KYUNGPOOK Math. J. 62(2022), 657-671
https://doi.org/10.5666/KMJ.2022.62.4.657
pISSN 1225-6951 eISSN 0454-8124

On Generators in the Category of Actions of Pomonoids on Posets and its Slices

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Abstract. Where S is a pomonoid, let **Pos**-S be the category of S-posets and S-poset maps. We start off by characterizing the pomonoids S for which all projectives in this category are either generators or free. We then study the notions of regular injectivity and weakly regularly d-injectivity in this category. This leads to homological classification results for pomonoids. Among other things, we get find relationships between regular injectivity in the slice category **Pos**-S/B_S, for any S-poset B_S, and generators and cyclic projectives in **Pos**-S.

1. Introduction and Preliminaries

General ordered algebraic structures play a key role in a wide range of areas, including analysis, logic, theoretical computer science, and physics [2]. One of these structures, which is of interest to mathematicians, is the category of representations of a pomonoid by order preserving maps of partially ordered sets (see for example [3, 4, 5, 6, 7, 8, 9, 14, 16, 18, 19]). Although there exist many papers which investigate various properties of generator acts over a fixed monoid (see [10, 11, 12, 17] for example), among them there seems to be very little known on generator S-posets, where S is a pomonoid. In [14], V. Laan investigated some properties of generator S-posets. Furthermore, in [9] some homoligical characterizations of pomonoids by properties of generators were presented. Continuing this study, in this paper, after some introductory results in Section 1, we attempt in Section 2 to collect new results on generators in **Pos**-S to apply to the question of the homological classification of pomonoids.

$\mathcal{M}$-injective objects in the slice category $\mathcal{C}/B$, for any B in $\mathcal{C}$, form the right part of a weak factorization system that has morphisms of $\mathcal{M}$ as the left part (see [1]). Here, we consider the same case in the slice category **Pos**-S/B_S of right S-poset

* Corresponding Author.
Received November 29, 2021; revised May 22, 2022; accepted October 10, 2022.
2020 Mathematics Subject Classification: 06F05, 18A25, 18G05, 20M30, 20M50..
Key words and phrases: S-poset, generator, projective, slice category, regular injective.

maps over B_S, where B_S is an arbitrary S-poset. In Section 3, we first find conditions for when all generators are regular d-injective or weakly regularly injective. Then, we prove that every $\mathcal{M}$-injective object in **Pos**-S/B_S is a split epimorphism, where $\mathcal{M}$ = Emb is the class of all order-embeddings of S-posets. Also, we investigate the relationship between regular injectivity in **Pos**-S and **Pos**-S/B_S and generators and cyclic projectives which becomes evident when passing to acts over their endomorphism monoids.

For the rest of this section, we give some preliminaries about S-acts, S-posets and slice category which we will need in the sequel. The reader is referred to [13] and [3], respectively, for information on general properties of S-acts and S-posets that are not fully explained here.

Let S be a monoid with identity 1. Recall that a (right) S-*act* is a set A equipped with a map $\mu : A \times S \to A$ called its action, such that, denoting $\mu(a, s)$ by as, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. The category of all S-acts, with action-preserving (S-act) maps ($f : A \to B$ with $f(as) = f(a)s$, for $s \in S, a \in A$), is denoted by **Act**-S. For instance, take any monoid S and a non-empty set A. Then A becomes a right S-act by defining $as = a$ for all $a \in A$, $s \in S$; we call that A an S-act with trivial action. Clearly S itself is an S-act with its operation as the action.

On a monoid S we define the following relations: for every $s, t \in S$

1. $s\mathcal{R}t$ iff $sS = tS$.
2. $s\mathcal{J}t$ iff $SsS = StS$.
3. $s\mathcal{D}t$ iff there exists $u \in S$ with $sS = uS$ and $St = Su$.

These relations are called Green's relations on S (see [13]). Here, we consider these notions for a pomonoid S and supply some suitable results. A monoid S is said to be a *partially ordered monoid* (briefly a *pomonoid*) if it is also a poset whose partial order $\leq$ is compatible with the binary operation, i.e., $s \leq t, s' \leq t'$ imply $ss' \leq tt'$ (see [2]). In this paper S denotes a pomonoid with an arbitrary order, unless otherwise stated.

Let S be a pomonoid and A be a poset. Then $A \times S$ becomes a poset with componentwise order. A poset A is said to be a (*right*) S-*poset* over a pomonoid S if it is an S-act and the action is monotone ($(a_1, s_1) \leq (a_2, s_2)$ implies, $a_1s_1 \leq a_2s_2$, where $a_1, a_2 \in A$ and $s_1, s_2 \in S$). We denote it by A_S. The category of all S-posets with action preserving monotone maps is denoted by **Pos**-S. Clearly S itself is an S-poset with its operation as the action. A left S-poset A can be defined analogously (see [3]) and denoted by ${}_SA$. Also, we denote the category of all left S-posets with action preserving monotone maps by S-**Pos**. As in the unordered case, the coproduct in **Pos**-S is simply the disjoint union, with S-action and order given componentwise, and as usual the coproduct of a family $\{A_i \mid i \in I\}$ will be denoted by $\coprod_{i\in I} A_i$. Let T and S be pomonoids. Then a poset A is called a T-S-*biposet* if it is a left T-poset and a right S-poset and $(ta)s = t(sa)$ for every $s \in S, t \in T$ and $a \in A$. We denote it by ${}_TA_S$.

