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Group Orders That Imply a Nontrivial p-Core

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Abstract. Given a prime number p and a natural number m not divisible by p, we propose the problem of finding the smallest number r_0 such that for $r \geq r_0$, every group G of order $p^r m$ has a non-trivial normal p-subgroup. We prove that we can explicitly calculate the number r_0 in the case where every group of order $p^r m$ is solvable for all r, and we obtain the value of r_0 for a case where m is a product of two primes.

1. Introduction

Throughout this note, p will be a fixed prime number. We use $O_p(G)$ to denote the p-core of G, that is, its largest normal p-subgroup.

We propose the following optimization problem: Given a number m not divisible by p, find the smallest r_0 such that every group having order $n = p^r m$, with $r \geq r_0$, has a nontrivial p-core $O_p(G)$. Denote such number r_0 by $\Lambda(p, m)$. In Theorem 2.1, we will prove that $\Lambda(p, m)$ is well-defined for any prime p and number m (with $p \nmid m$). In Theorem 2.3 we explicitly determine the value of $\Lambda(p, m)$ in the case that all groups whose order have the form $p^r m$ are solvable (for example, if m is prime or if both p and m are odd). Finally, in Section 3, we calculate $\Lambda(2, 15)$, a case that is not covered by the previous theorem.

We remark that the motivation for this research came from the search for examples of finite groups G such that the Brown complex $\mathcal{S}_p(G)$ of nontrivial p-subgroups of G (see for example [5] for the definition and properties) is connected but not contractible. It is known that $\mathcal{S}_p(G)$ is contractible when G has a nontrivial normal p-subgroup, and Quillen conjectured in [3] that the converse is also true.

2. Theorems

Theorem 2.1. *For any prime number p and natural number m such that $p \nmid m$,*

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there is a number $\Lambda(p,m)$ such that if $r \geq \Lambda(p,m)$, any group of order $p^r m$ has a non-trivial p-core $O_p(G)$.

Proof. Let G be a group of order $p^r m$ with $O_p(G)=1$. Let P be a Sylow p-subgroup of G. Since the kernel of the action of G on the set of cosets of P is precisely $O_p(G)$, we obtain that G embeds in S_m, and so p^r divides $(m-1)!$. Hence, if p^{r_0} is the largest power of p dividing ((m-1)!), we obtain that $\Lambda(p,m) \leq r_0+1$. □

For t,q natural numbers, let $\gamma(t,q)$ be the product

$$\gamma(t,q) = (q^t-1)(q^{t-1}-1)\cdots(q^2-1)(q-1), \tag{2.1}$$

(note that $\gamma(t,q)$ can also be defined as $(t)!_q(q-1)!$, where $(t)!_q$ is the q-factorial of t), and if $m = q_1^{t_1} q_2^{t_2}\cdots q_k^{t_k}$ is a prime factorization of m, with the q_i pairwise distinct and $t_i>0$ for each i, we let $\Gamma(m) = \gamma(t_1,q_1)\cdots\gamma(t_k,q_k)$. We prove that if p^{s_0} is the largest power of p dividing $\Gamma(m)$, then $\Lambda(p,m)\geq s_0+1$.

Theorem 2.2. *Let $n=p^s m$ where $p\nmid m$ and $s>0$. If $p^s \mid \Gamma(m)$, then there is a group of order n with $O_p(G)=1$.*

Proof. Let K be the group $C_{q_1}^{t_1}\times\cdots\times C_{q_k}^{t_k}$, that is, a product of elementary abelian groups, where $m=q_1^{t_1}\cdots q_k^{t_k}$ and $q_1,\dots,q_k$ are distinct primes and C_q denotes the cyclic group of order q. Then $\Gamma(m)$ divides the order of $\mathrm{Aut}(K)$, and hence so does p^s. Let H be a subgroup of $\mathrm{Aut}(K)$ of order p^s. For every $S\in H$ and $k\in K$ define the map $T_{S,k}\colon K\to K$ by $T_{S,k}(x)=Sx+k$. Then $G=\{\,T_{S,k}\mid S\in H, k\in K\,\}$ is a subgroup of $\mathrm{Sym}(K)$. If we identify H with the subgroup of maps of the form $T_{S,0}$ and K with the subgroup of maps of the form $T_{1_K,k}$, then G is just the semidirect product of K by H. Hence $|G|=n$. We have that G acts transitively on K in a natural fashion, and the stabilizer of $0\in K$ is H, a p-Sylow subgroup of G. Hence the stabilizers of points in K are precisely the Sylow subgroups of G, so their intersection $O_p(G)$ contains only the identity $K\to K$, as we wanted to prove. □

The next theorem will show that the lower bound given by Theorem 2.2 is tight in some cases.

Theorem 2.3. *Let $n=p^s m$, where $p\nmid m$. If G is a group of order n and p^s does not divide $\Gamma(m)$ then either:*

