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NEW THEOREM ON SYMMETRIC FUNCTIONS AND THEIR APPLICATIONS ON SOME (p, q)-NUMBERS†

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Abstract. In this paper, we present and prove an new theorem on symmetric functions. By using this theorem, we derive some new generating functions of the products of (p, q)-Fibonacci numbers, (p, q)-Lucas numbers, (p, q)-Pell numbers, (p, q)-Pell Lucas numbers, (p, q)-Jacobsthal numbers and (p, q)-Jacobsthal Lucas numbers with Chebyshev polynomials of the first kind.

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1. Introduction and preliminaries

Lucas, Pell and Pell Lucas numbers have been studied in many different research for centuries. Recently, several researchers have dedicated their works to the study of the properties of the sequences of (p, q)-Lucas, (p, q)-Pell and (p, q)-Pell Lucas numbers (see for example [9], [13], [17] and [20]), and also many identities of these sequences have been established.

It is well known that, for any positive real numbers p and q, the (p, q)-Lucas, (p, q)-Pell and (p, q)-Pell Lucas numbers $\{L_{p,q,n}\}_{n\in\mathbb{N}}$, $\{P_{p,q,n}\}_{n\in\mathbb{N}}$ and $\{Q_{p,q,n}\}_{n\in\mathbb{N}}$ are defined respectively by the following recurrence relations:

$$L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}, \text{ for } n \geq 2, \text{ with } L_{p,q,0} = 2,\ L_{p,q,1} = p,$$

$$P_{p,q,n} = 2pP_{p,q,n-1} + qP_{p,q,n-2}, \text{ for } n \geq 2, \text{ with } P_{p,q,0} = 0,\ P_{p,q,1} = 1,$$

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and

$$Q_{p,q,n} = 2pQ_{p,q,n-1} + qQ_{p,q,n-2}, \text{ for } n \geq 2, \text{ with } Q_{p,q,0} = 2, \ Q_{p,q,1} = 2p.$$

G. B. Djordjević and H. M. Srivastava in [11] introduced and investigated some properties and relations involving two sequences of the numbers $\left\{C_{n,3}(a,b,r) \equiv C_{n,3}\right\}$ and $\left\{C_{n,4}(a,b,c,r) \equiv C_{n,4}\right\}$. The same authors in [12] presented a systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal Lucas numbers. Also, in [26], Uygun introduced the (p,q)-Jacobsthal and (p,q)-Jacobsthal Lucas numbers $\{J_{p,q,n}\}_{n\in\mathbb{N}}$ and $\{j_{p,q,n}\}_{n\in\mathbb{N}}$, which are defined by the second order linear recurrence sequences, for any positive real numbers p and q,

$$J_{p,q,n} = pJ_{p,q,n-1} + 2qJ_{p,q,n-2}, \text{ for } n \geq 2, \text{ with } J_{p,q,0} = 0, \ J_{p,q,1} = 1,$$

and

$$j_{p,q,n} = pj_{p,q,n-1} + 2qj_{p,q,n-2}, \text{ for } n \geq 2, \text{ with } j_{p,q,0} = 2, \ j_{p,q,1} = p,$$

respectively. Associated with the sequences of (p,q)-Jacobsthal and (p,q)-Jacobsthal Lucas numbers the characteristic equation is $x^2 - px - 2q = 0$, where $e_1 = \frac{p+\sqrt{p^2+8q}}{2}$ and $e_2 = \frac{p-\sqrt{p^2+8q}}{2}$ are the roots of this equation. We note that

$$e_1 + e_2 = p, \ e_1e_2 = -2q \text{ and } e_1 - e_2 = \sqrt{p^2+8q}.$$

An important formula associated with these sequences is the well-known Binet's formula. The Binet's formulas for (p,q)-Jacobsthal and (p,q)-Jacobsthal Lucas numbers are given by:

$$J_{p,q,n} = \frac{e_1^n - e_2^n}{e_1 - e_2} \text{ and } j_{p,q,n} = e_1^n + e_2^n.$$

