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MERSENNE PRIME FACTOR AND SUM OF BINOMIAL COEFFICIENTS†

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Abstract. Let $M_p := 2^p - 1$ be a Mersenne prime. In this article, we find integers a, b, c, d, e and n satisfying $\sum_{t=0}^{n} \binom{an+b}{ct+d} = M_p e$ given a Mersenne prime number M_p. In order to find a special case that satisfies the above results, we reprove an well-known relation of a certain sum of binomial coefficients and a divisor function.

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1. Introduction

The study of divisor functions and binomial coefficients is important and diverse field in number theory. The sum of the l-th powers of the divisors of n will be denoted by $\sigma_l(n)$. The traditional results associated with divisor functions are [1], [7] and so on. Although Mersenne primes are closely related to perfect numbers, it can be seen that they are also useful for iteration of divisor functions [4]. We will examine the problem in this article by examining examples of whether the sum of any binomial coefficients includes a Mersenne prime. We can easily see that $\binom{5}{1} = 3 \times \frac{5}{3}$, $\binom{7}{2} = 7 \times \frac{9}{3}$, $\binom{11}{1} + \binom{11}{4} = 31 \times \frac{33}{3}$, $\binom{15}{0} + \binom{15}{3} + \binom{15}{6} = 127 \times \frac{129}{3}$ and $\binom{27}{0} + \binom{27}{3} + \binom{27}{6} + \binom{27}{9} + \binom{27}{12} = 8191 \times \frac{8193}{3}$. The sum of the binomial coefficients satisfying the above conditions was found by examining ten Mersenne primes using Mathematica 11.2. Since Mersenne primes themselves are very large numbers, we present only six Mersenne primes in Table 1. Based on these examples, we suggest the following question.

Question 1.1. *Given a Mersenne prime number M_p, can one find integers a, b, c, d, e and n such that $\sum_{t=0}^{n} \binom{an+b}{ct+d} = M_p e$?*

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p	M_p	e	$\sum_{t=0}^{n}\binom{an+b}{ct+d}$
3	7	3	$\sum_{t=0}^{\frac{3-2-1}{3}}\binom{2\times3+1}{3t+2}$
5	31	11	$\sum_{t=0}^{\frac{5-1-1}{3}}\binom{2\times5+1}{3t+1}$
7	127	43	$\sum_{t=0}^{\frac{7-0-1}{3}}\binom{2\times7+1}{3t}$
13	8191	2731	$\sum_{t=0}^{\frac{13-0-1}{3}}\binom{2\times13+1}{3t}$
17	131071	43691	$\sum_{t=0}^{\frac{17-1-1}{3}}\binom{2\times17+1}{3t+1}$
19	524287	174763	$\sum_{t=0}^{\frac{19-0-1}{3}}\binom{2\times19+1}{3t}$

TABLE 1. Six examples for sums of binomial coefficients.

The problem of completely solving Question 1.1 seems difficult. This article aims to find and prove the following special case that satisfies Question 1.1.

Theorem 1.2. *Let $i = 0, 1, 2$, let $M'_p := \frac{M_p+2}{3}$ and $p = 3n + 1 + i$. Then*

$$\sum_{t=0}^{\frac{p-i-1}{3}} \binom{2p+1}{3t+i} = M_p M'_p. \tag{1}$$

If $i = 2$, then there exist only one Mersenne prime $7 = 2^3 - 1$ satisfying (1).

Remark 1.3. In fact, if $M_p > 3$ then $2^p + 1 \equiv 0 \pmod 3$ and $M'_p = \frac{2^p+1}{3}$ is a positive integer. Therefore, if $M_p > 3$ then Theorem 1.2 is a special case that satisfies Question 1.1. If $i = 0$ in Theorem 1.2, then $M_p = 127$, 8191, 524287, 2147483647, 2305843009213693951, $\cdots$. If $i = 1$ in Theorem 1.2, then $M_p = 31$, 131071, 618970019642690137449562111, $\cdots$.