We recall the following results from [14]:

(i) For every A_S in **Pos**-S, consider the set $\mathrm{End}(A_S) = \mathbf{Pos}_S(A, A)$ as a pomonoid with respect to composition and pointwise order. We define the left $\mathrm{End}(A_S)$-action on A by $f \cdot a = f(a)$, for every $f \in \mathrm{End}(A_S)$, $a \in A$. Note that this action is monotone because if $f, g \in \mathrm{End}(A_S)$ and $a, b \in A$ are such that $f \leq g$ and $a \leq b$ then we have $f \cdot a = f(a) \leq f(b) \leq g(b) = g \cdot b$. Thus one has ${}_{\mathrm{End}(A_S)}A_S$.

(ii) The following two mappings are pomonoid homomorphisms:

$$\rho : S \to \mathrm{End}(A_S); \quad s \mapsto \rho_s,$$

$$\lambda : T \to \mathrm{End}({}_TA); \quad t \mapsto \lambda_t. \tag{1.1}$$

Here, $\rho_s : A_S \to A_S,\ a \mapsto as$ and $\lambda_t :\ {}_TA \to\ {}_TA,\ a \mapsto ta$ are morphisms in **Pos**-S and T-**Pos**, respectively.

(iii) For every T-S-biposet ${}_TA_S$ recall that if $B \in$ **Pos**-S then the set $\mathbf{Pos}_S(B, A)$ of all S-poset maps from B_S to A_S is an object in T-**Pos** with the action defined by $t \cdot f = \lambda_t f$ for every $t \in T, f \in \mathbf{Pos}_S(B, A)$. Consequently, we have a functor

$$\mathbf{Pos}_S(-, A) : \mathbf{Pos}\text{-}S \to T\text{-}\mathbf{Pos}$$

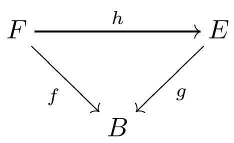
by taking

$$\mathbf{Pos}_S(-, A)(B) = \mathbf{Pos}_S(B, A)$$

for every $B \in$ **Pos**-S.

An S-poset G_S is a generator in the category **Pos**-S if for any distinct S-poset maps $\alpha, \beta : X_S \to Y_S$ there exists an S-poset map $f : G_S \to X_S$ such that $\alpha f \neq \beta f$.

For any category $\mathcal{C}$ and an object B of $\mathcal{C}$, there is a *slice category* (also called *comma category*) $\mathcal{C}/B$. The objects of $\mathcal{C}/B$ are morphisms of $\mathcal{C}$ with codomain B, and morphisms in $\mathcal{C}/B$ from one such object $f : F \to B$ to another $g : E \to B$ are commutative triangles in $\mathcal{C}$:



i.e, $gh = f$. We write $h : f \to g$. The composition in $\mathcal{C}/B$ is defined from the composition in $\mathcal{C}$, in the obvious way– the triangles are pasted together (for more details see [15]).

A poset is said to be *complete* if each of its subsets has an infimum and a supremum, in particular, a complete poset is bounded, that is, it has a least (bottom) element $\bot$ and a greatest (top) element $\top$.

2. Some Homological Classifications for Pomonoids by Generators in Pos-S

In this section, we discuss the properties of generators and projective generators in **Pos**-S. Recall that a projective S-poset A_S which is also a generator is called a projective generator S-poset. A cyclic S-poset is an S-poset A for which there exists an element $a \in A$ such that $A = aS$. By a cyclic projective S-poset we mean a cyclic S-poset which is also projective.

As we mentioned in the introduction, generators for the category **Pos**-S were characterized in [14] with the following two propositions.

Proposition 2.1. *Cyclic projectives in* **Pos**-S *are precisely retracts of* S_S.

Proposition 2.2. *An* S*-poset* A_S *is a cyclic projective generator in* **Pos**-S *if and only if* $A_S \cong eS_S$ *for an idempotent* $e \in S$ *with* $e\mathcal{J}1$.

The following is immediate from Proposition 2.2:

Proposition 2.3. *Let* S *be a commutative pomonoid. Then all cyclic projective generators in* **Pos**-S *are isomorphic to* S_S.

We will also need the following characterization of cyclic projective S-posets from [19, Proposition 4.2].

Proposition 2.4. *Let* A_S *be an* S*-poset and* $a \in A$. *Then the following statements are equivalent:*
(i) aS_S *is projective.*
(ii) $aS_S \cong eS_S$ *for some idempotent* $e \in S$.

We state the following two facts about projectives and generators from [19] and [14] respectively. They will be used throughout the paper.

Theorem 2.5. *An* S*-poset* P_S *is projective if and only if* $P_S \cong \coprod_{i\in I} e_iS$ *where* $e_i^2 = e_i \in S, i \in I$.

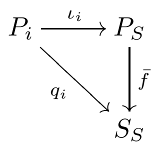
Theorem 2.6. *The following assertions are equivalent for a right* S*-poset* A_S.

