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SOME PROPERTIES AND IDENTITIES FOR (p, q)-GENOCCHI POLYNOMIALS COMBINING (p, q)-COSINE FUNCTION

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ABSTRACT. The purpose of this paper is to find some properties and identities for (p, q)-cosine Genocchi polynomials. This polynomials which is one of Appell polynomials, have multifarious relations of (p, q)-other polynomials.

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1. Introduction

We begin by introducing several definitions related to (p, q)-number used in this paper(see [1-3, 8, 15-16]). For any $n \in \mathbb{N}$, the (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \qquad \text{where} \quad p \neq q, \tag{1.1}$$

which is a natural generalization of the q-number. From equation (1.1), we note that $[n]_{p,q} = [n]_{q,p}$.

Definition 1.1. For $n \geq k$, the Gaussian binomial coefficients are defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}, \tag{1.2}$$

where m and r are non-negative integers.

We note $[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q} \cdots [2]_{p,q}[1]_{p,q}$, where $n \in \mathbb{N}$. For $r = 0$, the value is 1 since the numerator and the denominator are both empty products. There are (p, q)-analogues of the binomial formula and this definition has a great number of properties, see [4].

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Definition 1.2. The (p, q)-analogues of $(x-a)^n$ and $(x+a)^n$ are defined by

(i) $(x \ominus a)^n_{p,q} = \begin{cases} 1, & \text{if } n = 0 \\ (x-a)(px-qa)\cdots(p^{n-1}x - q^{n-1}a), & \text{if } n \geq 1 \end{cases}$

(ii) $(x \oplus a)^n_{p,q}$

$$= \begin{cases} 1, & \text{if } n = 0 \\ (x+a)(px+qa)\cdots(p^{n-2}x + q^{n-2}a)(p^{n-1}x + q^{n-1}a), & \text{if } n \geq 1 \end{cases}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}. \tag{1.3}$$

Definition 1.3. Two forms of (p, q)-exponential functions can be expressed as

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}, \qquad E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!}. \tag{1.4}$$

From Definition 1.3, we can find an important property, $e_{p,q}(x)E_{p,q}(-x) = 1$, see [5-6, 8, 11, 15-16]. Moreover, U. Duran, M. Acikgos and S. Araci define $\widetilde{e}_{p,q}(x)$ in [7] as the follows:

$$\widetilde{e}_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!}. \tag{1.5}$$

Definition 1.4. Let $i = \sqrt{-1} \in \mathbb{C}$. Then the (p, q)-cosine functions are defined by

$$\cos_{p,q}(x) = \frac{e_{p,q}(ix) + e_{p,q}(-ix)}{2}, \qquad \mathrm{COS}_{p,q}(x) = \frac{E_{p,q}(ix) + E_{p,q}(-ix)}{2}, \tag{1.6}$$

where, $\mathrm{COS}_{p,q}(x) = \cos_{p^{-1},q^{-1}}(x)$.

From equation (1.5), Definitions 1.3 and 1.4, we can remark

(i) $E_{p,q}(ity) = COS_{p,q}(ty) + iSIN_{p,q}(ty)$

(ii) $E_{p,q}(-ity) = COS_{p,q}(ty) - iSIN_{p,q}(ty)$,

(i) $\widetilde{e}_{p,q}((x \oplus y)_{p,q}) = \sum_{n=0}^{\infty} \frac{(x \oplus y)^n_{p,q}}{[n]_{p,q}!} = e_{p,q}(x)E_{p,q}(y)$

(ii) $\widetilde{e}_{p,q}((x \ominus y)_{p,q}) = \sum_{n=0}^{\infty} \frac{(x \ominus y)^n_{p,q}}{[n]_{p,q}!} = e_{p,q}(x)E_{p,q}(-y)$

Definition 1.5. For $x \neq 0$, the (p, q)-derivative of a function f with respect to x is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \tag{1.7}$$

and $(D_{p,q}f)(0) = f'(0)$, prove that f is differentiable at 0, and it is clear that $D_{p,q}x^n = [n]_{p,q}x^{n-1}$.