2. Proof of Theorem 1.2

To prove Theorem 1.2, the result of Lemma 2.8 is necessary. Therefore, we prove this first. There are various ways to prove the relation between the binomial coefficient and the divisor function. In this section, we demonstrate four polynomials and recurrence relations. Lemma 2.8 and the inner result of [3, 0.152] are the same, but the proof is different. We assume that $n(\geq 3)$, m, t and k are positive integers, unless otherwise specified. We define a sequence

$(a_k(n))_{1\le k\le 2n+1}$ and polynomials $F(x,n)$, $G(x)$, $H(x,n)$, $\Gamma(x,n)$, as follows:

$$a_k(n) := \begin{cases} \binom{2n+1}{1} & k=1, \\ -\binom{2n+1}{2} + 2a_1(n) & k=2, \\ \binom{2n+1}{3} - 2a_1(n) + 2a_2(n) & k=3, \\ (-1)^{k-1}\binom{2n+1}{k} + a_{k-3}(n) - 2a_{k-2}(n) + 2a_{k-1}(n) & 4 \le k \le 2n-1, \\ -\binom{2n+1}{2n} + a_{2n-3}(n) - 2a_{2n-2}(n) & k=2n, \\ a_{2n-2}(n) & k=2n+1 \end{cases} \tag{2}$$

and $F(x,n) := (x-1)^n - x^n + 1$, $G(x) := x^3 - 2x^2 + 2x - 1$, $H(x,n) := -\sum_{k=1}^{2n-2} a_k(n)x^{2n-2-k}$, $\Gamma(x,n) := -\left(a_{2n-1}(n)x^2 + a_{2n}(n)x + a_{2n+1}(n)\right)$.

Lemma 2.1. *Let $n \ge 3$ be an integer. Then*

$$F(x,2n+1) = G(x)H(x,n) + \Gamma(x,n). \tag{3}$$

Proof. It is obtained by comparing the coefficients of both sides in (3). □

To investigate some properties of the sequence $(a_k(n))$, we introduce a sequence $(b_k(n))_{3\le k\le 2n+1}$ as follows:

$$b_k(n) := (-1)^{k-1}\left(\binom{2n+1}{k} - 2\binom{2n+1}{k-1} + 2\binom{2n+1}{k-2} - \binom{2n+1}{k-3}\right). \tag{4}$$

The following Lemma 2.2 is derived from the symmetric property of binomial coefficients, that is, $\binom{m}{t} = \binom{m}{m-t}$ for $0 \le t \le m$.

Lemma 2.2. *Let $1 \le t \le 2n-3$. Then $b_{(2n+1)-t}(n) = -b_{3+t}(n)$. In particular, $b_{n+2}(n) = 0$.*

Lemma 2.3. *Let $4 \le k \le 2n-1$. Then*

$$b_k(n) = \begin{cases} a_k(n) & 4 \le k \le 6, \\ a_k(n) - a_{k-6}(n) & 7 \le k \le 2n-1. \end{cases} \tag{5}$$

Proof. This lemma is proved by using the induction on k. □

Lemma 2.4. *Let $n \ge 3$ be a positive integer. Then*

(*a*) $a_{2n-2}(n) = a_{2n+1}(n) = 0$,

(*b*) $a_{2n-1}(n) = -a_{2n}(n)$.

Proof. Note that $F(x,2n+1) = F(1-x,2n+1)$ for all positive integers n. By Lemma 2.1, we have that

$$F(1-x,2n+1) = G(1-x)H(1-x,n) + \Gamma(1-x,n). \tag{6}$$

Expand the right-hand side of (6) then we obtain

$G(1-x)H(1-x,n) + \Gamma(1-x,n) =$

$$x(x^2-x+1)\left(\sum_{k=1}^{2n-2} a_k(n)(1-x)^{2n-2-k}\right) \tag{7}$$
$$-\left(a_{2n-1}(n)x^2-(2a_{2n-1}(n)+a_{2n}(n))x+a_{2n-1}(n)+a_{2n}(n)+a_{2n+1}(n)\right).$$