1. *($O_p(G)\neq 1$), or*
2. *G is not solvable.*

Proof. Let G be solvable with order $n=p^s m$ and $O_p(G)=1$. Let $F(G)$ be the Fitting subgroup of G. Consider the map $c: G\to \mathrm{AutF(G)}$, sending g to $c_g: F(G)\to F(G)$ given by conjugation by g. The restriction of c to P, a p-Sylow subgroup of G, has kernel $P\cap C_G(F(G))$. Since $C_G(F(G))\leq F(G)$ (Theorem 7.67 from [4]), and $F(G)$ does not contain elements of order p by our assumption on $O_p(G)$, we have $P\cap C_G(F(G))=1$ and so P acts faithfully on $F(G)$. If $m=q_1^{t_1}\cdots q_k^{t_k}$ is the prime factorization of m, we have that $F(G)$ is the direct product of the $O_{q_i}(G)$

for $i = 1, \cdots, k$. Hence $P \leq \mathrm{Aut}(F(G)) \cong \mathrm{Aut}(O_{q_1}(G)) \times \cdots \times \mathrm{Aut}(O_{q_k}(G))$. Let $g \in P$ such that the action induced by c_g on $\prod_i O_{q_i}(G)/\Phi(O_{q_i}(G))$, is the identity. Since c_g acts on each factor $O_{q_i}(G)/\Phi(O_{q_i}(G))$ as the identity, then by Theorem 5.1.4 from [2], we have that it acts as the identity on each $O_{q_i}(G)$. By the faithful action of P on $F(G)$, we have that $g = 1$. This implies that P acts faithfully on $\prod_i O_{q_i}(G)/\Phi(O_{q_i}(G))$. But then $|P|$ divides the order of the automorphism group of $\prod_i O_{q_i}(G)/\Phi(O_{q_i}(G))$, which is a product of elementary abelian groups of respective orders $q_i^{s_i}$ with $s_i \leq t_i$ for all i. Hence $p^s = |P|$ divides $\Gamma(m)$ □

Corollary 2.4. *Let p^s be the largest power of p that divides $\Gamma(m)$. If m is prime, or if both p, m are odd, then $\Lambda(p, m) = s + 1$.*

Proof. By Burnside's p, q-theorem, and the Odd Order Theorem, we have that all groups that have order of the form $p^r m$ for some r are solvable. Therefore, for all $r > s$, by Theorem 2.3 we have that all groups of order $p^r m$ have non-trivial p-core. □

At this moment, we can prove that in some cases, the group constructed in 2.2 is unique.

Theorem 2.5. *Let $n = p^s m$ where $p \nmid m$ and $s > 0$. If $p^s \mid \Gamma(m)$, but $p^s \nmid \Gamma(m')$ for all proper divisors m' of m, then up to isomorphism, the group constructed in the proof of Theorem 2.2 is the only solvable group of order n with $O_p(G) = 1$.*

Proof. With the notation of the argument of the proof of 2.3, if G is a solvable group of order n with $O_p(G) = 1$, we must have that $|O_{q_i}(G)| = q_i^{t_i}$ and $\Phi(O_{q_i}(G)) = 1$ for all i in order to satisfy the divisibility conditions. Hence $O_{q_i}(G)$ is elementary abelian and a q_i-Sylow subgroup for all i, and so G is the semidirect product of a p-Sylow subgroup P of $F(G) = C_{q_1}^{t_1} \times \cdots \times C_{q_k}^{t_k}$ with $F(G)$, where the action of P on $F(G)$ by conjugation is faithful. Hence G is isomorphic to the group constructed in the proof of Theorem 2.3. □

One case in that we may apply Theorem 2.5 is when $n = 864$. There are 4725 groups of order $864 = 2^5 3^3$, but only one of them has the property of having a trivial 2-core.

3. An Example

An example that cannot be tackled with the previous results is the case $p = 2$, $m = 3 \cdot 5 = 15$. In this case, $\Gamma(15) = (3-1)(5-1) = 2^3$. Not all groups with order of the form $2^r \cdot 3 \cdot 5$ are solvable, however, we will prove that $\Lambda(2, 15)$ is actually 4. (The group S_5 attests that $\Lambda(2, 15) > 3$.)

Theorem 3.1. *Every group G of order $2^r \cdot 3 \cdot 5$ for $r \geq 4$ is such that $O_2(G) \neq 1$.*

Proof. Let G be a group of order $2^r \cdot 3 \cdot 5$ for $r \geq 4$. Suppose that $O_2(G) = 1$. From Theorem 2.3, we obtain that G is not solvable. We will prove then that $O_3(G) = 1$. Suppose otherwise, and let $T = O_3(G)$. Then $|G/T| = 2^r \cdot 5$, and so G/T is solvable. Since $2^r \nmid \Gamma(5)$, from Theorem 2.3, we have that $O_2(G/T) \neq 1$. Let $L \lhd G$ such that $O_2(G/T) = L/T$. Suppose $|L/T| = 2^j$. Since $O_2(G/L) = 1$, $|G/L| = 2^{r-j} \cdot 5$ and G/L is solvable, we have that 2^{r-j} divides $\Gamma(5) = 2^2$, that is, $r - j \leq 2$. Now, L is also solvable and $\Gamma(3) = 3 - 1 = 2$, hence if we had $j \geq 2$ we would have $O_2(L) \neq 1$, and G would have a non-trivial subnormal 2-subgroup, which contradicts our assumption that $O_2(G) = 1$. Hence $j = 1$. But then $r - 1 \leq 2$, which contradicts that $r \geq 4$. Hence $O_3(G) = 1$. By a similar argument, we get that $O_5(G) = 1$.

From [1] we obtain that G is not simple. Hence G has a proper minimal normal subgroup M. From the previous paragraph, we obtain that M is not abelian, since in that case we would have that $M \leq F(G)$. The only possibility is that $M = A_5$. We have then a morphism $c\colon G \to \mathrm{Aut}(A_5)$ sending g to c_g, the conjugation by g. Since $\mathrm{Aut}(A_5) = S_5$, and $|c(G)| = |\mathrm{Inn}(G)| \geq |\mathrm{Inn}(A_5)| = 60$, in any case the kernel of c is a nontrivial normal 2-subgroup. □

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