Definition 1.6. Let $|p/q| < 1$ and $x, y \in \mathbb{R}$. (p,q)-cosine Bernoulli polynomials ${}_CB_{n,p,q}(x,y)$ and (p,q)-cosine Euler polynomials ${}_C\mathcal{E}_{n,p,q}(x,y)$ are respectively defined by the following, see [13-14]:

$$\sum_{n=0}^{\infty} {}_CB_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \frac{t}{e_{p,q}(t)-1}e_{p,q}(tx)COS_{p,q}(ty),$$
$$\sum_{n=0}^{\infty} {}_C\mathcal{E}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \frac{2}{e_{p,q}(t)+1}e_{p,q}(tx)COS_{p,q}(ty).$$

Definition 1.7. The q-cosine Genocchi polynomials is defined by

$$\sum_{n=0}^{\infty} {}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}e_q(tx)COS_q(ty).$$

From the Definition 1.7, we note that ${}_CG_{n,q}(x,y) = {}_CG_n(x,y)$ when $q \to 1$, see [9]. The main goal of this paper is to find some properties of (p,q)-cosine Genocchi polynomials. In Section 2, we define the (p,q)-cosine Genocchi polynomials and find some properties. Moreover, we derive some relation between (p,q)-cosine Genocchi polynomials and (p,q)-other polynomials.

2. Main results

Definition 2.1. Let $|q/p| < 1$ with $x, y \in \mathbb{R}$. Then, (p,q)-cosine Genocchi polynomials is defined by the following.

$$\sum_{n=0}^{\infty} {}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \frac{2t}{e_{p,q(t)}+1}e_{p,q}(tx)COS_{p,q}(ty).$$

From the generating function of (p,q)-cosine Genocchi polynomials, we can note that

(i) $$\lim_{q\to 1}\sum_{n=0}^{\infty} {}_CG_{n,1,q}(x,y)\frac{t^n}{[n]_{1,q}!} = \sum_{n=0}^{\infty} {}_CG_n(x,y)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{tx}cos(ty),$$

where ${}_CG_n(x,y)$ is the cosine Genocchi polynomials.

(ii) $$\sum_{n=0}^{\infty} {}_CG_{n,1,q}(x,y)\frac{t^n}{[n]_{1,q}!} = \sum_{n=0}^{\infty} {}_CG_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{2t}{e_q(t)+1}e_q(tx)COS_q(ty),$$

where ${}_CG_{n,q}(x,y)$ is the q-cosine Genocchi polynomials, see [9].

Theorem 2.2. *For $|q/p| < 1$, we obtain*

$$ {}_CG_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{(x \oplus iy)^k_{p,q} + (x \ominus iy)^k_{p,q}}{2} G_{n-k,p,q},$$

where $G_{n,p,q}$ is the (p,q)-Genocchi numbers.

Proof. In [6], we can see (p,q)-Genocchi numbers such as

$$\sum_{n=0}^{\infty} G_{n,p,q}\frac{t^n}{[n]_{p,q}!}=\frac{2t}{e_{p,q}(t)+1}.$$

If we multiple $\widetilde{e}_{p,q}(t(x\oplus iy)_{p,q})$ in the generating function of (p,q)-Genocchi numbers, then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,p,q}\frac{t^n}{[n]_{p,q}!}\widetilde{e}_{p,q}(t(x\oplus iy)_{p,q})\\
&=\sum_{n=0}^{\infty} G_{n,p,q}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}(x\oplus iy)_{p,q}^n\frac{t^n}{[n]_{p,q}!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}(x\oplus iy)_{p,q}^k G_{n-k,p,q}\right)\frac{t^n}{[n]_{p,q}!},
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
\frac{2t}{e_{p,q}(t)+1}\widetilde{e}_{p,q}(t(x\oplus iy)_{p,q})&=\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tx)E_{p,q}(ity)\\
&=\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tx)(COS_{p,q}(ty)+iSIN_{p,q}(ty)).
\end{aligned}\tag{2.2}$$

From Equations (2.1) and (2.2), we derive the following:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}(x\oplus iy)_{p,q}^k G_{n-k,p,q}\frac{t^n}{[n]_{p,q}!}\\
&=\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tx)(COS_{p,q}(ty)+iSIN_{p,q}(ty)).
\end{aligned}\tag{2.3}$$

By applying a similar process, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}(x\ominus iy)_{p,q}^k G_{n-k,p,q}\frac{t^n}{[n]_{p,q}!}\\
&=\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tx)(COS_{p,q}(ty)-iSIN_{p,q}(ty)).
\end{aligned}\tag{2.4}$$

From Equations (2.3) and (2.4), we find the required result. □

To find some identities of the (p,q)-cosine Genocchi polynomials, we note

$$\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}=e_{p,q}(tx)COS_{p,q}(ty),\quad \text{see [13, 14]}.\tag{2.5}$$

Theorem 2.3. *Let k be a non-negative integer. Then we find*

$${}_{C}G_{n,p,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}G_{k,p,q}C_{n-k,p,q}(x,y),$$

where $G_{n,p,q}$ is the (p,q)-Genocchi numbers, see [6].