Hence, the constant term of $F(1-x,2n+1)$ is $a_{2n-1}(n)+a_{2n}(n)+a_{2n+1}(n)$. By the definition of $F(x,2n+1)$, we obtain

$$F(0,2n+1)=a_{2n-1}(n)+a_{2n}(n)+a_{2n+1}(n)=0. \tag{8}$$

On the other hand, if we substitute x to 1 in (7), then we have

$$a_{2n+1}(n)=a_{2n-2}(n)=0. \tag{9}$$

By (8) and (9),

$$a_{2n-1}(n)+a_{2n}(n)=0.$$

□

It is well-known [6, p.315] that

$$\binom{m}{t}+\binom{m}{t+1}=\binom{m+1}{t+1} \text{ for } 0\le t\le m.$$

Using the Pascal's rule, we obtain

Lemma 2.5. *Let n be a positive integer with $n\ge 4$. If $6\le k\le 2n-1$ then*

$$b_k(n)=b_k(n-1)-2b_{k-1}(n-1)+b_{k-2}(n-1). \tag{10}$$

Lemma 2.6. *If n is a positive integer then*

$$a_{2n}(n)=-a_{2n-1}(n)=\begin{cases}-3 & \textit{if } n\equiv 1(\textit{mod } 3),\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. By (2) and Lemma 2.4, we have that $a_{2n}(n)=-\binom{2n+1}{2n}+a_{2n-3}(n)$ and

$$\begin{aligned} a_{2n-1}(n) &= \binom{2n+1}{2n-1}+a_{2n-4}(n)-2a_{2n-3}(n)\\ &= \binom{2n+1}{2n-1}+a_{2n-4}(n)-2\left(a_{2n}(n)+\binom{2n+1}{2n}\right). \end{aligned} \tag{11}$$

(2) and (11) imply

$$a_{2n}(n)=\binom{2n+1}{2n-1}-2\binom{2n+1}{2n}+a_{2n-4}(n)=-a_2(n)+a_{2n-4}(n). \tag{12}$$

To prove this lemma, we compute $a_{2n-4}(n)$ for three cases $n\equiv 0,1,2(\text{mod } 3)$.

First, we consider the case for $n\equiv 0(\text{mod } 3)$. It is divided into $n\equiv 0,3(\text{mod } 6)$. Let $n=6m$ with $m\ge 1$. Then, by Lemma 2.2 and 2.3, we have that

$$a_{2n-4}(n)=a_{12m-4}(6m)=a_2(6m)+\sum_{t=1}^{2m-1} b_{6t+2}(6m)$$

$$= a_2(6m) + \sum_{t=1}^{m-1} b_{6t+2}(6m) + \sum_{t=m+1}^{2m-1} b_{6t+2}(6m) = a_2(6m).$$

It is readily checked that $\sum_{t=m+1}^{2m-1} b_{6t+2}(6m) = -\sum_{t=1}^{m-1} b_{6t+2}(6m)$ by using Lemma 2.2 and 2.3. Combining (12) and $a_{12m-4}(6m) = a_2(6m)$ leads to $a_{12m}(6m) = 0$. Similarly, we can verify that $a_{12m+2}(6m+3) = a_2(6m+3)$ and $a_{12m+6}(6m+3) = 0$. Thus, if $n \equiv 0 \pmod 3$, then

$$a_{2n}(n) = a_{2n-1}(n) = 0. \tag{13}$$

Next, consider the case for $n \equiv 1 \pmod 3$. It is also divided into $n \equiv 1, 4 \pmod 6$. The Proofs of case $n \equiv 1 \pmod 6$ and case $n \equiv 4 \pmod 6$ are almost identical. So we will check only the case for $n \equiv 1 (\mathrm{mod}\ 6)$. Let $n = 6m+1$ with $m \geq 1$. Then, by Lemma 2.3 and 2.5, we have

$$a_{2n-4}(n) = a_{12m-2}(6m+1) = a_4(6m+1) + \sum_{t=1}^{2m-1} b_{6t+4}(6m+1)$$

$$= a_4(6m+1) + \sum_{t=1}^{2m-1} \left(b_{6t+4}(6m) - 2b_{6t+3}(6m) + b_{6t+2}(6m)\right).$$

By Lemma 2.3, $\sum_{t=1}^{2n-1} b_{6t+4-j}(6n+1) = a_{12n-(j+2)}(6n) - a_{4-j}(6n)$ for $j = 0, 1, 2$. Thus, the following holds that

$$\begin{aligned} a_{12m-2}(6m+1) =& a_4(6m+1) + (a_{12m-2}(6m) - a_4(6m)) \\ &- (2a_{12m-3}(6m) - 2a_3(6m)) + (a_{12m-4}(6m) - a_2(6m)) \\ =& a_4(6m+1) + (a_{12m-4}(6m) - 2a_{12m-3}(6m)) \\ &- a_4(6m) + 2a_3(6m) - a_2(6m). \end{aligned}$$

