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On the Paneitz-Branson Operator in Manifolds with Negative Yamabe Constant

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Abstract. This paper deals with the Paneitz-Branson operator in compact Riemannian manifolds with negative Yamabe invariant. We start off by providing a new criterion for the positivity of the Paneitz-Branson operator when the Yamabe invariant of the manifold is negative. Another result stated in this paper is about the existence of a metric on a manifold of dimension 5 such that the Paneitz-Branson operator has multiple negative eigenvalues. Finally, we provide new inequalities related to the upper bound of the mean value of the Q-curvature.

1. Introduction

Let (M, g) be a smooth 4-dimensional Riemannian manifold. The Paneitz operator discovered in [18] is the fourth order operator defined for all smooth functions u by

$$P_g^4(u) = \Delta_g^2(u) - div_g\left(\frac{2}{3}Sc_g.g - 2Ric_g\right)du$$

where $\Delta_g(u) = -div_g(\nabla u)$ is the Laplacian of u with respect to the metric g, and Sc_g and Ric_g denote the scalar and the Ricci curvatures of g respectively (we will use this notation throughout the paper).

This operator is conformally covariant in the following sense: if $\tilde{g}$ is $e^{2\varphi}g$ where φ is a smooth function, then the following holds

$$P_{\tilde{g}}^4(u) = e^{-4\varphi}P_g^4(u), \quad \forall u \in C^\infty(M). \tag{1.1}$$

Moreover, this operator is intimately related to a conformally invariant Q_g^4- *curvature*

$$Q_g^4 := \frac{1}{6}\left(\Delta_g Sc_g - Sc_g^2 - 3|Ric_g|^2\right).$$

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Originally the Paneitz operator was introduced for physical motivations and has many applications in mathematical physics. This operator was generalized to manifolds of greater dimension ($n \geq 5$) by Branson.

Given a smooth compact Riemannian manifold (M, g) of dimension $n \geq 5$, the Paneitz-Branson operator is defined as

$$P_g^n(u) = \Delta^2(u) - div_g\left(a_n Sc_g.g + b_n Ric_g\right) du + \frac{n-4}{2} Q_g u \tag{1.2}$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}\ , \quad b_n = \frac{-4}{n-2}, \quad \text{and} \tag{1.3}$$

$$Q_g = \frac{1}{2(n-1)}\Delta Sc_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} Sc_g^2 - \frac{2}{(n-2)^2}|Ric_g|_g^2. \tag{1.4}$$

The Paneitz-Branson operator is also conformally covariant in this sense: if $\tilde{g} = \varphi^{\frac{4}{n-4}} g$ is a metric conformal to the metric g where φ is a smooth positive function, then

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u), \quad \forall u \in C^\infty(M). \tag{1.5}$$

Of course, as an object in conformal geometry, a lot of research has been devoted to this operator; see [9], [4], [14], [8], and [21], and the references therein.

Our aim in this paper is to investigate the influence of the geometry on the sign of the eigenvalues of this operator. As a first result we give conditions sufficient to ensure the positivity of the Paneitz-Branson operator even when the scalar curvature is negative. In particular we prove the following.

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ with negative scalar curvature Sc_g not necessarily constant. If the following three conditions hold*

$$-4\frac{(n-1)(n-2)}{n^3-2n^2-2n+8} < Sc_g \tag{1.6}$$

$$\frac{1}{n} Sc_g.g \leq Ric_g \leq \frac{1}{2n} Sc_g.g \tag{1.7}$$

$$Q_g \geq \frac{2}{n(n-4)} \tag{1.8}$$

then the Paneitz-Branson operator P_g^n is positive.

The proof of this theorem is based on a new inequality which we give in the next section. Observe that in this theorem we allow the Yamabe constant of M to be negative (for the definition of the Yamabe constant one can see Section 4). Now, we give an outline of the rest of the paper. In Section 2 we prove Theorem 1.1 and give another result for the positivity of the operator P_g^n when the scalar curvature is positive. In Section 3 we investigate the negativity of the eigenvalues of P_g^n in an Einstein manifolds, in particular we prove the following.

Theorem 1.2. *Given (M, g) an Einstein manifold of dimension $n \geq 5$ with negative scalar curvature. If*

$$\frac{(n^2-2n-8)}{4n(n-1)}|Sc_g| < \lambda_1 < \frac{(n^2-2n)}{4n(n-1)}|Sc_g| \tag{1.9}$$

where λ_1 is the first eigenvalue of the Laplacian operator Δ_g, then the first eigenvalue μ_1 of the Paneitz-Branson operator is strictly negative.

