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FINDING A ZERO OF THE SUM OF TWO MAXIMAL MONOTONE OPERATORS WITH MINIMIZATION PROBLEM

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Abstract. The aim of this paper is to construct a new method for finding the zeros of the sum of two maximally monotone mappings in Hilbert spaces. We will define a simple function such that its set of zeros coincide with that of the sum of two maximal monotone operators. Moreover, we will use the Newton-Raphson algorithm to get an approximate zero. In addition, some illustrative examples are given at the end of this paper.

1. Introduction

Let $\mathbb{X}$ be a real Hilbert space and $A : \mathbb{X} \rightrightarrows \mathbb{X}$, $B : \mathbb{X} \rightrightarrows \mathbb{X}$ be two maximal monotone operators on $\mathbb{X}$. We consider the following principal problem: find an element x in $\mathbb{X}$ such that

$$0 \in A(x) + B(x). \tag{1.1}$$

We denote the set of solutions of (1.1) by $(A+B)^{-1}(0)$.

The problem (1.1) was studied by various authors for example, see, [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 14, 15].

For all $\lambda > 0$, we define the following function $f_\lambda : \mathbb{X}^2 \to \mathbb{X}^2$ by

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$$f_{\lambda}(x,y) = \begin{pmatrix} J_{\lambda}^{A}(x) - \dfrac{x+y}{2} \\ J_{\lambda}^{B}(y) - \dfrac{x+y}{2} \end{pmatrix},$$

where J_{λ}^{A} and J_{λ}^{B} are respectively the resolvents of A and B, which are defined as follows:

$$J_{\lambda}^{A}(x) := (I + \lambda A)^{-1}(x) \quad \text{and} \quad J_{\lambda}^{B}(x) := (I + \lambda B)^{-1}(x),$$

where I is the identity operator in $\mathbb{X}$.

For all $\lambda > 0$, we consider the problem:

$$\min_{(x,y)\in\mathbb{X}\times\mathbb{X}} \|f_{\lambda}(x,y)\| = 0. \tag{1.2}$$

It is well known that the second problem (1.2) has many algorithms to obtain one of its solutions(see [13]).

This paper is organized as follows: in the second section, we establish some notations and some properties of maximal monotone operator. In the third section, we will explain the relationship between the problems (1.1) and (1.2). In the last section, we use the Newton-Raphson algorithm to approximate a solution of (1.1).

2. Preliminaries

Let $\mathbb{X}$ be a real Hilbert space and $A : \mathbb{X} \rightrightarrows \mathbb{X}$ a set-valued map. We denote by $\operatorname{dom} A$ the domain of A that is, $\operatorname{dom} A := \{x \in \mathbb{X} : Ax \neq \emptyset\}$, and the graph of A is given by:

$$\operatorname{graph} A := \{(x,y) \in \mathbb{X}\times\mathbb{X} : x \in \operatorname{dom} A \text{ and } y \in Ax\}.$$

Definition 2.1. ([6]) The operator A is said to be monotone, if:

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq 0 \quad \text{for all} \quad (x_i, y_i) \in \operatorname{graph} A, i = 1, 2.$$

Definition 2.2. ([6]) A monotone operator A is called maximal, if its graph has not an extension to a graph of another monotone operator.

We can rephrase the definition of a maximal monotone operator in terms of its graph. If A is a monotone operator, then A is maximal if and only if the following conditions are equivalents:

(1) $u \in A(x)$,
(2) for all $(y,v) \in \operatorname{graph}(A)$, $\langle u - v, x - y\rangle \geq 0$.