By (2) and Lemma 2.4,

$$a_{12m-1}(6m) = \binom{12m+1}{12m-1} + a_{12m-4}(6m) - 2a_{12m-3}(6m) = 0.$$

Hence we have that

$$a_{12m-2}(6m+1) = a_4(6m+1) - \binom{12m+1}{2} - a_4(6m) + 2a_3(6m) - a_2(6m). \tag{14}$$

From (12), (14) and Lemma 2.3, we obtain

$$\begin{aligned} a_{12m+2}(6m+1) =& -a_2(6m+1) + a_{12m-2}(6m+1) \\ =& -a_2(6m+1) + a_4(6m+1) - \binom{12m+1}{2} \\ &- a_4(6m) + 2a_3(6m) - a_2(6m) \\ =& -\binom{12m+3}{1} - \binom{12m+3}{2} + 2\binom{12m+3}{3} - \binom{12m+3}{4} \end{aligned}$$

$$+\binom{12m+1}{1}-2\binom{12m+1}{2}+\binom{12m+1}{4}=-3.$$

Thus, if $n\equiv 1 (\mathrm{mod}\ 3)$, then

$$a_{2n}(n)=-a_{2n-1}(n)=-3. \tag{15}$$

Finally, let $n=6m+2$ with $m\geq 1$. Then, by Lemma 2.3, 2.4 and 2.5, we have

$$\begin{aligned} a_{2n-4}(n)=&a_{12m}(6m+2)=\sum_{t=1}^{2m} b_{6t}(6m+2)\\ =&\sum_{t=1}^{2m}\left(b_{6t}(6m+1)-2b_{6t-1}(6m+1)+b_{6t-2}(6m+1)\right)\\ =&-2a_{12m-1}(6m+1)+a_{12m-2}(6m+1). \end{aligned} \tag{16}$$

By (15),

$$a_{12m+1}(6m+1)=\binom{12m+3}{12m+1}+a_{12m-2}(6m+1)-2a_{12m-1}(6m+1)=3. \tag{17}$$

Combine (12), (16) and (17) then we have

$$\begin{aligned} a_{12m+4}(6m+2)=&-a_2(6m+2)+3-\binom{12m+3}{2}\\ =&-\binom{12m+5}{2}+2\binom{12m+5}{1}+3-\binom{12m+3}{2}=0. \end{aligned}$$

Similarly, we can compute $a_{12m+10}(6m+5)=0$. This completes the proof. □

Lemma 2.7. *Let $n\equiv 0,2 \pmod 3$. If $1\leq k\leq n$, then $a_k(n)=a_{2n-2-k}(n)$.*

Proof. Let $n\equiv 0,2 (\mathrm{mod}\ 3)$. By (2) and Lemma 2.6, we obtain

$$a_{2n}(n)=-\binom{2n+1}{2n}+a_{2n-3}(n)-2a_{2n-2}(n)=0.$$

It induces that $a_{2n-3}(n)=\binom{2n+1}{1}=a_1(n)$. Similarly, we have $a_{2n-4}(n)=a_2(n)$ and $a_{2n-5}(n)=a_3(n)$. To use the mathematical induction, suppose that, for fixed $k\geq 3$,

$$a_{2n-2-k}(n)=a_k(n),\ a_{2n-1-k}(n)=a_{k-1}(n),\ a_{2n-k}(n)=a_{k-2}(n).$$

By (2), we obtain

$$a_{2n-k}(n)=(-1)^{2n-1-k}\binom{2n+1}{2n-k}+a_{2n-3-k}(n)-2a_{2n-2-k}(n)+2a_{2n-1-k}(n)$$

and

$$a_{2n-3-k}(n)=(-1)^k\binom{2n+1}{k+1}+a_{k-2}(n)-2a_{k-1}(n)+2a_k(n)=a_{k+1}(n).$$

□

Lemma 2.8. *Let n be a nonnegative integer. If $i = 0, 1, 2$ then*

$$\sum_{t=0}^{n} \binom{6n+3+2i}{3t+i} = \sigma_2(2^{3n+i}). \tag{18}$$

Furthermore, $\sum_{t=0}^{n} \binom{6n+7}{3t+2} = 5\sum_{t=0}^{n} \binom{6n+5}{3t+1} - 4\sum_{t=0}^{n} \binom{6n+3}{3t}$.