Moreover, we also prove in this section, that it is possible to choose a metric g on the manifold M such that P_g^n has multiple negative eigenvalues. Finally, in the last section we give several inequalities related to the mean value of the Q-curvature, as an example we prove the following result

Theorem 1.3. *Let M be a compact Riemannian manifold endowed with a Yamabe metric g. If the scalar curvature $Sc_g \geq n(n-1)$, then*

$$\int_M Q_g dv_g \leq \left(\frac{\lambda_1}{n}\right)^2 \int_{\mathbb{S}^n} Q_h dv_h \tag{1.10}$$

where λ_1 is the first eigenvalue of the Laplacian operator Δ_g and $(\mathbb{S}^n, h)$ denotes the round sphere endowed with its canonical metric.

2. Positivity of the Paneitz-Branson operator

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, we say that the Paneitz-Branson operator P_g^n is positive [8] if

$$\int_M u P_g^n(u) dv_g \geq 0, \quad \forall u \in H_2^2(M).$$

The conditions under which the operator P_g^n is positive have been intensively studied. For example it was considered by Gursky [11] for dimension $n = 4$, Yang and Xu [22] for dimension $n \geq 5$, Hebey and Robert [14], and recently by Gursky and Malchiodi [12].

Yang and Xu in [22] showed that when the dimension of (M, g) is $n \geq 6$, if the Yamabe invariant of g is non negative and the Q-curvature is positive, then with respect to any conformal metric of positive scalar curvature, the Paneitz-Branson operator is positive.

In this section, we are concerned with finding sufficient conditions on the curvature of g, to ensure the positivity of the Paneitz-Branson operator P_g^n.

As a first result we have

Theorem 2.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$ with positive non constant scalar curvature and positive Q-curvature. If*

$$\frac{1}{n} Sc_g.g \leq Ric_g \leq \frac{n-2}{2n} Sc_g.g, \tag{2.1}$$

then the Paneitz-Branson operator is positive.

Proof. To prove this theorem, we use an idea from [22].

First, we multiply both sides of (1.2) by u, and integrate by parts,

$$\begin{aligned}\int_M u P_g^n(u) dv_g = &\int_M (\Delta_g u)^2 dv_g + a_n \int_M Sc_g |\nabla_g u|^2 dv_g \\ &- \frac{4}{n-2}\int_M Ric_g(\nabla_g u, \nabla_g u) dv_g + \frac{n-4}{2}\int_M Q_g u^2 dv_g\end{aligned} \tag{2.2}$$

where a_n as in (1.3). An application of the Bochner formula together with condition (2.1) gives

$$\begin{aligned}\int_M u P_g^n(u) dv_g \geq &\int_M |\nabla_g^2 u|^2 dv_g + \left(a_n - \frac{1}{n}\right)\int_M Sc_g |\nabla_g u|^2 dv_g \\ &+ \frac{n-4}{2}\int_M Q_g u^2 dv_g.\end{aligned} \tag{2.3}$$

which implies the positivity of P_g^n. □

Now, to prove Theorem 2.1, we need the following lemma.

Lemma 2.2. $\forall u \in H_2^2(M)$ *we have*

$$\frac{2}{n}\int_M |\nabla_g u|^2 dv_g \leq \frac{1}{n}\int_M u^2 dv_g + \int_M |\nabla_g^2 u|^2 dv_g \tag{2.4}$$

Proof. Let u be a smooth function, and consider the following tensor with local coordinates

$$T_{ij} := \left(\nabla_g^2 u\right)_{ij} + \frac{1}{n} u(x) g_{ij}.$$

The tensorial norm of T with respect to the metric g is

$$|T|_g^2 = |\nabla_g^2 u|^2 - \frac{2}{n} u \Delta_g(u) + \frac{1}{n} u^2.$$

Since $|T|_g^2 \geq 0$, it follows that

$$\frac{2}{n} u \Delta_g(u) \leq |\nabla_g^2 u|^2 + \frac{1}{n} u^2.$$

Therefore, an integration by parts give us

$$\frac{2}{n}\int_M |\nabla_g u|^2 dv_g \le \frac{1}{n}\int_M u^2 dv_g + \int_M |\nabla_g^2 u|^2 dv_g, \forall u \in C^\infty(M). \tag{2.5}$$

Now, by the density of $C^\infty(M)$ in $H_2^2(M)$ with respect to the norm

$$\|u\|^2 = \int_M |\nabla_g^2 u|^2 dv_g + \int_M |\nabla_g u|^2 dv_g + \int_M u^2 dv_g,$$

one can assume that $u \in H_2^2(M)$ (instead of $C^\infty(M)$) and inequality (2.5) remains valid.