1. *For all* $X_S, Y_S \in$ **Pos**-S *and* $f, g \in \mathbf{Pos}_S(X, Y)$, $f \leq g$ *whenever* $fk \leq gk$ *for all* $k \in \mathbf{Pos}_S(A; X)$.
2. A_S *is a generator.*
3. *For every* $X_S \in$ **Pos**-S *there exists a set* I *and an epimorphism* $h : \coprod_I A \to X$ *in* **Pos**-S.
4. *There exists an epimorphism* $\pi : A \to S$ *in* **Pos**-S.
5. S_S *is a retract of* A_S.

Now we can prove the following result.

Theorem 2.7. *Every S-poset P_S is projective generator if and only if $P_S = \coprod_{i\in I} P_i$ where $P_i \cong e_iS$ for every $i \in I$, and at least one P_j, $j \in I$ is a generator with $e_j\mathfrak{J}1$.*

Proof. On the one hand, let the S-poset P_S be a projective generator. By Theorem 2.5 we have $P_S \cong \coprod_{i\in I} e_iS$ where $e_i^2 = e_i \in S, i \in I$. And by Theorem 2.6 there exists a surjective S-poset epimorphism $\pi : P_S \longrightarrow S_S$, so $1 = \pi(a)$ for some $a \in e_jS, j \in I$. Now $\pi|_{e_jS} : e_jS \longrightarrow S_S$ is also an epimorphism in **Pos**-S, because for any $s \in S$ we have $s = 1s = \pi(a)s = \pi(as)$ and $as \in e_jS$. Hence, e_jS is a generator and by Proposition 2.2, $e_j\mathfrak{J}1$.
On the other hand, assume that P_S has the factorisation in the statement of the theorem. By Theorem 2.5, P_S is projective. That P_j is generator, implies that there exists an S-poset epimorphism $\pi_j : P_j \longrightarrow S_S$. Now, for the following diagram



take $q_j = \pi_j$ and q_i the composite S-poset map $P_i \cong e_iS \hookrightarrow S$ for every $i \in I$, $i \neq j$. By the property of the coproduct S-poset $P_S = \coprod_{i\in I} P_i$, corresponding to the S-poset epimorphisms $\{q_i \mid i \in I\}$, there exists a unique S-poset map $\pi : P \longrightarrow S_S$ such that $\pi|_{P_i} = q_i$ for all $i \in I$. In particular, $\pi|_{P_j} = \pi_j$ and π_j is an S-poset epimorphism, so π is also an S-poset epimorphism. Hence, P_S is generator. □

Notice that for every pomonoid S and idempotent $e \in S$, the sub S-poset eS_S of S_S is projective according to Proposition 2.4, but it is not a generator because $e\mathfrak{J}1$ does not necessarily hold. For example, if we take a periodic monoid S endowed it with discrete order then we have a pomonoid. Now if we take an idempotent $1 \neq e \in S$, then $e\mathfrak{J}1$ does not hold (see [13, Proposition I.3.26 on page 32] for more details).

Next, we have the following result.

Theorem 2.8. *For any pomonoid S the following statements are equivalent:*
(i) *All projective right S-posets are generators in* **Pos**-S.
(ii) *All cyclic projective right S-posets are generators in* **Pos**-S.
(iii) $e\mathfrak{J}1$ *for every idempotent $e \in S$.*

Proof. That (i) implies (ii) is clear. To see that (ii) implies (iii) observe that for any idempotent $e \in S$, the right S-poset eS_S is cyclic, hence it is a genreator by assumption. The result thus follows by Proposition 2.2. .

For the implication (iii) $\Rightarrow$ (i), let P_S be an S-poset. By Theorem 2.5 we have $P_S \cong \coprod_{i\in I} e_iS$ where $e_i^2 = e_i \in S, i \in I$. By the assumption we have $e_i\mathfrak{J}1$ for every $i \in I$ and so P_S is a generator by Theorem 2.7. □

Recall [4] that a right *poideal* of a pomonoid S is a (possibly empty) subset I of S if it is both a monoid right ideal ($IS \subseteq I$) and a down set ($a \leq b, b \in I$ imply that $a \in I$). It is *principal* if it is generated (as a monoid right ideal of S) by a single element. For example

$$\downarrow rS = \{t \in S \ : \ \exists s \in S, \ t \leq rs\}$$

is a principal poideal of S, for every $r \in S$.

In the following we shall characterize pomonoids for which all principal right poideals are generators.

Proposition 2.9. *Let S be a pomonoid and $e \in S$ satisfy $e^2 = e$. If the cyclic projective sub S-poset eS_S of S_S is a generator in **Pos**-S, then $\downarrow eS$ is also a generator.*

Proof. By assumption there exists an S-poset epimorphism $f\colon eS_S \to S_S$. Define the mapping $g\colon \downarrow eS \to S_S$ by $g(x) := f(ex)$ for every $x \in \downarrow eS$. It is easy to see that g is an S-poset map. Also, for every $s \in S$ there exists $u \in S$ such that $f(eu) = s$. Then we have

$$g(eu) = f(eeu) = f(eu) = s.$$

This means that g is an epimorphism. By Theorem 2.6 we conclude that $\downarrow eS$ is a generator, as required. □