In the last years, there is huge interest of natural science in the applications of Fibonacci numbers. It is known that the Fibonacci sequence is defined by the following equation:

$$F_0 = 0, \ F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2,$$

for more informations of Fibonacci numbers and their properties see the papers [8, 11, 15], the Fibonacci sequence is a special case of the sequence:

$$F_{k,0} = 0, \ F_{k,1} = 1 \text{ and } F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } n \geq 2 \text{ and } k \geq 1,$$

which is defined by Falcon and Plaza in [25], this sequence is called k-Fibonacci sequence. The k-Fibonacci numbers are studied by many authors, for example in [23], N. Saba and A. Boussayoud calculated the generating function of the product of (p,q)-modified Pell numbers with k-Fibonacci numbers, and in [16], the authors defined a new class of q-starlike functions associated with k-Fibonacci numbers.

In Binet's formula, the k-Fibonacci sequence is given by:

$$F_{k,n} = \frac{e_1^n - e_2^n}{e_1 - e_2}, \text{ with } e_{1,2} = \frac{k \pm \sqrt{k^2+4}}{2},$$

where e_1 and e_2 are roots of the characteristic equation $x^2 - kx - 1 = 0$. Alternative, the k-Fibonacci sequence is given by the symmetric function as:

$$F_{k,n} = S_{n-1}\left(e_1 + [-e_2]\right), \text{ with } e_{1,2} = \frac{k \pm \sqrt{k^2+4}}{2}.$$

In the negative extension, is also given by:

$$F_{k,-n} = (-1)^{n+1} F_{k,n}, \text{ for all } n \geq 0, \text{ (see [3, 23])}. \tag{1.1}$$

As a generalization of k-Fibonacci sequence, Suvarnamani and Tatong in [10] introduced the (p,q)-Fibonacci sequence, denoted by $\{F_{p,q,n}\}_{n\in\mathbb{N}}$. They also proved some interesting properties of them, this sequence is given by:

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2} \text{ for } n \geq 2, \text{ with } F_{p,q,0} = 0,\ F_{p,q,1} = 1.$$

Definition 1.1. [7] The Pell polynomials, denoted by $\{P_n(x)\}_{n\in\mathbb{N}}$ are defined by the following recurrence relation:

$$\begin{cases} P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \text{ for } n \geq 2 \\ P_0(x) = 0,\ P_1(x) = 1 \end{cases}.$$

Corollary 1.2. [22] *For $n \in \mathbb{N}$, we have:*

$$P_n(x) = S_{n-1}\left(e_1 + [-e_2]\right), \text{ with } \begin{cases} e_1 = x + \sqrt{x^2+1} \\ e_2 = x - \sqrt{x^2+1} \end{cases}.$$

Definition 1.3. [2] The Chebyshev polynomials of the first kind, denoted by $\{T_n(x)\}_{n\in\mathbb{N}}$ are defined recursively by:

$$\begin{cases} T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{ for } n \geq 2 \\ T_0(x) = 1,\ T_1(x) = x \end{cases}.$$

Corollary 1.4. [21] *For $n \in \mathbb{N}$, we have:*

$$T_n(x) = S_n\left(2e_1 + [-2e_2]\right) - xS_{n-1}\left(2e_1 + [-2e_2]\right), \text{ with } \begin{cases} e_1 = x + \sqrt{x^2-1} \\ e_2 = x - \sqrt{x^2-1} \end{cases}.$$

Definition 1.5. [1, 18] Let A and E be any two alphabets. We define $S_n(A-E)$ by the following form:

$$\frac{\prod_{e\in E}(1-ez)}{\prod_{a\in A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-E)z^n, \tag{1.2}$$

with the condition $S_n(A-E) = 0$ for $n < 0$.