□

We are now in position to prove Theorem 1.1.

Proof. We begin with (2.2)

$$\begin{aligned}\int_M uP_g^n(u)dv_g = & \int_M (\Delta_g u)^2 dv_g + a_n \int_M Sc_g |\nabla_g u|^2 dv_g \\ & - \frac{4}{n-2}\int_M Ric_g(\nabla_g u, \nabla_g u) dv_g + \frac{n-4}{2}\int_M Q_g u^2 dv_g\end{aligned} \tag{2.6}$$

An application of the Bochner formula, together with condition (1.7) gives

$$\begin{aligned}\int_M uP_g^n(u)dv_g \ge & \int_M |\nabla_g^2 u|^2 dv_g + \left(a_n + \frac{1}{n} - \frac{2}{n(n-2)}\right)\int_M Sc_g|\nabla_g u|^2 dv_g \\ & + \frac{n-4}{2}\int_M Q_g u^2 dv_g.\end{aligned} \tag{2.7}$$

Now, we apply inequality (2.4) with condition (1.8), to obtain

$$\int_M uP_g^n(u)dv_g \ge \int_M \left(\frac{2}{n} + \frac{n^3-2n^2-2n+8}{2n(n-1)(n-2)}Sc_g\right)|\nabla_g u|^2 dv_g$$

which implies by assumption (1.6), that P_g^n is positive. □

3. Multiple Negative Eigenvalues for the Paneitz-Branson Operator

Our first observation is that the negativity of the quantity

$$\kappa_g := \int_M Q_g dv_g \tag{3.1}$$

is a sufficient condition for the negativity of the first eigenvalue μ_1 of the operator P_g^n.

Indeed by the variational definition of μ_1, we have

$$\mu_1 = \inf_{u \in H_2^2(M)\setminus\{0\}} \frac{\int_M u P_g^n(u) dv_g}{\int_M u^2 dv_g}, \tag{3.2}$$

where $H_2^2(M)$ is the Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|^2 = \int_M |\nabla_g^2 u|^2 dv_g + \int_M |\nabla_g u|^2 dv_g + \int_M u^2 dv_g.$$

It follows, in the particular case of $u \equiv 1$, that

$$\mu_1 \leq \frac{n-4}{2V_g(M)} \int_M Q_g dv_g.$$

So, if

$$\int_M Q_g dv_g < 0 \tag{3.3}$$

then $\mu_1 < 0$. Thus it is clear that though the scalar curvature of the manifold is positive; the Paneitz-Branson operator can have negative eigenvalues. The following is a typical example.

Example 3.1. Let $(\mathbb{S}^4, h)$ be the standard round sphere of dimension 4 and (Σ^3, g_0) a hyperbolic manifold. We equip the product manifold

$$M^7 := \mathbb{S}^4 \times \Sigma^3$$

with the product metric

$$g := h \otimes 1 + 1 \otimes g_0.$$

Thus (M^7, g) is a conformally flat manifold, since it is the product of two manifolds with sectional curvatures of opposite sign $K_p(\mathbb{S}^4) = +1$, $K_p(\Sigma^3) = -1$. Moreover, the Ricci curvatures of $\mathbb{S}^4$ and Σ^3 are respectively

$$Ric_h = 3h, \quad \text{and } Ric_{g_0} = -2g_0;$$

and consequently, the scalar curvatures are

$$Sc_h = 12, \quad \text{and } Sc_{g_0} = -6.$$

Finally, the scalar curvature of M^7 is $Sc_g = 6$ and $|Ric_g|_g^2 = 48$.

Thus, as a conclusion, the manifold M^7 is conformally flat manifold with positive scalar curvature ($Sc_g = 6$) but according to formula (1.4) with negative $Q-$curvature ($Q_g = -\frac{21}{8}$), which implies that the first eigenvalue μ_1 of the Paneitz-Branson operator is strictly negative.

However there is an important class of manifolds for which condition (3.3) is not satisfied, for example the Einstein manifolds (note that the previous example is not an Einstein manifold). Indeed, if the scalar curvature Sc_g of an Einstein manifold is a non zero constant, then the Q-curvature is strictly positive constant and satisfied

$$Q_g = \frac{n^2-4}{8n(n-1)^2} Sc_g^2. \tag{3.4}$$

But still in this situation, the Paneitz-Branson operator has negative eigenvalues; Theorem 1.2 gives us an example. In what follow we prove this theorem.