Lemma 2.10. *Let S be a pomonoid and $z \in S$. If the principal right poideal $\downarrow zS$ is a generator in **Pos**-S, then there exist $x, y \in S$ such that $1 \leq yx$, and $za \leq zb$, $a, b \in S$ implies $ya \leq yb$.*

Proof. Since $\downarrow zS$ is a generator in **Pos**-S, by Theorem 2.6, there exists an epimorphism $g : \downarrow zS \to S_S$. Hence, there are elements $u \in \downarrow zS$ and $t \in S$ such that $u \leq zt$ and $g(u) = 1$. Let $y = g(z)$ and $x = t$. Then $yx = g(z)x = g(zx)$. Since $u \leq zx$, the monotonicity of g implies that $g(u) \leq g(zx)$. Consequently, $1 = g(u) \leq g(zx) = yx$. Now, suppose $za \leq zb$, $a, b \in S$. Then $ya = g(z)a = g(za) \leq g(zb) = g(z)b = yb$. □

Next we answer the question about the conditions under which the assumptions of Proposition 2.9 are satisfied.

Proposition 2.11. *Let S be a pomonoid in which the identity element is the top element. If all poideals of S are generators in **Pos**-S, then the sub S-poset eS_S of S_S is a generator in **Pos**-S, for every idempotent $e \in S$.*

Proof. Assume that all poideals of S are generators in **Pos**-S. Then for every idempotent $e \in S$, $\downarrow eS$ is a generator in **Pos**-S. By Lemma 2.10, there exist $x, y \in S$ such that $1 \leq yx$, and $ea \leq eb$, $a, b \in S$, always implies $ya \leq yb$. In particular, since $e1 \leq ee$ we have $y \leq ye$, so $1 \leq yx \leq yex$. As we have $yex \leq 1$ by the hypothesis, we get $yex = 1$, which means that $e\mathcal{J}1$. So eS_S is a projective generator by Proposition 2.2, as needed. □

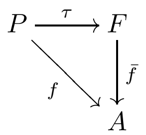
Theorem 2.12. *Let S be a pomonoid in which the identity element is the top element. The following statements are equivalent:*
(i) *All projective right S-posets are generators in* **Pos**-S.
(ii) *All cyclic projective right S-posets are generators in* **Pos**-S.
(iii) $e\mathcal{J}1$ *for every idempotent $e \in S$.*
(iv) *All principal right poideals of S which are generated by an idempotent, are generators in* **Pos**-S.

Proof. The equivalence of the first three statements is Theorem 2.8.

That (iii) implies (iv) is easy. Indeed, by Proposition 2.2 we get that eS_S is a cyclic projective generator, and Proposition 2.9 shows that $\downarrow eS$ is a generator in **Pos**-S.

To finish off, we show that (iv) implies (iii). Consider the principal right poideal $\downarrow eS$ for every idempotent $e \in S$ which is a generator in **Pos**-S. By a proof similar to that of Proposition 2.11, the cyclic projective sub S-poset eS_S of S_S is a generator. Using Proposition 2.2, we conclude that $e\mathcal{J}1$. □

By a *free S-poset on a poset P* we mean an S-poset F together with a poset map $\tau : P \to F$ with the universal property that given any S-poset A and any poset map $f : P \to A$ there exists a unique S-poset map $\bar{f} : F \to A$ such that $\bar{f} \circ \tau = f$, i.e, the diagram



commutes. The S-poset F (up to isomorphism) is given by $F = P \times S$ with componentwise order and the action $(x, s)t = (x, st)$, for $x \in P$ and $s, t \in S$ (see [3] for example). Furthermore, by a *free S-poset* we mean an S-poset which is free on some poset.

Example 2.13. Let S be a pomonoid generated by the elements e, k, k' and with discrete order such that $kk' = 1, e^2 = e$ and $ek = k'$. Then eS_S is a cyclic projective generator in **Pos**-S. But eS_S is not free (see Lemma 2.14 below). □

Now, we present some condition under which the sub S-posets eS_S of S_S are free for idempotent elements $e \in S$. The proof of the following result is similar to the proof for the unordered case in [13, Proposition 3.17.17], so we omit it. Moreover, we conclude when projectivity (or cyclic projectivity) implies freeness in **Pos**-S.

Lemma 2.14. *Let e be an idempotent of a pomonoid S. Then the sub S-poset eS_S of S_S is a free right S-poset if and only if $e\mathcal{D}1$.* □

This allows us to prove the following.

Theorem 2.15. *For any pomonoid S the following statements are equivalent:*
(i) *All projective right S-posets are free.*

(ii) *All projective generators in* **Pos**-S *are free.*
(iii) *All cyclic projective right S-posets are free.*
(iv) $e\mathcal{D}1$ *for every idempotent* $e \in S$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial.

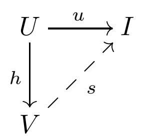
To see (ii) $\Rightarrow$ (iii), observe that by Proposition 2.4, all cyclic projective S-posets are isomorphic to eS_S for some idempotent $e \in S$. Let $A = S_S \coprod eS_S$. By Proposition 2.7, A_S is a projective generator in **Pos**-S. By hypothesis A_S is free which implies that eS_S is free.