Equation (1.2) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} S_n(A-E)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-E)z^n\right), \tag{1.3}$$

where

$$S_n(A-E) = \sum_{j=0}^{n} S_{n-j}(-E)S_j(A).$$

Remark 1.1. Taking $A = \{0\}$ in (1.2) gives:

$$\sum_{n=0}^{\infty} S_n(-E)z^n = \prod_{e\in E}(1-ez).$$

Definition 1.6. [24, 19] Let n be positive integer and $E = \{e_1, e_2\}$ are set of given variables. Then, the n^{th} symmetric function $S_n(e_1+e_2)$ is defined by:

$$S_n(E) = S_n(e_1+e_2) = \frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2},$$

with

$$\begin{aligned} S_0(E) &= S_0(e_1+e_2) = 1, \\ S_1(E) &= S_1(e_1+e_2) = e_1 + e_2, \\ S_2(E) &= S_2(e_1+e_2) = e_1^2 + e_1e_2 + e_2^2, \\ &\vdots \end{aligned}$$

Definition 1.7. [5, 6] Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta_{e_1e_2}^k$ is defined by:

$$\delta_{e_1e_2}^k(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2}, \text{ for all } k \in \mathbb{N}_0 := \{\mathbb{N} \cup \{0\}\} = \{0, 1, 2, ...\}. \quad (1.4)$$

Remark 1.2. If we get $k = 0$ and $E = \{q, qz\}$ in the Eq. (1.4), we get (see [14, 16]):

$$D_q f(z) = \frac{f(q) - f(qz)}{q - qz}, \ (z \neq 0).$$

2. Main results

In this part, we are now in a position to provide new theorem.

Theorem 2.1. *Given two alphabets $A = \{a_1, a_2, a_3, ..., a_n\}$ and $E = \{e_1, e_2\}$, we have:*

$$\sum_{n=0}^{\infty} S_n(A) S_{n-l}(E) z^n = \frac{S_{-l}(E) - e_1^{1-l} e_2^{1-l} z^{2-l} \sum_{n=0}^{\infty} S_{n-l+2}(-A) S_n(E) z^n}{\left(\sum_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)},$$

for all $n \in \mathbb{N}_0$ and $l \in \{0, 1\}$. (2.1)

Proof. By applying the operator $\delta_{e_1e_2}^{1-l}$ to the series $f(e_1z) = \sum_{n=0}^{\infty} S_n(A)\, e_1^n z^n$, we have:

$$\begin{aligned}
\delta_{e_1e_2}^{1-l} f(e_1z) &= \frac{e_1^{1-l}\sum_{n=0}^{\infty} S_n(A)\, e_1^n z^n - e_2^{1-l}\sum_{n=0}^{\infty} S_n(A)\, e_2^n z^n}{e_1 - e_2} \\
&= \sum_{n=0}^{\infty} S_n(A)\left(\frac{e_1^{n-l+1} - e_2^{n-l+1}}{e_1 - e_2}\right) z^n \\
&= \sum_{n=0}^{\infty} S_n(A)\, S_{n-l}(E) z^n.
\end{aligned}$$

On the other hand, by applying the operator $\delta_{e_1e_2}^{1-l}$ to the series $f(e_1z) = \frac{1}{\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n}$, we have:

$$\begin{aligned}
\delta_{e_1e_2}^{1-l}\left(\frac{1}{\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n}\right) &= \frac{\frac{e_1^{1-l}}{\sum_{n=0}^{\infty} S_n(-A)e_1^n z^n} - \frac{e_2^{1-l}}{\sum_{n=0}^{\infty} S_n(-A)e_2^n z^n}}{e_1 - e_2} \\
&= \frac{e_1^{1-l}\sum_{n=0}^{\infty} S_n(-A)\, e_2^n z^n - e_2^{1-l}\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n}{(e_1 - e_2)\left(\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, e_2^n z^n\right)} \\
&= \frac{\sum_{n=0}^{\infty} S_n(-A)\, e_1^n e_2^n \frac{e_1^{1-l-n} - e_2^{1-l-n}}{e_1 - e_2} z^n}{\left(\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, e_2^n z^n\right)} \\
&= \frac{\sum_{n=0}^{\infty} S_n(-A)\, e_1^n e_2^n S_{-n-l}(E)\, z^n}{\left(\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, e_2^n z^n\right)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{n=0}^{-l} S_n(-A)\, e_1^n e_2^n S_{-n-l}(E)\, z^n + \sum_{n=2-l}^{\infty} S_n(-A)\, e_1^n e_2^n S_{-n-l}(E)\, z^n}{\left(\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, e_2^n z^n\right)} \\
&= \frac{\sum_{n=0}^{-l} S_n(-A)\, e_1^n e_2^n S_{-n-l}(E)\, z^n - \sum_{n=2-l}^{\infty} S_n(-A)\, e_1^{1-l} e_2^{1-l}\left(\frac{e_1^{n+l-1} - e_2^{n+l-1}}{e_1 - e_2}\right) z^n}{\left(\sum_{n=0}^{\infty} S_n(-A)\, e_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, e_2^n z^n\right)}
\end{aligned}$$

$$= \frac{S_{-l}(E) - e_1^{1-l} e_2^{1-l} \sum\limits_{n=2-l}^{\infty} S_n(-A) S_{n+l-2}(E) z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}$$

$$= \frac{S_{-l}(E) - e_1^{1-l} e_2^{1-l} z^{2-l} \sum\limits_{n=0}^{\infty} S_{n-l+2}(-A) S_n(E) z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}.$$

Therefore

$$\sum_{n=0}^{\infty} S_n(A) S_{n-l}(E) z^n = \frac{S_{-l}(E) - e_1^{1-l} e_2^{1-l} z^{2-l} \sum\limits_{n=0}^{\infty} S_{n-l+2}(-A) S_n(E) z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}.$$

Thus, this completes the proof. □

- For $A = \{a_1, a_2\}$, $E = \{e_1, e_2\}$ and $l \in \{0, 1\}$ in the Theorem 2.1 we deduce the following lemmas.

Lemma 2.2. [5] *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(A) S_n(E) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}. \tag{2.2}$$

Based on the relationship (2.2), we get:

$$\sum_{n=0}^{\infty} S_{n-1}(A) S_{n-1}(E) z^n = \frac{z - a_1 a_2 e_1 e_2 z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}. \tag{2.3}$$

Lemma 2.3. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(A) S_{n-1}(E) z^n = \frac{(a_1 + a_2) z - a_1 a_2 (e_1 + e_2) z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}. \tag{2.4}$$

From (2.4), we get:

$$\sum_{n=0}^{\infty} S_{n-1}(A) S_n(E) z^n = \frac{(e_1 + e_2) z - e_1 e_2 (a_1 + a_2) z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}. \tag{2.5}$$

3. Ordinary generating functions of the products of (p, q)-numbers with Chebyshev polynomials of the first kind

In this part, we now derive the new generating functions of the products of some (p, q)-numbers with Chebyshev polynomials of the first kind.

For the case $A = \{a_1, -a_2\}$ and $E = \{2e_1, -2e_2\}$ with replacing a_2 by $(-a_2)$, e_1 by $(2e_1)$ and e_2 by $(-2e_2)$ in the Eqs. (2.2), (2.3), (2.4) and (2.5), we have:

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_n\left(2e_1 + [-2e_2]\right) z^n$$
$$= \frac{1 - 4a_1a_2e_1e_2z^2}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)}. \tag{3.1}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_{n-1}\left(2e_1 + [-2e_2]\right) z^n$$
$$= \frac{z - 4a_1a_2e_1e_2z^3}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)}. \tag{3.2}$$