Proof. Let (M, g) be an Einstein manifold of dimension $n \geq 5$ with negative scalar curvature, and $u \in H_2^2(M)$.

$$\begin{aligned} \int_M u P_g^n(u) dv_g &= \int_M (\Delta_g u)^2 dv_g + \frac{n^2-2n-4}{2n(n-1)} Sc_g \int_M |\nabla_g u|^2 dv_g \\ &+ \frac{(n-4)(n^2-4)}{16n(n-1)^2} Sc_g^2 \int_M u^2 dv_g. \end{aligned} \tag{3.5}$$

In the particular case when $u \equiv \varphi_1$, where φ_1 is an eigenfunction corresponding to the first eigenvalue λ_1 of Δ_g; (3.5) becomes

$$\int_M \varphi_1 P_g^n(\varphi_1) dv_g = \left(\lambda_1^2 + \frac{n^2-2n-4}{2n(n-1)} Sc_g \lambda_1 + \frac{(n-4)(n^2-4)}{16n(n-1)^2} Sc_g^2 \right) \int_M \varphi_1^2 dv_g.$$

So, if λ_1 is satisfied (1.9), then it is obvious that the quantity

$$\lambda_1^2 + \frac{n^2-2n-4}{2n(n-1)} Sc_g \lambda_1 + \frac{(n-4)(n^2-4)}{16n(n-1)^2} Sc_g^2$$

is strictly negative, which implies immediately the negativity of the first eigenvalue μ_1 of the operator P_g^n by the variational definition (3.2) of μ_1. □

It turns out that Riemannian manifolds with negative scalar curvature are the most favoured example for the Paneitz-Branson operator to have negative eigenvalues. In fact, in [7] Canzani et al provided several examples of manifolds of negative curvature for which the Yamabe operator

$$L_g(u) := \Delta_g(u) + \frac{n-2}{4n(n-1)} Sc_g u \tag{3.6}$$

has multiple negative eigenvalues. In particular, they proved the following

Theorem 3.2. *Let (M, g) be a compact connected Riemannian manifold. Then, for every m there is a metric g on M such that the Yamabe operator L_g has at least m negative eigenvalues counted with multiplicity.*

Below, we extend this result to the Paneitz-Branson operator on compact connected

Einstein manifolds of dimension $n = 5$.

Theorem 3.3. *Let (M, g) be a compact connected Einstein manifold of dimension 5. For every $m \in \mathbb{N}^\star$ there is a metric g_0 on M such that the Paneitz-Branson operator $P^n_{g_0}$ has at least m negative eigenvalues counted with multiplicity.*

The proof of this result is based on an idea of Canzani et al [7].

Proof. Let (M, g) be a compact connected Einstein manifolds of dimension $n = 5$. By a result of Lohkamp [16], for any $\lambda \in \mathbb{R}$, there exist a metric g_0 such that λ is the first eigenvalue of Δ_{g_0} of multiplicity m, the volume $V_{g_0}(M)$ of M with respect to g_0 satisfies $V_{g_0}(M) = 1$, and $Ric_{g_0} \le -m^2 g$.

Since (M, g) is an Einstein manifold, so we may assume

$$Ric_{g_0} = -m^2 g, \quad \text{hence } Sc_{g_0} = -5m^2.$$

Thus, $\forall u \in H^2_2(M)$, the Paneitz-Branson operator $P^n_{g_0}$ take the form

$$P^n_{g_0}(u) := \Delta^2_{g_0}(u) - div_{g_0}\left[\left(\frac{13}{24}(-5m^2)g_0 - \frac{4}{3}(-m^2)g_0\right) du\right] + \frac{1}{2}Q_{g_0}u$$

which implies

$$\int_M uP^n_{g_0}(u)dv_{g_0} = \int_M (\Delta_{g_0}u)^2\, dv_{g_0} - \frac{11}{8}m^2 \int_M |\nabla_{g_0}u|^2 dv_{g_0} + \frac{105}{256}m^4 \int_M u^2 dv_{g_0}. \tag{3.7}$$

In particular, for $u \equiv \varphi_1$ the eigenfunction associated to the first eigenvalue λ of Δ_{g_0} with multiplicity m, (3.7) yields

$$\int_M \varphi_1 P^n_{g_0}(\varphi_1)dv_{g_0} = \left(\lambda^2 - \frac{11}{8}m^2\lambda + \frac{105}{256}m^4\right)\int_M \varphi_1^2 dv_{g_0}. \tag{3.8}$$

So, if we set $\lambda = \frac{1}{2}m^2$, then it follows from (3.8), that $\mu_1 = -\frac{7}{256}m^4$ is a negative eigenvalue for the Paneitz-Branson operator with multiplicity m. □

An other example of manifolds such that the Paneitz-Branson operator admits multiple negative eigenvalues is also presented by Canzani et al in [7]. Before stating this result, we need to introduce the notion of the GJMS operator P^k_g.