Now we show that (iii) $\Rightarrow$ (i). By decomposition theorem in [19], every projective S-poset is isomorphic to a coproduct of cyclic projective S-posets which are free by assumption. Now since the coproducts of free S-posets being free we get the result.

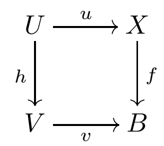
By the characterization of cyclic projective S-posets in Proposition 2.4 and Lemma 2.14 we get the equivalence of (iii) and (iv), which completes the proof. □

3. Regular Injectivity in Pos-S and Pos-S/B_S and Generators

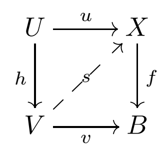
Let $\mathcal{C}$ be a category and $\mathcal{M}$ a class of its morphisms. An object I of $\mathcal{C}$ is called $\mathcal{M}$-injective if for each $\mathcal{M}$-morphism $h : U \to V$ and morphism $u : U \to I$ there exists a morphism $s : V \to I$ such that $sh = u$. That is, the following diagram is commutative:



In particular, this means that, in the slice category $\mathcal{C}/B$, an object $f : X \to B$ is $\mathcal{M}$-injective if, for any commutative diagram in $\mathcal{C}$



with $h \in \mathcal{M}$, there exists an arrow $s : V \to X$ such that $sh = u$ and $fs = v$.



Recall that regular monomorphisms (morphisms which are equalizers) in **Pos**-S (and also in **Pos**-S/B_S) are exactly order-embeddings (see [3] and [6]). By Emb-injectivity in **Pos**-S we mean $\mathcal{M}$-injectivity in **Pos**-S, where $\mathcal{M} = \text{Emb}$ is the class of all order-embeddings of S-posets. In the following we shall deal with Emb-injectivity in **Pos**-S and **Pos**-S/B_S, where Emb is the class of all order-embeddings of S-posets.

Theorem 3.1. *All generators in* **Pos**-S *are Emb-injective if and only if all* S*-posets are Emb-injective.*

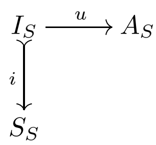
Proof. Clearly it is enough to show the forward implication. Let A_S be an S-poset. Consider the product S-poset $A_S \times S_S$ which is a generator in **Pos**-S by Theorem 2.6 and so is Emb-injective. By a general category-theoretic result which states that a product of a family of injective objects in a category is injective if and only if each component of the product is injective, we get that A_S is Emb-injective in **Pos**-S. □

Note that the class of all embeddings of right poideals into S_S is a subclass of all down-closed embeddings in **Pos**-S, i.e. all embeddings $g : B_S \to C_S$ with the property that $g(B)$ is down-closed in C, and hence is a subclass of all embeddings.

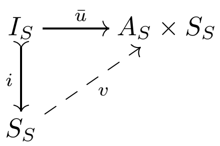
Definition 3.2. An S-poset A_S is called (*principally*) *weakly regularly d-injective* if it is injective with respect to all embeddings of (principal) right poideals into S_S.

Proposition 3.3. *If all generators in* **Pos**-S *are weakly regularly d-injective then all* S*-posets are weakly regularly d-injective.*

Proof. Let A_S be an S-poset. Since $A_S \times S_S$ is a generator in **Pos**-S it is a weakly regularly d-injective. To show that A_S is weakly regularly d-injective consider the following diagram



where I is a poideal of S. Define S-poset map $\bar{u} : I_S \to A_S \times S_S$ by $\bar{u}(s) = (u(s), s)$ for each $s \in I_S$. By the assumption, there exists an S-poset map $v : S_S \to A_S \times S_S$ such that $vi = \bar{u}$.



Now by composition v with the projection $\pi_A : A_S \times S_S \to A_S$, we get A_S is a weakly regularly d-injective. □

For a pomonoid S recall that an element $s \in S$ is called *regular* if there exists $t \in S$ such that $sts = s$. One calls S a *regular pomonoid* if all its elements are regular.

Theorem 3.4. *Let S be a pomonoid whose identity element is the top element. Then the following statements are equivalent:*
(i) *All S-posets are principally weakly regularly d-injective.*
(ii) *All principal right poideals of S are principally weakly regularly d-injective.*
(iii) *All generators in* **Pos**-*S are principally weakly regularly d-injective.*
(iv) *S is a regular pomonoid.*

Proof. The equivalence of (i) and (iii) comes from (the proof of) Proposition 3.3. The implication (iv) $\Rightarrow$ (i) is in [18, Theorem 3.6] and the implication (i) $\Rightarrow$ (ii) is trivial, so it is enough for us to show the implication (ii) $\Rightarrow$ (iv).

So assume (ii). For every $s \in S$, consider the down-closed embedding $i : \downarrow sS \to S_S, x \mapsto x$. It has a left inverse f, as $\downarrow sS$ is principally weakly regularly d-injective. Taking $f(1) = z$, we have $z \leq st$ for some $t \in S$ and

$$s = f(s) = f(1)s = zs \leq sts.$$

On the other hand, $sts \leq s$, as 1 is the top element of S. Therefore $sts = s$, showing that s is a regular element. As this was for any s, S is a regular pomonoid. □

Recall from [4] that a pomonoid S which has no proper non-empty left (right) poideal is said to be left (right) *simple*.