Proof. Using the generating function of the (p,q)-cosine Genocchi polynomials and Equation (2.5), we have a relation as

$$\sum_{n=0}^{\infty} {}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} G_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} G_{k,p,q}C_{n-k,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!}.$$

From the above equation, we can derive the desired result. □

Corollary 2.4. *Setting $p=1$ in Theorem 2.3, the following equation hold*

$${}_CG_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q} G_{k,q}C_{n-k,q}(x,y),$$

where $G_{n,q}$ is the q-Genocchi numbers and $\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!} = e_q(tx)COS_q(ty)$, see [6, 9-10, 12].

Theorem 2.5. *Let n be a non-negative integer. Then we have*

$$2[n]_{p,q}C_{n-1,p,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}{}_CG_{k,p,q}(x,y) + {}_CG_{k,p,q}(x,y).$$

Proof. Consider that $e_{p,q}(t) \neq -1$ in the generating function of the (p,q)-cosine Genocchi polynomials. Then, we find

$$\sum_{n=0}^{\infty} {}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}(e_{p,q}(t)+1) = 2te_{p,q}(tx)COS_{p,q}(ty). \tag{2.6}$$

From Equation (2.6), we transform the left-hand side into the following:

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}(e_{p,q}(t)+1) \\
&= \sum_{n=0}^{\infty} {}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty} p^{\binom{n}{2}}\frac{t^n}{[n]_{p,q}!}+1\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}{}_CG_{k,p,q}(x,y) + {}_CG_{n,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!},
\end{aligned} \tag{2.7}$$

and we write the right-hand side of Equation (2.6) as follows:

$$\begin{aligned}
2te_{p,q}(tx)COS_{p,q}(ty) &= 2\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^{n+1}}{[n]_{p,q}!} \\
&= 2\sum_{n=0}^{\infty}[n]_{p,q}C_{n-1,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{2.8}$$

By comparing Equations (2.7) and (2.8), we can derive the required result. □

In order to find some relations between the (p,q)-cosine Genocchi polynomials and (p,q)-other polynomials, we note the following equations:

$$[n]_{p,q}C_{n-1,p,q}(x,y)=\sum_{k=0}^{n-1}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}{}_{C}B_{k,p,q}(x,y), \tag{2.9}$$

where ${}_{C}B_{n,p,q}(x,y)$ is the (p,q)-cosine Bernoulli polynomials, and

$$2C_{n,p,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}{}_{C}\mathcal{E}_{k,p,q}(x,y)+{}_{C}\mathcal{E}_{n,p,q}(x,y), \tag{2.10}$$

where ${}_{C}\mathcal{E}_{n,p,q}(x,y)$ is the (p,q)-cosine Euler polynomials, see [14].

Corollary 2.6. *From Equations (2.9), (2.10) and Theorem 2.5, the following holds:*

(i) $$\sum_{k=0}^{n-1}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}{}_{C}B_{k,p,q}(x,y)=\frac{1}{2}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}{}_{C}G_{k,p,q}(x,y)+{}_{C}G_{n,p,q}(x,y)\right),$$

(ii) $${}_{C}\mathcal{E}_{n-1,p,q}(x,y)=\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}_{p,q}\left(\frac{p^{\binom{n-k}{2}}{}_{C}G_{k,p,q}(x,y)}{[n-k]_{p,q}}-p^{\binom{n-k-1}{2}}{}_{C}\mathcal{E}_{k,p,q}(x,y)\right).$$

Theorem 2.7. *For $|q/p|<1$, we get*

$${}_{C}G_{n,p,q}(1,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(-1)^{k}q^{\binom{k}{2}}\left(2[n-k]_{p,q}C_{n-k-1,p,q}(x,y)+{}_{C}G_{n-k,p,q}(x,y)\right)x^{k}.$$