Let (M, g) be a compact Riemannian manifold of dimension $n \ge 3$. Let k be a positive integer such that $n > 2k$. In [10], Graham-Jenne-Mason-Sparling defined a differential operator denoted by P^k_g. From the conformal geometric point of view, this operator can be considered as a generalization of both the Yamabe operator given in (3.6) and the Paneitz-Branson operator given in (1.2). More precisely, this operator is conformally covariant in the sense that if $\tilde{g} := \varphi^{\frac{4}{n-2k}} g$ where $\varphi \in C^\infty(M), \varphi > 0$, then

$$P^k_g(\varphi u) = \varphi^{\frac{n+2k}{n-2k}}.P^k_{\tilde{g}}(u), \quad \forall u \in C^\infty(M).$$

Moreover, P_g^k is self-adjoint with respect to the L^2-scalar product. This operator is intimately related to the geometric quantity of Q-curvature, denoted as Q_g^k and satisfying

$$Q_g^k = \frac{2}{n-2k} P_g^k(1).$$

A lot of work has been devoted to the study of the GJMS operator, see, for example, [17],[3],[19],[15], and [5], and the references therein.
Having defined the GJMS operator, we can now state the result of Canzani et al.

First, Let (M, g) be a compact hyperbolic product manifold

$$M = N \times \Sigma$$

of dimension n, equipped with a product metric

$$g := g_1 \otimes 1 + 1 \otimes g_2,$$

where (N, g_1) is an hyperbolic manifold of dimension $n-2$, and (Σ, g_2) is an hyperbolic surface with genus $s \geq 2$. Notice that (M, g) is an Einstein manifold with $Ric_g = -g$.

Theorem 3.4. *For every $m \in \mathbb{N}$, we can choose the hyperbolic metric g_2 on Σ so that the GJMS operator P_g^k has at least m negative eigenvalues for all odd integers $k \leq \frac{n-1}{2}$.*

If we further assume that $n = 4l$ or $n = 4l+1$ for some $l \in \mathbb{N}$, then the same conclusion holds for all integers $k \geq \frac{n}{2}$.

It seems that this result does not cover the case of the Paneitz-Branson operator in manifolds of five dimension.

Indeed, if $n = 5$, then either k is an odd integer less then 2, i.e, $k = 1$ and in this case we have only the Yamabe operator P_g^1, or $l = 1$, i.e., $n = 4l+1$, and in this case $k \geq \frac{5}{2}$. This means all GJMS operators are of order greater than 3, and as is obvious, the Paneitz-Branson operator does not appears in this sequence of operators. So, in this section and by the same technique, we investigate the case of the Paneitz-Branson operator on manifolds of dimension 5, and we prove the following.

Theorem 3.5. *Let (M, g) be a compact hyperbolic product manifold*

$$M = N \times \Sigma$$

of dimension 5, equipped with a product metric

$$g := g_1 \otimes 1 + 1 \otimes g_2,$$

where (N, g_1) is an hyperbolic manifold of dimension 3, and (Σ, g_2) is an hyperbolic surface with genus $s \geq 2$. For any $m \in \mathbb{N}$, there exists a metric g_2 on Σ, such that the Paneitz-Branson operator has at least m negative eigenvalues.

It should be mentioned that the basic idea underlying the presented proof was borrowed from [7].

Proof. Let (N, g_1) be a hyperbolic manifold of dimension 3, and (Σ, g_2) be a hyperbolic surface with genus $s \geq 2$. Consider the product manifold $M = N \times \Sigma$ equipped with the product metric $g := g_1 \otimes 1 + 1 \otimes g_2$.

After scaling by a positive constant we may assume that the scalar curvature Sc_g of the manifold M is

$$Sc_g = -2, \quad \text{and } Ric_g = -\frac{2}{5}g.$$

In other words, (M, g) is an Einstein manifold; therefore, the operator P_g^n has the following formula

$$P_g^n(u) = \Delta_g^2(u) - \frac{11}{20} Sc_g \Delta_g(u) + \frac{21}{320}u. \tag{3.9}$$

Let λ be an eigenvalue of Δ_{g_2} and let φ be an eigenfunction associated to λ. If we regard φ as a function on M, then

$$\Delta_g(\varphi) = \Delta_{g_2}(\varphi) = \lambda\varphi.$$

Combining this with (3.9), we then see that φ is an eigenfunction of P_g^n with eigenvalue

$$\mu(\lambda) = \lambda^2 - \frac{11}{20}\lambda + \frac{21}{320}. \tag{3.10}$$

Now, let $m \in \mathbb{N}$; Since $\frac{1}{4} \in \left]\frac{7}{40}, \frac{3}{8}\right[$ and Σ has genus $s \geq 2$, so it follows by a result of Buser [6] that one can find an hyperbolic metric g_2 on Σ, such that the Laplacian Δ_{g_2} has at least m eigenvalues which belong to $\left]\frac{7}{40}, \frac{3}{8}\right[$.