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{n-1}\left(2e_1 + [-2e_2]\right) z^n$$
$$= \frac{(a_1 - a_2)z + 2a_1a_2(e_1 - e_2)z^2}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)}. \tag{3.3}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) S_n\left(2e_1 + [-2e_2]\right) z^n$$
$$= \frac{2(e_1 - e_2)z + 4e_1e_2(a_1 - a_2)z^2}{(1 - 2a_1e_1z)(1 + 2a_2e_1z)(1 + 2a_1e_2z)(1 - 2a_2e_2z)}. \tag{3.4}$$

This case consists of three related parts. **First**, the substitutions

$$\begin{cases} a_1 - a_2 = p \\ a_1a_2 = q \end{cases} \text{and} \begin{cases} e_1 - e_2 = x \\ 4e_1e_2 = -1 \end{cases},$$

in the Eqs. (3.1), (3.2), (3.3) and (3.4), we give:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(2e_1 + [-2e_2])z^n$$
$$= \frac{1 + qz^2}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2z^4}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(2e_1 + [-2e_2])z^n$$
$$= \frac{z + qz^3}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2z^4}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n$$
$$= \frac{pz+2qxz^2}{1-2pxz-(4qx^2-p^2-2q)z^2+2pqxz^3+q^2z^4}, \quad (3.7)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n$$
$$= \frac{2xz-pz^2}{1-2pxz-(4qx^2-p^2-2q)z^2+2pqxz^3+q^2z^4}, \quad (3.8)$$

respectively, and we have the following theorems.

Theorem 3.1. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with Chebyshev polynomials of the first kind is given by:*

$$\sum_{n=0}^{\infty} F_{p,q,n}T_n(x)z^n = \frac{xz-pz^2-qxz^3}{1-2pxz-(4qx^2-p^2-2q)z^2+2pqxz^3+q^2z^4}. \quad (3.9)$$

Proof. By [20], we have $F_{p,q,n} = S_{n-1}(a_1+[-a_2])$. Then, we can see that:

$$\sum_{n=0}^{\infty} F_{p,q,n}T_n(x)z^n$$
$$= \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])\left(S_n(2e_1+[-2e_2]) - xS_{n-1}(2e_1+[-2e_2])\right)z^n$$
$$= \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n$$
$$- x\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n,$$

by using the relationships (3.6) and (3.8), we obtain:

$$\sum_{n=0}^{\infty} F_{p,q,n}T_n(x)z^n$$
$$= \frac{2xz-pz^2}{1-2pxz-(4qx^2-p^2-2q)z^2+2pqxz^3+q^2z^4}$$
$$- \frac{x\left(z+qz^3\right)}{1-2pxz-(4qx^2-p^2-2q)z^2+2pqxz^3+q^2z^4}$$
$$= \frac{xz-pz^2-qxz^3}{1-2pxz-(4qx^2-p^2-2q)z^2+2pqxz^3+q^2z^4}.$$

This completes the proof. □

Theorem 3.2. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Lucas numbers with Chebyshev polynomials of the first kind is given by:*

$$\sum_{n=0}^{\infty} L_{p,q,n} T_n(x) z^n = \frac{2 - 3pxz + \left(2q - 4qx^2 + p^2\right) z^2 + pqxz^3}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2 z^4}. \quad (3.10)$$

Proof. By referred to [20], we have:

$$L_{p,q,n} = 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2]).$$

We see that:

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n} T_n(x) z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} (2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2])) \\ \times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2])) \end{pmatrix} z^n \\
&= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad -2x \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&\quad -p \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad +px \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n.
\end{aligned}$$