Proof. If we put 1 instead of x in Definition 2.1, we find the following:

$$\begin{aligned}
&\sum_{n=0}^{\infty}{}_{C}G_{n,p,q}(1,y)\frac{t^{n}}{[n]_{p,q}!}\\
&=\frac{2t}{e_{p,q}(t)+1}(e_{p,q}(t)+1)COS_{p,q}(ty)-\frac{2t}{e_{p,q}(t)+1}COS_{p,q}(ty)\\
&=2tCOS_{p,q}(ty)-\frac{2t}{e_{p,q}(t)+1}COS_{p,q}(ty)\\
&=\left(\sum_{n=0}^{\infty}2C_{n,p,q}(x,y)\frac{t^{n+1}}{[n]_{p,q}!}+\sum_{n=0}^{\infty}{}_{C}G_{n,p,q}(x,y)\frac{t^{n}}{[n]_{p,q}!}\right)\sum_{n=0}^{\infty}q^{\binom{n}{2}}(-x)^{n}\frac{t^{n}}{[n]_{p,q}!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(-1)^{k}q^{\binom{k}{2}}\left(2[n-k]_{p,q}C_{n-k-1,p,q}(x,y)+{}_{C}G_{n-k,p,q}(x,y)\right)x^{k}\right)\frac{t^{n}}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of both sides in the above equation, we can derive the desired result. □

Corollary 2.8. *Setting $p=1$ in Theorem 2.7, one holds*

$${}_{C}G_{n,q}(1,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}(-1)^{k}q^{\binom{k}{2}}\left(2[n-k]_{q}C_{n-k-1,q}(x,y)+{}_{C}G_{n-k,q}(x,y)\right)x^{k}.$$

Theorem 2.9. *For nonzero integers a and b, we find*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}a^{n-k}b^{k}{}_{C}G_{n-k,p,q}(bx,by)_{C}G_{k,p,q}(aX,aY)$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}b^{n-k}a^{k}{}_{C}G_{n-k,p,q}(ax,ay)_{C}G_{k,p,q}(bX,bY).$$

Proof. Assume form A as follows:

$$A:=\frac{4t^{2}e_{p,q}(abtx)e_{p,q}(abtX)COS_{p,q}(abty)COS_{p,q}(abtY)}{(e_{p,q}(at)+1)(e_{p,q}(bt)+1)}.$$

From form A, we get

$$\begin{aligned}A&=\frac{2t}{e_{p,q}(at)+1}e_{p,q}(abtx)COS_{p,q}(abty)\frac{2t}{e_{p,q}(bt)+1}e_{p,q}(abtx)COS_{p,q}(abtY)\\&=\sum_{n=0}^{\infty}{}_{C}G_{n,p,q}(bx,by)\frac{(at)^{n}}{[n]_{p,q}!}\sum_{n=0}^{\infty}{}_{C}G_{n,p,q}(aX,aY)\frac{(bt)^{n}}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}a^{n-k}b^{k}{}_{C}G_{n-k,p,q}(bx,by)_{C}G_{k,p,q}(aX,aY)\right)\frac{t^{n}}{[n]_{p,q}!},\end{aligned}\tag{2.11}$$

and form A also can be transformed into the following:

$$\begin{aligned}A&=\sum_{n=0}^{\infty}{}_{C}G_{n,p,q}(ax,ay)\frac{(bt)^{n}}{[n]_{p,q}!}\sum_{n=0}^{\infty}{}_{C}G_{n,p,q}(bX,bY)\frac{(at)^{n}}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}b^{n-k}a^{k}{}_{C}G_{n-k,p,q}(ax,ay)_{C}G_{k,p,q}(bX,bY)\right)\frac{t^{n}}{[n]_{p,q}!}.\end{aligned}\tag{2.12}$$

From Equation (2.11) and (2.12), we know (p, q)-cosine Genocchi polynomials have symmetric property and find the required result. □

Corollary 2.10. *Setting $b = 1$ in Theorem 2.9, one holds*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}a^{n-k}{}_{C}G_{n-k,p,q}(x,y)_{C}G_{k,p,q}(aX,aY)$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}a^{k}{}_{C}G_{n-k,p,q}(ax,ay)_{C}G_{k,p,q}(X,Y).$$

Corollary 2.11. *Setting $p = 1$ in Theorem 2.9, the following holds*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}a^{n-k}b^{k}{}_{C}G_{n-k,q}(bx,by)_{C}G_{k,q}(aX,aY)$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}b^{n-k}a^{k}{}_{C}G_{n-k,q}(ax,ay)_{C}G_{k,q}(bX,bY).$$