Therefore, we infer by (3.10) that, for each eigenvalue λ of Δ_{g_2} in the interval $\left]\frac{7}{40}, \frac{3}{8}\right[$; there exists a negative eigenvalue $\mu(\lambda)$ for the Paneitz-Branson operator P_g^n. As a conclusion, the operator P_g^n has m negative eigenvalues. This completes the proof of Theorem 3.5. □

We finish this section with a classification result which comes as a consequence of the negativity of the first eigenvalue of the Paneitz-Branson operator.

Proposition 3.6. *Let (M, g) be a compact Riemannian manifold of dimension 5 with non negative Q-curvature, and negative scalar curvature. Assume that*

$$Ric_g \leq \frac{13}{2} Sc_g g. \tag{3.11}$$

If the first eigenvalue μ_1 of the Paneitz-Branson operator is non positive, then (M, g) is conformally diffeomorphic to the round sphere $(\mathbb{S}^5, h)$.

To prove this result we need the following lemma of Tashiro [20]. For the following version one can see [13] page 291.

Lemma 3.7. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 2$. If there exist a non constant function $f \in C^\infty(M)$; such that*

$$\nabla^2 f + \frac{1}{n}\left(\Delta_g f\right) g = 0, \tag{3.12}$$

then (M,g) is conformally diffeomorphic to the round sphere $(\mathbb{S}^n, h)$.

Proof. Let (M,g) be a compact Riemannian manifold of dimension 5; and let $u \in H_2^2(M)$. By formula (2.2) and since $Q_g \geq 0$, we have

(3.13)
$$\int_M u P_g^n(u) dv_g \geq \int_M (\Delta_g u)^2 dv_g + \frac{13}{24}\int_M Sc_g |\nabla_g u|^2 dv_g - \frac{4}{3}\int_M Ric_g(\nabla_g u, \nabla_g u) dv_g.$$

using the formula

$$|\nabla^2 u|_g^2 = \left|\nabla^2 u + \frac{1}{5}\left(\Delta_g u\right) g\right|_g^2 + \frac{1}{5}\left(\Delta_g u\right)^2$$

with the Bochner formula

$$\int_M \left(\Delta_g u\right)^2 dv_g = \int_M |\nabla^2 u|^2 dv_g + \int_M Ric_g(\nabla_g u, \nabla_g u) dv_g$$

we obtain

$$\begin{aligned} \int_M u P_g^n(u) dv_g \quad &\geq \quad \frac{5}{4}\int_M \left|\nabla^2 u + \frac{1}{5}\left(\Delta_g u\right) g\right|_g^2 - \frac{1}{12}\int_M Ric_g(\nabla_g u, \nabla_g u) dv_g \\ &+ \quad \frac{13}{24}\int_M Sc_g |\nabla_g u|^2 dv_g. \end{aligned} \tag{3.14}$$

So, if (3.11) holds, then

$$\int_M u P_g^n(u) dv_g \geq \frac{5}{4}\int_M \left|\nabla^2 u + \frac{1}{5}\left(\Delta_g u\right) g\right|_g^2, \quad \forall u \in H_2^2(M). \tag{3.15}$$

Now, if the first eigenvalue μ_1, of the Paneitz-Branson operator P_g^n is non positive, then it follows from (3.15) in the particular case when $u \equiv \varphi$, where φ is the eigenfunction corresponding to the first eigenvalue μ_1 of P_g^n, that

$$\int_M \left|\nabla^2 \varphi + \frac{1}{5}\left(\Delta_g \varphi\right) g\right|_g^2 dv_g = 0$$

which implies that

$$\left|\nabla^2 \varphi + \frac{1}{5}\left(\Delta_g \varphi\right) g\right|_g^2 = 0$$

or, equivalently

$$\nabla^2\varphi + \frac{1}{5}\left(\Delta_g\varphi\right) g = 0$$

Thus, (M,g) is conformally diffoemorphic to the round sphere $(\mathbb{S}^5,h)$ by Lemma 3.7. $\square$

4. Upper Bound for the Mean Value of the Q-curvature

Our purpose in this section is to get an upper bound for the quantity κ_g, where

$$\kappa_g = \int_M Q_g dv_g$$

with

$$Q_g = \frac{1}{2(n-1)}\Delta Sc_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}Sc_g^2 - \frac{2}{(n-2)^2}|Ric_g|_g^2. \tag{4.1}$$

in terms of curvatures such as scalar curvature and Ricci curvature. A famous result in this subject is the result of Gursky in [11] for four-dimensional Riemannian manifolds.