Using the relationships (3.5)-(3.8), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n} T_n(x) z^n &= \frac{2\left(1 + qz^2\right)}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2 z^4} \\
&\quad - \frac{2x\left(pz + 2qxz^2\right)}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2 z^4} \\
&\quad - \frac{p\left(2xz - pz^2\right)}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2 z^4} \\
&\quad + \frac{px\left(z + qz^3\right)}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2 z^4} \\
&= \frac{2 - 3pxz + \left(2q - 4qx^2 + p^2\right) z^2 + pqxz^3}{1 - 2pxz - (4qx^2 - p^2 - 2q)z^2 + 2pqxz^3 + q^2 z^4}.
\end{aligned}$$

This completes the proof. □

Second, the substitutions

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = 2q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = x \\ 4e_1 e_2 = -1 \end{cases},$$

in the Eqs. (3.1)-(3.4), we give:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n = \frac{1+2qz^2}{1-2pxz-(8qx^2-p^2-4q)z^2+4pqxz^3+4q^2z^4}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n = \frac{z+2qz^3}{1-2pxz-(8qx^2-p^2-4q)z^2+4pqxz^3+4q^2z^4}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n = \frac{pz+4qxz^2}{1-2pxz-(8qx^2-p^2-4q)z^2+4pqxz^3+4q^2z^4}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n = \frac{2xz-pz^2}{1-2pxz-(8qx^2-p^2-4q)z^2+4pqxz^3+4q^2z^4}, \tag{3.14}$$

respectively, and we have the following theorems.

Theorem 3.3. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with Chebyshev polynomials of the first kind is given by:*

$$\sum_{n=0}^{\infty} J_{p,q,n}T_n(x)z^n = \frac{xz-pz^2-2qxz^3}{1-2pxz-(8qx^2-p^2-4q)z^2+4pqxz^3+4q^2z^4}. \tag{3.15}$$

Proof. By [20], we have $J_{p,q,n} = S_{n-1}(a_1+[-a_2])$. Then, we can see that:

$$\begin{aligned}
&\sum_{n=0}^{\infty} J_{p,q,n}T_n(x)z^n \\
&= \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])\left(S_n(2e_1+[-2e_2]) - xS_{n-1}(2e_1+[-2e_2])\right)z^n \\
&= \sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(2e_1+[-2e_2])z^n \\
&\quad - x\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(2e_1+[-2e_2])z^n
\end{aligned}$$

$$= \frac{2xz - pz^2}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4} - \frac{x\left(z + 2qz^3\right)}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4}$$
$$= \frac{xz - pz^2 - 2qxz^3}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4}.$$

This completes the proof. □

Theorem 3.4. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Jacobsthal Lucas numbers with Chebyshev polynomials of the first kind is given by:*

$$\sum_{n=0}^{\infty} j_{p,q,n} T_n(x) z^n = \frac{2 - 3pxz + \left(p^2 + 4q - 8qx^2\right) z^2 + 2pqxz^3}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4}. \quad (3.16)$$

Proof. By [20], we have $j_{p,q,n} = 2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2])$. Then, we can see that:

$$\begin{aligned}
\sum_{n=0}^{\infty} j_{p,q,n} T_n(x) z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} (2S_n(a_1 + [-a_2]) - pS_{n-1}(a_1 + [-a_2])) \\ \times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2])) \end{pmatrix} z^n \\
&= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad -2x \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&\quad -p \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad +px \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n,
\end{aligned}$$

by using the relationships (3.11)-(3.14), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} j_{p,q,n} T_n(x) z^n &= \frac{2\left(1 + 2qz^2\right)}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4} \\
&\quad - \frac{2x\left(pz + 4qxz^2\right)}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4} \\
&\quad - \frac{p\left(2xz - pz^2\right)}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4} \\
&\quad + \frac{px\left(z + 2qz^3\right)}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4} \\
&= \frac{2 - 3pxz + \left(p^2 + 4q - 8qx^2\right) z^2 + 2pqxz^3}{1 - 2pxz - (8qx^2 - p^2 - 4q)z^2 + 4pqxz^3 + 4q^2z^4}.
\end{aligned}$$