Proposition 2.3. ([6]) *Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be a maximal monotone operator. Then for all $\lambda > 0$, $(I + \lambda A)$ is surjective.*

Proposition 2.4. ([6]) *Let A be a maximal monotone operator. For each $y \in \mathbb{X}$, there is a unique $x \in \operatorname{dom} A$ such that $y \in x + \lambda A(x)$.*

For $\lambda > 0$, the Yoshida approximation of a maximal monotone A is given by

$$A_\lambda(x) := \frac{1}{\lambda}\left(\mathrm{id} - J_\lambda^A\right)(x).$$

Proposition 2.5. ([5, 6]) *For all element $y \in \mathbb{X}$, we have*

$$A_\lambda(y) \in A(J_\lambda(y)).$$

Definition 2.6. The sum $A + B$ is defined as:

$$A(x) + B(x) = \{u + v / u \in A(x) \text{ and } v \in B(x)\}.$$

Clearly $\operatorname{dom}(A+B) = \operatorname{dom} A \cap \operatorname{dom} B$.

Let $f : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous function. The subdifferential of f, $\partial f : \mathbb{X} \rightrightarrows \mathbb{X}$ is defined by

$$\partial f(x) = \{x^* \in \mathbb{X} : f(y) - f(x) \geq \langle y - x, x^* \rangle, \ \forall\, y \in \mathbb{X}\}.$$

It is known that $\partial f : \mathbb{X} \rightrightarrows \mathbb{X}$ is a maximal monotone operator on $\mathbb{X}$, and $0 \in \partial f(x^*)$ if and only if x^* is a minimizer of f. Setting $\partial f = A$, it follows that solving the inclusion $0 \in Ax$, in this case, is equivalent to solving for a minimizer of f.

3. Main results

Our goal in this section is to find the relation between the problems (1.1) and (1.2), we start with the following theorem.

Theorem 3.1. *For all $\lambda > 0$, the problem* (1.2) *has a solution if and only if* (1.1) *has a solution.*

Proof. For all $\lambda > 0$, if (x, y) is a solution of (1.2), then

$$\begin{cases} J_\lambda^A(x) = \dfrac{x+y}{2}, \\ J_\lambda^B(y) = \dfrac{x+y}{2}. \end{cases}$$

Therefore,

$$\begin{cases} y = 2J_\lambda^A(x) - x, \\ 2J_\lambda^B(y) = x + y. \end{cases}$$

Replacing y by $2J_\lambda^A(x) - x$ in $2J_\lambda^B(y) = x + y$, we get,

$$J_\lambda^B\left(2J_\lambda^A(x) - x\right) = J_\lambda^A(x).$$

So

$$2J_{\lambda}^{A}(x) - x \in J_{\lambda}^{A}(x) + \lambda B\left(J_{\lambda}^{A}(x)\right).$$

That means

$$-A_{\lambda}(x) \in B\left(J_{\lambda}^{A}(x)\right).$$

Since $A_{\lambda}(x) \in A\left(J_{\lambda}^{A}(x)\right)$, we obtain

$$0 \in A\left(J_{\lambda}^{A}(x)\right) + B\left(J_{\lambda}^{A}(x)\right).$$

Consequently

$$J_{\lambda}^{A}(x) \in (A+B)^{-1}(0).$$

Conversely, if $z \in (A+B)^{-1}(0)$, then $z \in \operatorname{dom} A$. This means that there exists an element $x' \in \mathbb{X}$ such that $x' \in A(z)$. So,

$$x' + z \in z + A(z),$$

and by the definition of the resolvent of A, we get

$$z = J_{\lambda}^{A}\left(x' + z\right).$$

We conclude that, for all $z \in (A+B)^{-1}(0)$, there exists $s = x' + z \in \mathbb{X}$ such that $z = J_{\lambda}^{A}(s)$. Thus we have

$$J_{\lambda}^{A}(s) \in (A+B)^{-1}(0).$$

Then

$$0 \in A\left(J_{\lambda}^{A}(s)\right) + B\left(J_{\lambda}^{A}(s)\right).$$

Therefore, there exists $a \in \mathbb{X}$ such that

$$\begin{cases} a \in A\left(J_{\lambda}^{A}(s)\right), \\ -a \in B\left(J_{\lambda}^{A}(s)\right). \end{cases}$$

Hence

$$\begin{cases} a + J_{\lambda}^{A}(s) \in J_{\lambda}^{A}(s) + A\left(J_{\lambda}^{A}(s)\right), \\ -a + J_{\lambda}^{A}(s) \in J_{\lambda}^{A}(s) + B\left(J_{\lambda}^{A}(s)\right). \end{cases}$$