Corollary 3.5. *If all generators in* **Pos**-*S are Emb-injective then S is left simple.*

Proof. From the hypothesis and Theorem 3.1, we conclude that all complete S-posets are Emb-injective. It follows then from [4, Theorem 3.9] that S is left simple. □

Proposition 3.6. *For any pomonoid S the following statements are equivalent:*
(i) *All generators in* **Pos**-*S are complete S-posets.*
(ii) *All S-posets are complete.*

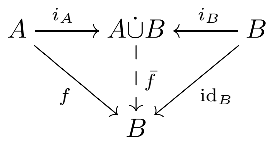
Proof. First assume (i). Let A_S be an S-poset. Consider the generator $A_S \times S_S$, which is a complete S-poset by assumption. Since the order on the product $A_S \times S_S$ is the componentwise order, joins are computed componentwise in the product as well. That is, for a subset $T \subseteq A_S \times B_S$ we have $\bigvee T = (\bigvee \pi_A(T), \bigvee \pi_B(T))$ where π_A and π_B are canonical projections on A_S and B_S, respectively. Therefore, for any subset $B \subseteq A$, $\bigvee B$ exists and so A_S is complete, giving (ii).

The converse implication is trivial. □

We state the following result from [7, Proposition 3.17] that will be used later on. We give a direct proof of it here, for the convenience of the reader .

Proposition 3.7. *Let S be a pomonoid and $B_S \in$* **Pos**-*S. Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-*S/B_S. Then f is a split epimorphism in* **Pos**-*S.*

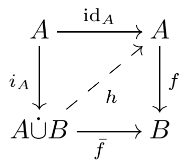
Proof. By the universal property of the coproduct S-poset $A\dot{\cup}B$ (the disjoint union of A and B) there exists a unique S-poset map $\bar{f} : A\dot{\cup}B \to B$ such that the following diagram commutes where i_A and i_B are injection S-poset maps.



In fact,

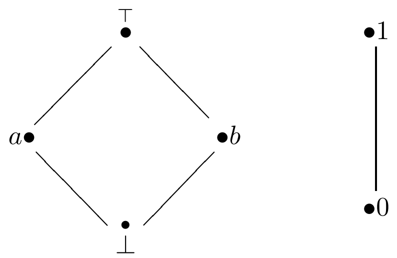
$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in B. \end{cases}$$

Now, let us consider the following commutative square



Since f is an Emb-injective object in **Pos**-S/B_S, there exists a unique S-poset map $h : A\dot{\cup}B \to A$ such that $fh = \bar{f}$ and $hi_A = \mathrm{id}_A$. So $fhi_B = \bar{f}i_B = \mathrm{id}_B$, which shows that f is a split epimorphism in **Pos**-S. □

Remark 3.8. There exist split epimorphisms in **Pos**-S which are not Emb-injective in **Pos**-S/B_S. To present an example, take an arbitrary pomonoid S and let X and B be, respectively, the first and second lattices shown in the following diagram:



Evidently, X is an S-poset with the action defined by $\top s = \top$ and $as = bs = \bot s = a$ for all $s \in S$, also we consider B with the trivial action as an S-poset. Define the S-poset map $f : X_S \to B_S$, by $f(a) = f(b) = f(\bot) = 0$ and $f(\top) = 1$. Then f is a convex map. We show that it is not a regular injective object in **Pos**-S/B_S. Since $f^{-1}(0) = \{\bot, a, b\}$ is not a complete lattice, the authors in [6] showed that it is not Emb-injective in **Pos**-S/B_S.

On the other hands, define the S-poset map $g : B_S \to X_S$ by $g(0) = \bot, g(1) = \top$. Then we have $fg = \mathrm{id}_B$, so f is a split epimorphism. Therefore, the converse of the above proposition is not true generally. □

Next recall that for a given poset P and a pomonoid S, the cofree S-poset on P is the set $P^{(S)}$ of all monotone maps from S to P, with pointwise order and action given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in P^{(S)}$ (see also [3, Theorem 13]).

Corollary 3.9. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S. *If A is a complete lattice which is also a retract of the cofree S-poset $A^{(S)}$, then A_S and B_S are Emb-injective object in* **Pos**-S.

Proof. By hypothesis we conclude that $A^{(S)}$ is an Emb-injective S-poset (see [4, Theorem 3.3]). Also it is straightforward to see that a retract of a Emb-injective S-poset is Emb-injective and so we get A_S is an Emb-injective S-poset. Also, by Proposition 3.7 the S-poset map f is a split epimorphism. Consequently B_S being a retract of an Emb-injective S-poset is an Emb-injective S-poset. □

At the rest of this section, we investigate some connections between Emb-injectivity in **Pos**-S/B_S and generators and cyclic projectives in **Pos**-S.

Theorem 3.10. *If $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S *and B_S is a generator in* **Pos**-S *then A_S is a generator. Further, ${}_{\mathrm{End}(A_S)}A$ is a cyclic projective in* $\mathrm{End}(A_S)$-**Pos**.

