3) \rightarrow 3TB\ (\downarrow^3)$
RSEM labeling of $TB\ (\downarrow^3)$ was given as shown in Figure 11.

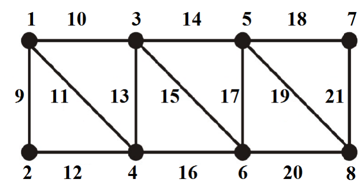

Figure 11. RSEM labeling for $TB\ (\downarrow^3)$

In applying theorem 1.3 to triangular belt $TB\ (\downarrow^3)$ we treat the graph as a collection of paths, without considering cycles that are contained in it. Thus for the second column of our method table we can also just use . The tables for the method are

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
9	1	3	2	28	30	29
2	1	2	3	7	8	9
4	2	3	1	14	15	13
13	1	3	2	40	42	41
4	1	2	3	13	14	15
6	2	3	1	20	21	19
17	1	3	2	52	54	53

Λ	κ or κ'			θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
11	1	3	2	34	36	35
3	1	2	3	10	11	12
5	2	3	1	17	18	16
15	1	3	2	46	48	47
5	1	2	3	16	17	18
3	2	3	1	23	24	22
19	1	3	2	58	60	59

Table 9: Tables for $TB\ (\downarrow^3) \rightarrow 3TB\ (\downarrow^3)$ (horizontal paths)

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
1	2	3	1	5	6	4
9	1	3	2	25	27	26

Λ	κ or κ'			θ_i		
2	1	2	3	7	8	9
3	2	3	1	11	12	10
12	1	3	2	37	39	38

Λ	κ or κ'			θ_i		
4	1	2	3	13	14	15
5	2	3	1	17	18	16
16	1	3	2	49	51	50

Λ	κ or κ'			θ_i		
6	1	2	3	19	20	21
7	2	3	1	23	24	22
20	1	3	2	61	63	62

Table 10: Tables for $TB\ (\downarrow^3) \rightarrow 3TB\ (\downarrow^3)$ (Vertical paths)

For the diagonal paths, define new matrix $\kappa^{''}$ as the matrix obtained by switching the second and third rows from $\kappa^{'}$

$$\kappa^{''} = \begin{bmatrix} 1 & 2 \cdots r+1 & r+1 \ \ldots \ 2r & 2r+1 \\ 2r+1 & 2r \ \ldots \ 1 & 2r-1 \ \ldots \ 4 & 2 \\ r+1 & r+2 \ \ldots \ 2r+1 & 2r \ \ldots \ r-1 & r \end{bmatrix}$$

Hence the table for the diagonal paths are

Λ	κ or κ'			θ_i		
0	1	2	3	1	2	3
3	3	1	2	12	10	11
10	2	1	3	32	31	33

Λ	κ or κ'			θ_i		
2	1	2	3	7	8	9
5	3	1	2	18	16	17
14	2	1	3	44	43	45

Λ	κ or κ'			θ_i		
4	1	2	3	13	14	15
7	3	1	2	24	22	23
18	2	1	3	56	55	57

Table 11: Tables for $TB\ (\downarrow^3) \rightarrow 3TB\ (\downarrow^3)$ (diagonal paths)

From the tables we get an RSEM labeling for $3TB\ (\downarrow^3)$ with $k = 19$.

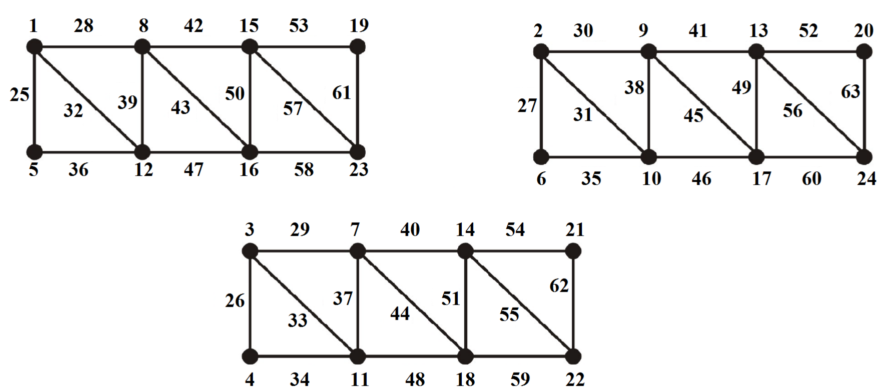

Figure 12. RSEM labeling for $3TB\ (\downarrow^3)$

Applying theorem 1.3 to triangular belts in general, we have the following theorem for union of triangular belts.

Theorem 5.1. *For any* $\propto \in S^n$, $S = \{\uparrow, \downarrow\}$,$n > 1$, *the triangular belt* $TB(\propto)$ *has an RSEM labeling.*

Theorem 5.2. *For any* $\propto \in S^n$, $S = \{\uparrow, \downarrow\}$,$n > 1$, *and odd* m, *the union of triangular belts* $mTB(\propto)$ *has an RSEM labeling.*

Proof. The result follows from applying Method 4 theorem 1.3 to the graph in Theorem 4.2. □

6. Conclusion

In this paper, we given the construction of REM labeling for the Cartesian Product, Unions of Braids and Unions of Triangular Belts. We will describe new method that later applied to construct a REM labeling. This method preserves the REM (RSEM) properties as we extend the length of cycles, or multiplying the number of paths, by a factor of an odd number.

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Kotte Amaranadha Reddy received M.Sc. from Sri Venkateswara University and doing Ph.D. at Vellore Institute of Technology, Vellore.

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, India.
e-mail: amar.anil159@gmail.com

S. Sharief Basha received M.Sc. from Sri Venkateswara University, and Ph.D. from Sri Venkateswara University. He is currently a Assistant professor(Sr) at Vellore Institute of Technology, Vellore since 2009. His research interests are discrete mathematics and fazzy graph theory.

Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore-632014, India.
e-mail: shariefbasha.s@vit.ac.in