By the definition of the resolvent of A, we have

$$\begin{cases} J_{\lambda}^{A}(s) = J_{\lambda}^{A}\left(J_{\lambda}^{A}(s) + a\right), \\ J_{\lambda}^{A}(s) = J_{\lambda}^{B}\left(J_{\lambda}^{A}(s) - a\right). \end{cases}$$

We can write $J_{\lambda}^{A}(s)$ as

$$J_{\lambda}^{A}(s) = \frac{x+y}{2},$$

where

$$\begin{cases} x = J_\lambda^A(s) + a, \\ y = J_\lambda^A(s) - a. \end{cases}$$

Then

$$\begin{cases} J_\lambda^A(x) = \dfrac{x+y}{2}, \\ J_\lambda^B(y) = \dfrac{x+y}{2}. \end{cases}$$

Consequently, the problem (1.2) has a solution. □

Remark 3.2. Let $\lambda > 0, \mu > 0$, $S_\lambda = \{(x,y) \in \mathbb{X} \times \mathbb{X} : \|f_\lambda(x,y)\| = 0\}$ and $S_\mu = \{(x,y) \in \mathbb{X} \times \mathbb{X} : \|f_\mu(x,y)\| = 0\}$. Then

$$\left\{J_\lambda^A(x) : (x,y) \in S_\lambda\right\} = \left\{J_\mu^A(x) : (x,y) \in S_\mu\right\}.$$

that is,

$$(A+B)^{-1}(0) = \left\{J_\lambda^A(x) : (x,y) \in S_\lambda\right\}, \quad \forall\, \lambda > 0.$$

Theorem 3.3. *For all $\lambda > 0$, if $\theta_\lambda(a) = J_\lambda^B(2J_\lambda^A(a) - a) - J_\lambda^A(a) + a$, then*

$$J_\lambda^A(S(\theta_\lambda)) = (A+B)^{-1}(0),$$

where $S(\theta_\lambda)$ denotes the set of all fixed points of θ_λ.

Proof. If a is a fixed point for the function θ_λ, we have

$$\begin{aligned} a = J_\lambda^B(2J_\lambda^A(a) - a) - J_\lambda^A(a) + a &\Longleftrightarrow J_\lambda^B(2J_\lambda^A(a) - a) = J_\lambda^A(a) \\ &\Longleftrightarrow 2J_\lambda^A(a) - a \in J_\lambda^A(a) + \lambda B\left(J_\lambda^A(a)\right) \\ &\Longleftrightarrow \frac{1}{\lambda}\left(J_\lambda^A(a) - a\right) \in B\left(J_\lambda^A(a)\right) \\ &\Longleftrightarrow -A_\lambda(a) \in B\left(J_\lambda^A(a)\right). \end{aligned}$$

Since $A_\lambda(a) \in A\left(J_\lambda^A(a)\right)$, we get $0 \in A\left(J_\lambda^A(a)\right) + B\left(J_\lambda^A(a)\right)$. □

Example 3.4. Consider $A(x) = \{2x\}$, $B(x) = \{4x\}$ and $\mathbb{X} = \mathbb{R}$. We will find a solution of the following problem:

$$0 \in A(x) + B(x). \tag{3.1}$$

Clearly, for all $\lambda > 0$,

$$J_\lambda^A(x) = \frac{x}{2\lambda + 1} \quad \text{and} \quad J_\lambda^B(x) = \frac{x}{4\lambda + 1}.$$

We can define the function: $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_\lambda(x,y) = \begin{pmatrix} J_\lambda^A(x) - \dfrac{x+y}{2} \\ J_\lambda^B(y) - \dfrac{x+y}{2} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{2\lambda+1} - \dfrac{x+y}{2} \\ \dfrac{y}{4\lambda+1} - \dfrac{x+y}{2} \end{pmatrix}.$$

If $\|f_\lambda(x,y)\| = 0$, then

$$\begin{cases} \dfrac{x}{2\lambda+1} - \dfrac{x+y}{2} = 0, \\ \dfrac{y}{4\lambda+1} - \dfrac{x+y}{2} = 0. \end{cases}$$

Hence $(x,y) = (0,0)$. Apply Theorem (3.1), we obtain $J_\lambda^A(0) = 0$ is a solution of (3.1).