Proof. Since $f : A_S \to B_S$ is Emb-injective object in **Pos**-S/B_S, by Proposition 3.7, there exists $g : B_S \to A_S$ in **Pos**-S such that $fg = \mathrm{id}_B$. As B_S is a generator in **Pos**-S and f is an epimorphism, A_S is also a generator (see [14]). Now, applying this fact and [14, Theorem 2.2], we get that ${}_{\mathrm{End}(A_S)}A$ is a cyclic projective. □

Theorem 3.11. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S *where A_S is a cyclic projective S-poset. Then B_S is a cyclic projective S-poset. Moreover, ${}_{\mathrm{End}(B_S)}B$ is a generator in* $\mathrm{End}(B_S)$-**Pos**.

Proof. Since $f : A_S \to B_S$ is Emb-injective object in **Pos**-S/B_S, by Proposition 3.7, there exists $g : B_S \to A_S$ in **Pos**-S such that $fg = \mathrm{id}_B$. Also, A_S is a cyclic projective in **Pos**-S hence by Proposition 2.1, there exist two S-poset maps $S_S \underset{\gamma}{\overset{\pi}{\rightleftarrows}} A_S$ such that $\pi\gamma = \mathrm{id}_A$. This yields $f\pi\gamma g = \mathrm{id}_B$ which shows that B_S is a retract of S_S. We get B_S is a cyclic projective S-poset by Proposition 2.1, so by [14, Proposition 3.1], we conclude that ${}_{\mathrm{End}(B_S)}B$ is a generator in $\mathrm{End}(B_S)$-**Pos**. □

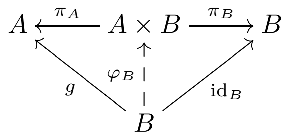
Theorem 3.12. *Suppose $f : A_S \to B_S$ is an Emb-injective object in* **Pos**-S/B_S. *Then all of the following hold.*
(i) $\mathbf{Pos}_S(B_S, A_S)$ *is a generator in* **Pos**-$\mathrm{End}(B_S)$.
(ii) $\mathbf{Pos}_S(A_S, B_S)$ *is a generator in* $\mathrm{End}(B_S)$-**Pos**.
(iii) $\mathbf{Pos}_S(B_S, A_S)$ *is a cyclic projective in* $\mathrm{End}(A_S)$-**Pos**.
(iv) $\mathbf{Pos}_S(A_S, B_S)$ *is a cyclic projective in* **Pos**-$\mathrm{End}(A_S)$.

Proof. Since $f : A_S \to B_S$ is Emb-injective object in **Pos**-S/B_S, in view of Proposition 3.7, there exists $g : B_S \to A_S$ such that $fg = \mathrm{id}_B$. Applying the functors $\mathbf{Pos}_S(B_S, -)$ and $\mathbf{Pos}_S(-, B_S)$ to the identity map id_{B_S} we can easily get the assertions (i) and (ii), respectively. Again by applying the functors $\mathbf{Pos}_S(-, A_S)$ and

$\mathbf{Pos}_S(A_S, -)$ to the above identity, in light of Proposition 2.1, we can deduce that the statements (iii) and (iv) are true. □

Proposition 3.13. *Let A_S be an S-poset. Then in any of the following cases $\mathbf{Pos}_S(A_S \times B_S, B_S)$ is a generator in* $\mathrm{End}(B_S)$-$\mathbf{Pos}$*, for every $B_S \in$*$\mathbf{Pos}$-S*:*
(i) *A_S is an Emb-injective S-poset.*
(ii) *$f : A_S \to B_S$ is an Emb-injective object in* $\mathbf{Pos}$-S/B_S.

Proof. (i) Consider the second projection S-poset map $\pi_B : A_S \times B_S \to B_S$. The authors in [6] have showed that it is an Emb-injective object in $\mathbf{Pos}$-S/B_S. Consequently, by Theorem 3.12(ii), we get the result.
(ii) By Proposition 3.7, there exists an S-poset map $g : B_S \to A_S$ such that $fg = \mathrm{id}_B$. By the universal property of the product S-poset $A \times B$ there exists a unique S-poset map $\varphi_B : B_S \to A \times B$ (indeed $b \mapsto (g(b), b)$) such that the following diagram commutes:



i.e., $\pi_B\varphi_B = \mathrm{id}_B$ and $\pi_A\varphi_B = g$. Applying the functor $\mathbf{Pos}_S(-, B_S)$ to the first identity above we obtain

$$\mathrm{End}(B_S) = \mathbf{Pos}_S(B, B) \underset{\bar{\varphi}_B}{\overset{\bar{\pi}_B}{\rightleftarrows}} \mathbf{Pos}_S(A \times B, B)$$

such that $\bar{\varphi}_B\bar{\pi}_B = \mathrm{id}_{\mathrm{End}(B_S)}$. This means that $\mathrm{End}(B_S)$ is a retract of $\mathbf{Pos}_S(A \times B, B)$ as we needed (see Theorem 2.6 again). □

Proposition 3.13. *Suppose that B_S is in* $\mathbf{Pos}$-S*, ${}_TA_S$ is a T-S-biposet, and $A \times B$ is a cyclic projective S-poset. If $f : A_S \to B_S$ is an Emb-injective object in* $\mathbf{Pos}$-S/B_S *and $\lambda : T \to \mathrm{End}(A_S)$, defined as in (1.1), is an isomorphism then ${}_TA$ is a generator in T-*$\mathbf{Pos}$.