This completes the proof. □

Third, the substitutions

$$\left\{\begin{array}{c} a_1 - a_2 = 2p \\ a_1 a_2 = q \end{array}\right. \quad \text{and} \left\{\begin{array}{c} e_1 - e_2 = x \\ 4e_1 e_2 = -1 \end{array}\right.,$$

in the Eqs. (3.1)-(3.4), we give:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{1 + qz^2}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2 z^4}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n = \frac{z + qz^3}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2 z^4}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n = \frac{2pz + 2qxz^2}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2 z^4}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n = \frac{2xz - 2pz^2}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2 z^4}, \tag{3.20}$$

respectively, and we have the following theorems.

Theorem 3.5. *For $n \in \mathbb{N}$, the new generating function of the product of (p, q)-Pell numbers with Chebyshev polynomials of the first kind is given by:*

$$\sum_{n=0}^{\infty} P_{p,q,n} T_n(x) z^n = \frac{xz - 2pz^2 - qxz^3}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2 z^4}. \tag{3.21}$$

Proof. We know that:

$$P_{p,q,n} = S_{n-1}(a_1 + [-a_2]), \text{ (see [20])}.$$

We see that:

$$\sum_{n=0}^{\infty} P_{p,q,n} T_n(x) z^n = \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) \left(S_n(2e_1 + [-2e_2]) - x S_{n-1}(2e_1 + [-2e_2]) \right) z^n$$

$$= \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n$$
$$- x \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n$$
$$= \frac{2xz - 2pz^2}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2z^4}$$
$$- \frac{x\left(z + qz^3\right)}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2z^4}$$
$$= \frac{xz - 2pz^2 - qxz^3}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2z^4}.$$

This completes the proof. □

Theorem 3.6. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with Chebyshev polynomials of the first kind is given by:*

$$\sum_{n=0}^{\infty} Q_{p,q,n} T_n(x) z^n = \frac{2 - 6pxz + \left(4p^2 + 2q - 4qx^2\right) z^2 + 2pqxz^3}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2z^4}. \quad (3.22)$$

Proof. We know that:

$$Q_{p,q,n} = 2S_n(a_1 + [-a_2]) - 2pS_{n-1}(a_1 + [-a_2]), \text{ (see [20]).}$$

We see that:

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n} T_n(x) z^n &= \sum_{n=0}^{\infty} \begin{pmatrix} (2S_n(a_1 + [-a_2]) - 2pS_{n-1}(a_1 + [-a_2])) \\ \times (S_n(2e_1 + [-2e_2]) - xS_{n-1}(2e_1 + [-2e_2])) \end{pmatrix} z^n \\
&= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad -2x \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&\quad -2p \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_n(2e_1 + [-2e_2]) z^n \\
&\quad +2px \sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(2e_1 + [-2e_2]) z^n \\
&= \frac{2\left(1 + qz^2\right)}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2z^4} \\
&\quad - \frac{2x\left(2pz + 2qxz^2\right)}{1 - 4pxz - (4qx^2 - 2q - 4p^2)z^2 + 4pqxz^3 + q^2z^4}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{2p\left(2xz-2pz^2\right)}{1-4pxz-(4qx^2-2q-4p^2)z^2+4pqxz^3+q^2z^4} \\
& +\frac{2px\left(z+qz^3\right)}{1-4pxz-(4qx^2-2q-4p^2)z^2+4pqxz^3+q^2z^4} \\
= \quad & \frac{2-6pxz+\left(4p^2+2q-4qx^2\right)z^2+2pqxz^3}{1-4pxz-(4qx^2-2q-4p^2)z^2+4pqxz^3+q^2z^4}.
\end{aligned}
$$

This completes the proof. □

4. Conclusion

In this paper, by making use of Theorem 2.1, we have derived some new generating functions of the products of Chebyshev polynomials of the first kind with several special numbers attached to p and q parameters. The derived theorems are based on symmetric functions and products of these numbers and polynomials.

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