Theorem 4.1. *Let (M^4,g) be a smooth compact four-dimensional Riemannian manifold. If the $Y_g(M)\geq 0$ then $\kappa_g \leq 8\pi^2$. Moreover, $Y_g(M)\geq 0$ and $\kappa_g = 8\pi^2$ if and only if (M^4,g) is conformally equivalent to the round sphere.*

There is no analogue to this theorem for manifolds of greater dimension ($n\geq 5$) or for manifolds with negative Yamabe constant. Almost all that is known in this direction is Theorem 1.3 in the case of manifolds with Yamabe metric.

Now, we begin by the following lemma which is useful for the sequel.

Lemma 4.2. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 5$ with negative scalar curvature, not necessarily constant. Its Yamabe constant $Y_g(M)$ satisfies*

$$Y_g(M) \geq -\frac{n-2}{4(n-1)}\|Sc_g\|_{\frac{n}{2}}. \tag{4.2}$$

It is worth noticing that the previous inequality give us a lower bound for the Yamabe constant, which is negative since the scalar curvature is negative. From now on, we use the following notations.

The quantity

$$\|u\|_p = \left[\int_M |u|^p dv_g\right]^{\frac{1}{p}}.$$

denotes the norm of a function u in the Lebesgue space $L^p(M)$, and $2^\star = \frac{2n}{n-2}$ the critical Sobolev exponent. Now we prove the lemma.
Proof. First let us recall the definition of the Yamabe constant:

$$Y_g(M) := \inf_{u \in H_2^2(M)\setminus\{0\}} \frac{\int_M |\nabla_g u|^2 dv_g + \frac{n-2}{4(n-1)} \int_M Sc_g u^2 dv_g}{\|u\|_{2^\star}^2}.$$

By the Hölder's inequality and since $Sc_g(x) < 0$, we have

$$\int_M |\nabla_g u|^2 dv_g + \frac{n-2}{4(n-1)} \int_M Sc_g u^2 dv_g \geq -\frac{n-2}{4(n-1)} \|Sc_g\|_{\frac{n}{2}} \|u\|_{2^\star}^2$$

which implies that the number $-\frac{n-2}{4(n-1)}\|Sc_g\|_{\frac{n}{2}}$ is a lower bound for

$$\frac{\int_M |\nabla_g u|^2 dv_g + \frac{n-2}{4(n-1)} \int_M Sc_g u^2 dv_g}{\|u\|_{2^\star}^2}.$$

Hence

$$-\frac{n-2}{4(n-1)} \|Sc_g\|_{\frac{n}{2}} \leq Y_g(M).$$

□

Now, we state our main result

Theorem 4.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$ with non constant negative scalar curvature ($Sc_g(x) < 0$, $\forall x \in M$). If the Q-curvature satisfies $Q_g \geq 0$, then*

$$\int_M Q_g dv_g \leq \alpha_n \frac{\|Sc_g\|_{\frac{n}{2}} \|Sc_g\|_{2^\star}^2}{|\max_{x \in M} Sc_g(x)|} \tag{4.3}$$

where

$$\alpha_n = \frac{n^2-4}{8n(n-1)^2}, \quad \text{and } 2^\star = \frac{2n}{n-2}. \tag{4.4}$$

Before proving this theorem, let us make the following remark.

If we integrate (4.1) over M, then it follows by the divergence theorem and the inequality $|Ric_g|_g^2 \geq \frac{1}{n} Sc_g^2$ that

$$\int_M Q_g dv_g \leq \frac{n^2-4}{8n(n-1)^2} \int_M Sc_g^2 dv_g. \tag{4.5}$$

So, it appears that (4.3) is an analogous inequality to (4.5), and with the same dimensional constant $\frac{n^2-4}{8n(n-1)^2}$.
Now, we prove Theorem 4.3.