Proof. Consider the second projection S-poset map $\pi_A : A \times B \to A_S$ and the unique S-poset map $\varphi_A : A_S \to A \times B$ for which $\pi_A\varphi_A = \mathrm{id}_A$. That is, let $\varphi_A(a) = (a, f(a))$. Since $A \times B$ is a cyclic projective S-poset by assumption, there exist S-poset maps $A \times B \underset{\pi}{\overset{\gamma}{\rightleftarrows}} S_S$ such that $\pi\gamma = \mathrm{id}_{A\times B}$. Applying the functor $\mathbf{Pos}_S(-, A_S)$ to the former identity and knowing that the composition $\pi_A\pi\gamma\varphi_A = \mathrm{id}_A$, we obtain

$$T \cong \mathbf{Pos}_S(A, A) \underset{\bar{\varphi}_A}{\overset{\bar{\pi}_A}{\rightleftarrows}} \mathbf{Pos}_S(A \times B, A) \underset{\bar{\gamma}}{\overset{\bar{\pi}}{\rightleftarrows}} \mathbf{Pos}_S(S, A) \cong_T A$$

in which $\bar{\varphi}_A\bar{\pi}_A = \mathrm{id}_{\mathbf{Pos}_S(A,A)}$ and $\bar{\gamma}\bar{\pi} = \mathrm{id}_{\mathbf{Pos}_S(S,A)}$. Thus, T is a retract of ${}_TA$ and hence ${}_TA$ is a generator in $\mathbf{Pos}$-S. □

Acknowledgements

The authors would like to express their sincere thanks to the anonymous referee for a careful reading of the manuscript and for invaluable comments which improved the exposition of the article. Parts of this research were completed while the second author was on sabbatical leave at the Department of Mathematics, Vanderbilt University (VU), Nashvile, TN, USA. This author expresses his thanks for the warm hospitality and facilities provided by Prof. Constantine Tsinakis and the Department of Mathematics of VU. He is greatly indebted to Semnan University for its financial support during the sabbatical.

References

[1] A. Bailey and J. Renshaw, *Weak factorization system for S-acts*, Semigroup Forum, **87(3)**(2013), 443–461.

[2] G. Birkhoff, *Lattice Theory*, Bull. Amer. Math. Soc., Providence(1973).

[3] S. Bulman-Fleming and M. Mahmoudi, *The category of S-posets*, Semigroup Forum, **71(3)**(2005), 443–461.

[4] M. M. Ebrahimi, M. Mahmoudi and H. Rasouli, *Banaschewski's theorem for S-posets: regular injectivity and completeness*, Semigroup Forum, **80(2)**(2010), 313–324.

[5] S. M. Fakhruddin, *On the category of S-posets* Acta Sci. Math. (Szeged), **52**(1988), 85–92.

[6] F. Farsad and A. Madanshekaf, *Regular injectivity and exponentiability in the slice categories of actions of pomonoids on posets*, J. Korean Math. Soc., **52(1)**(2015), 67–80.

[7] F. Farsad and A. Madanshekaf, *Weak factorization systems and fibrewise regular injectivity for actions of po-monoids on posets*, Algebra Discrete Math., **24(2)**(2017), 235–249.

[8] S. Irannezhad and A. Madanshekaf, *Projective S-posets and Hom functor*, Algebra Colloq., **22(Spec 1)**(2015), 975–984.

[9] S. Irannezhad and A. Madanshekaf, *On classification of pomonoids by properties of generators*, Asian-Eur. J. Math., **10(3)**(2017), 16 pp.

[10] M. Jafari, A. Golchin and H. Mohammadzadeh Saany, *On characterization of monoids by properties of generators II*, Ital. J. Pure Appl. Math., **40**(2018), 256–276.

[11] M. Jafari, A. Golchin and H. Mohammadzadeh Saany, *On characterization of monoids by properties of generators*, J. Math. Res. Appl., **40(4)**(2020), 367-380.

[12] M. Kilp and U. Knauer, *Characterization of monoids by properties of generators*, Comm. Algebra, **20(7)**(1992), 1841–1856.

[13] M. Kilp, U. Knauer and A. V. Mikhalev, *Monoids, Acts and Categories*, de Gruyter, Berlin(2000).

[14] V. Laan, *Generators in the category of S-posets*, Cent. Eur. Math. **6(3)**(2008), 357–363.

[15] C. McLarty, *Elementary Categories, Elementary Toposes*, Oxford University Press, Oxford(1991).

[16] A. Madanshekaf and J. Tavakoli, *Tiny objects in the category of M-***Sets**, Ital. J. Pure Appl. Math., **10**(2001), 153–162.

[17] M. Sedaghatjoo, *On monoids over which all generators satisfy a flatness property*, Semigroup Forum, **87(3)**(2013), 653–662.

[18] L. Shahbaz and M. Mahmoudi, *Injectivity of S-posets with respect to down closed regular monomorphisms*, Semigroup Forum, **91(3)**(2015), 584–600.

[19] X. Shi, Z. Liu F. Wang and S. Bulman-Fleming, *Indecomposable, projective and flat S-posets*, Comm. Algebra, **33(1)**(2005), 235–251.