4. Numerical example

4.1. **Algorithm.** We have:

$$\begin{aligned} f:\ & \mathbb{R}^2 \to \mathbb{R}^2 \\ & (x,y) \to f(x,y) = \begin{pmatrix} J_\lambda^A(x) - \dfrac{x+y}{2} \\ J_\lambda^B(y) - \dfrac{x+y}{2} \end{pmatrix}. \end{aligned}$$

The Jacobian matrix of f is :

$$J_f(x,y) = \begin{pmatrix} \dfrac{\partial J_\lambda^A}{\partial x}(x) - \dfrac{1}{2} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{\partial J_\lambda^B}{\partial y}(y) - \dfrac{1}{2} \end{pmatrix}.$$

We calculate the determinant of $J_f(x,y)$:

$$\begin{aligned} \det J_f(x,y) &= \left(\frac{\partial J_\lambda^A}{\partial x}(x) - \frac{1}{2}\right)\left(\frac{\partial J_\lambda^B}{\partial y}(y) - \frac{1}{2}\right) - \frac{1}{4} \\ &= \frac{\partial J_\lambda^A}{\partial x}(x)\frac{\partial J_\lambda^B}{\partial y}(y) - \frac{1}{2}\frac{\partial J_\lambda^A}{\partial x}(x) - \frac{1}{2}\frac{\partial J_\lambda^B}{\partial y}(y). \end{aligned}$$

Newton-Raphson method, or Newton Method, is a powerful technique for solving the equations numerically:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - J_f^{-1}(x_k, y_k)\, f(x_k, y_k), \quad k = 0, 1, 2, ...$$

4.2. **MATLAB programming.**

clear all; close all; clc ; $x = 10$;
$y = 10$;
$variable = [x; y]$;
$t = [0; 0]$;
$delta = 1$;
$iteration = 1$;
while $(abs(delta) > 1e - 20)$
$f = [J_\lambda^A(variable(1)) + 0.5 * (variable(1) * variable(2))$;
$J_\lambda^B(variable(1)) + 0.5 * (variable(1) * variable(2))]$;
$jacop{=}[\left(\frac{\partial J_\lambda^A}{\partial x}(variable(1)) - 0.5\right)(-0.5)\,;\left((-0.5)\frac{\partial J_\lambda^B}{\partial x}(variable(1)) - 0.5\right)]$;
$delta = (jacop) * (t - f)$;
$variable = variable + delta$;
$iteration = iteration + 1$;
end
format long
$x\_optimise = variable(1)$
$y\_optimise = variable(2)$

Example 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two convex functions defined as:

$$f(x) = (2x + 1)^2 \quad \text{and} \quad g(x) = (5x + 2)^2.$$

Then

$$\partial f(x) = \{8x + 4\} \quad \text{and} \quad \partial g(x) = \{50x + 20\}.$$

It is known that $A = \partial f : \mathbb{R} \rightrightarrows \mathbb{R}$ and $B = \partial g : \mathbb{R} \rightrightarrows \mathbb{R}$ are maximal monotone operators on $\mathbb{R}$. Our problem,

$$0 \in \partial f(x) + \partial g(x). \tag{4.1}$$

Then, for all $\lambda > 0$,

$$J_\lambda^A(x) = \frac{x - 4\lambda}{8\lambda + 1} \quad \text{and} \quad J_\lambda^B(x) = \frac{x - 20\lambda}{50\lambda + 1}.$$

If we choose $\lambda = 1$, we obtain by using the above program,

$$(x, y) = (0.246, -1.073)$$

and from Theorem 3.1, $J_1^A(0.246) = -0.417$ is an approximate solution of (4.1).

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