Proof. Multiply (4.1) by Sc_g, we have, after integration by parts,

$$\int_M Sc_g Q_g dv_g = \frac{1}{2(n-1)}\int_M |\nabla_g Sc_g|^2 dv_g + a_n \int_M Sc_g^3 dv_g - \frac{2}{(n-2)^2}\int_M Sc_g |Ric_g|^2 dv_g$$

where $a_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}$. Since $|Ric_g|^2 \geq \frac{1}{n}Sc_g^2$, we have

$$\int_M Sc_g Q_g dv_g \geq \frac{1}{2(n-1)}\left[\int_M |\nabla_g Sc_g|^2 dv_g + \frac{n^2-4}{4n(n-1)}\int_M Sc_g^3 dv_g\right]. \tag{4.6}$$

Now, by the definition of the Yamabe constant $Y_g(M)$ we get

$$\int_M |\nabla_g Sc_g|^2 dv_g \geq \|Sc_g\|_{2^\star}^2 Y_g(M) - \frac{n-2}{4(n-1)}\int_M Sc_g^3 dv_g,$$

and combining this with (4.6) one gets

$$\int_M Sc_g Q_g dv_g \geq \frac{1}{2(n-1)}\left[\|Sc_g\|_{2^\star}^2 Y_g(M) + \frac{n-2}{2n(n-1)}\int_M Sc_g^3 dv_g\right]. \tag{4.7}$$

Or equivalently, since $Sc_g < 0$

$$\int_M Sc_g Q_g dv_g \geq \frac{1}{2(n-1)}\left[\|Sc_g\|_{2^\star}^2 Y_g(M) - \frac{n-2}{2n(n-1)}\int_M |Sc_g|^3 dv_g\right]. \tag{4.8}$$

Now using Lemma 4.3 together with the following Hölder's inequality

$$\int_M |Sc_g|^3 dv_g \leq \|Sc_g\|_{2^\star}^2 \|Sc_g\|_{\frac{n}{2}}$$

we deduce that

$$\int_M Sc_g Q_g dv_g \geq -\frac{n^2-4}{8n(n-1)^2}\|Sc_g\|_{2^\star}^2 \|Sc_g\|_{\frac{n}{2}}. \tag{4.9}$$

Therefore, from (4.9) and the fact $Sc_g < 0$, it follows that

$$-|\max_{x\in M} Sc_g(x)| \int_M Q_g dv_g \geq -\frac{n^2-4}{8n(n-1)^2}\|Sc_g\|_{2^\star}^2 \|Sc_g\|_{\frac{n}{2}}$$

which proves (4.3). □

Now, it remains for us to prove Theorem 1.3.

Proof. Let (M,g) be a compact Riemannian manifold endowed with a Yamabe metric. As is well known, the scalar curvature Sc_g of M is constant and satisfied by the Aubin estimate [2]

$$Sc_g \leq (n-1)\lambda_1 \tag{4.10}$$

where λ_1 is the first eigenvalue of the Laplacian operator Δ_g.

Now, An integration by parts of the formula (4.1) over M gives us

$$\int_M Q_g dv_g = a_n \int_M Sc_g^2 dv_g - \frac{2}{(n-2)^2} \int_M |Ric_g|^2 dv_g$$

where a_n as in (1.3). Since $|Ric_g|^2 \geq \frac{1}{n} Sc_g^2$, we have

$$\int_M Q_g dv_g \leq \frac{n^2-4}{8n(n-1)^2} \int_M Sc_g^2 dv_g, \tag{4.11}$$

and with (4.10) we infer

$$\int_M Q_g dv_g \leq \frac{n^2-4}{8n} \lambda_1^2 V_g(M) \tag{4.12}$$

where $V_g(M)$ stands for the volume of M with respect to g.

Recall now from [1] that the scalar curvature satisfied

$$\begin{aligned} Sc_g &= Y_g(M) \left(V_g(M)\right)^{-\frac{2}{n}} \\ &\leq n(n-1) \left(V_h(\mathbb{S}^n)\right)^{\frac{2}{n}} \left(V_g(M)\right)^{-\frac{2}{n}} \end{aligned} \tag{4.13}$$

where $Y_g(M)$ is the Yamabe constant of M, and $V_h(\mathbb{S}^n)$ is the volume of $\mathbb{S}^n$ with respect to h.

If $Sc_g \geq n(n-1)$, then it follows from (4.13) that

$$V_g(M) \leq V_h(\mathbb{S}^n). \tag{4.14}$$

Combining (4.14) with (4.12), we infer

$$\int_M Q_g dv_g \leq \frac{n(n^2-4)}{8} \left(\frac{\lambda_1}{n}\right)^2 V_h(\mathbb{S}^n).$$

Recall that $Q_h = \frac{n(n^2-4)}{8}$, the inequality (1.10) follows. □

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