Proof. By Lemma 2.2, 2.3 and 2.4, the following holds, for $n \geq 1$,

$$\begin{aligned}
a_{12n-2}(6n) &= \sum_{t=0}^{2n-1} b_{4+6t}(6n) = \sum_{t=0}^{n-1} b_{4+6t}(6n) - \sum_{t=0}^{n-1} b_{6+6t}(6n) \\
&= \sum_{t=0}^{n-1} \left[\binom{12n+1}{6t+6} - 2\binom{12n+1}{6t+5} + \binom{12n+1}{6t+4} \right] \\
&\quad + \sum_{t=0}^{n-1} \left[\binom{12n+1}{6t+3} - 2\binom{12n+1}{6t+2} + \binom{12n+1}{6t+1} \right] \\
&= \sum_{t=1}^{6n} \binom{12n+1}{t} - 3\sum_{t=1}^{2n} \binom{12n+1}{3t-1} \\
&= (2^{12n} - 1) - 3\sum_{t=0}^{2n-1} \binom{12n+1}{3t+2} = 0.
\end{aligned}$$

Thus, we have $2^{12n} - 1 = 3\sum_{t=0}^{2n-1} \binom{12n+1}{3t+2}$. Similarly, if we compute $a_{12n+4}(6n+3)$, then we obtain $2^{12n+6} - 1 = 3\sum_{t=0}^{2n} \binom{12n+7}{3t+2}$. By combining them, we have

$$3\sum_{t=0}^{n-1} \binom{6n+1}{3t+2} = 2^{6n} - 1, \quad \sum_{t=0}^{n-1} \binom{6n+1}{3t+2} = \sigma_2(2^{3n-1}). \tag{19}$$

If we apply the Pascal's rule to the second of (19), then we have

$$\begin{aligned}
\sigma_2(2^{3n-1}) &= \sum_{t=0}^{n-1} \binom{6n+1}{3t+2} = \sum_{t=0}^{n-1} \left[\binom{6n-1}{3t+2} + 2\binom{6n-1}{3t+1} + \binom{6n-1}{3t} \right] \\
&= \sum_{t=0}^{3n-1} \binom{6n-1}{t} + \sum_{t=0}^{n-1} \binom{6n-1}{3t+1}.
\end{aligned}$$

Since $\sum_{t=0}^{3n-1} \binom{6n-1}{t} = 2^{6n-2}$ and $\sigma_2(2^{3n-1}) = \sigma_2(2^{3n-2}) + 2^{6n-2}$, we obtain $\sum_{t=0}^{n-1} \binom{6n-1}{3t+1} = \sigma_2(2^{3n-2})$. Repeat the above method again then we have $\sum_{t=0}^{n-1} \binom{6n+3}{3t} = \sigma_2(2^{3n-3})$. Using [7, p.26], the final result of Lemma 2.8 can be easily obtained. □

Proof of Theorem 1.2

The case of $M_p = 3, 7$ was shown in introduction. Therefore, we assume that $M_p > 7$. It is well-known that $\sigma_2(2^{3n+i}) = (2^{3n+1+i} - 1)(\frac{2^{3n+1+i}+1}{3})$. If $i = 2$ and $n \geq 1$ then $2^{3n+1+i} - 1$ is not a prime which is divided by 7. For the case of $i = 0$ or 1, if a Mersenne prime $2^p - 1$ is given, then it is possible to find a positive integer n which $3n + 1 + i = p$ is a prime number. Thus, for a given $M_p > 3$, we can find integers n and i such that $M_p = 2^p - 1 = 2^{3n+1+i} - 1$. This completes the proof of Theorem 1.2. □

Question 2.9. *It is a famous conjecture that Mersenne primes are infinite* [2]. *For the question of the number of extended Mersenne primes, see* [5]. *From Table 1, given* M'_p *are all prime numbers. From this, we raise the question of whether the number of primes* M'_p *is infinite or not.*

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