Theorem 2.12. *For* $0<\frac{q}{p}<1$*, we have*

$$D_{p,q,x\,C}G_{n,p,q}(x,y)=\frac{{}_CG_{n,p,q}(px,y)-{}_CG_{n,p,q}(qx,y)}{(p-q)x}.$$

Proof. Applying (p,q)-derivative in the generating function of (p,q)-cosine Genocchi polynomials, we have

$$\begin{aligned}
&D_{p,q,x}\sum_{n=0}^{\infty}{}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\\
&=D_{p,q,x}\left(\frac{2}{e_{p,q}(t)+1}e_{p,q}(tx)COS_{p,q}(ty)\right)\\
&=\frac{1}{(p-q)x}\left(\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tpx)COS_{p,q}(ty)-\frac{2t}{e_{p,q}(t)+1}e_{p,q}(tqx)COS_{p,q}(ty)\right)\\
&=\frac{1}{(p-q)x}\sum_{n=0}^{\infty}\left({}_CG_{n,p,q}(px,y)-{}_CG_{n,p,q}(qx,y)\right)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

From the above equation, we can finish the proof of Theorem 2.12. □

Corollary 2.13. *Let* $p=1$ *in Theorem 2.12. Then, the following equation holds*

$$D_{q,x\,C}G_{n,q}(x,y)=\frac{{}_CG_{n,q}(x,y)-{}_CG_{n,q}(qx,y)}{(1-q)x}.$$

Theorem 2.14. *Let* $x,y\in\mathbb{R}$*. Then we have*

$$\begin{aligned}
&{}_CG_{n,p,q}(x,y)+2{}_CB_{n,p,q}(x,y)\\
&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\left(2{}_CB_{n-k,p,q}(x,y)-{}_CG_{n-k,p,q}(x,y)\right),
\end{aligned}$$

where ${}_CB_{n,p,q}(x,y)$ *is the (p,q)-cosine Bernoulli polynomials, see [13].*

Proof. Transforming the generating function of (p,q)-cosine Genocchi polynomials and using (p,q)-cosine Bernoulli polynomials, we investigate

$$\begin{aligned}
2t&=\sum_{n=0}^{\infty}{}_CG_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{t^n}{[n]_{p,q}!}+1\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}{}_CG_{n-k,p,q}(x,y)+{}_CG_{n,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!}\\
&=2\sum_{n=0}^{\infty}{}_CB_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{t^n}{[n]_{p,q}!}-1\right)\\
&=2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}{}_CB_{n-k,p,q}(x,y)-{}_CB_{n,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

From the above equation, we can find the required result which is a relation between (p,q)-cosine Genocchi polynomials and (p,q)-cosine Bernoulli polynomials. □

Corollary 2.15. *From Theorem 2.14, one holds*

$$ {}_{C}G_{n,q}(x,y) + 2{}_{C}B_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} \left(2{}_{C}B_{n-k,q}(x,y) - {}_{C}G_{n-k,q}(x,y)\right), $$

where ${}_{C}B_{n,q}(x,y)$ is the q-cosine Bernoulli polynomials, see [10].

Theorem 2.16. *Let $x, y \in \mathbb{R}$. Then we get*

$$ {}_{C}G_{n,p,q}(x,y) = [n]_{p,q}\,{}_{C}\mathcal{E}_{n-1,p,q}(x,y), $$

where ${}_{C}\mathcal{E}_{n,p,q}(x,y)$ is the (p,q)-cosine Euler polynomials, see [14].

Proof. From Definition 2.1, we have a relation of (p,q)-cosine Genocchi polynomials and (p,q)-cosine Euler polynomials such as

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{C}G_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} t\,{}_{C}\mathcal{E}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} [n]_{p,q}\,{}_{C}\mathcal{E}_{n-1,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}. \end{aligned} $$

By comparing the coefficients of both sides in the above equation, we complete the proof of Theorem 2.16. □

Corollary 2.17. *Putting $p = 1$, the following holds*

$$ {}_{C}G_{n,q}(x,y) = [n]_{q}\,{}_{C}\mathcal{E}_{n-1,q}(x,y), $$

where ${}_{C}\mathcal{E}_{n,q}(x,y)$ is the q-cosine Euler polynomials, see [12].

Corollary 2.18. *Putting $p = 1$ and $q \to 1$, the following equation holds*

$$ {}_{C}G_{n}(x,y) = n\,{}_{C}\mathcal{E}_{n-1}(x,y), $$

where ${}_{C}\mathcal{E}_{n}(x,y)$ is the cosine Euler polynomials, see [12].

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