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AN EXISTENCE OF THREE DIFFERENT NON-TRIVIAL SOLUTIONS FOR DISCRETE ANISOTROPIC EQUATIONS WITH TWO REAL PARAMETERS

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Abstract. This study finds three different solutions (3-Sol's) for the fourth order non-linear discrete anisotropic equations (DAE) with real parameter. We use the variational method(VM) and ϕ_p-Laplacian operator (ϕ_p-LO) to prove the main results. In the following paper, we take the parameters λ, μ such that $\lambda > 0$ and $\mu \geq 0$ into consideration.

1. Introduction

Recently, there has been a surge in interest in discrete equations the existence results of boundary value problems(BVP) with (ϕ_p-LO). Fixed point theorems in cones are commonly used to get conclusions on this issue (see [1, 2, 3, 4, 5, 6, 16] and references therein). The upper and lower solution approach is another instrument for studying nonlinear difference equations (see,

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for instance, [8, 9, 10, 11] and references therein). The variational methodology and critical point theory have been good tools for dealing with differential equations issues. Variational approaches have recently been used to explore the presence and multiplicity of solutions for nonlinear discrete boundary value problems(DBVP) (see [12, 13, 14, 15, 17]).

The major results of the article are obtained the existence of (3-Sol's) for the nonlinear discrete fourth-order anisotropic equations(DAE) as follows:

$$\begin{cases} -\Delta^2(\phi_p \Delta^2 u(i-2)) - \Delta(\phi_p \Delta u(i-1)) + q(i)\phi_p(u(i)) \\ \quad = \lambda f(i,u(i)) + \mu g(i,u(i)) + h(u(i)), \ i \in [2,K]_{\mathbb{Z}}, \\ u(0) = u(1) = u(K+1) = u(K+2) = 0, \end{cases} \tag{1.1}$$

where Δ present the operator of forward difference and presented by

$$\Delta\left[u(i)\right] = \left[u(i+1) - u(i)\right],$$
$$\Delta^{(n)}u(i) = \Delta(\Delta^{(n-1)}u(i)),$$
$$\phi_p(u) = |u|^{p-2}u.$$

K is fixed positive integer and satisfy $K > 2$, $[2,K]_{\mathbb{Z}}$ is discrete interval $\{2,3,...,K\}$, $h : \mathbb{R} \to \mathbb{R}$ is a monotone strictly continuous Lipschitz function with Lipschitz constant $L > 0$ and $h(0)$=0 and $f, g : [2,K]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. $\lambda > 0$, $q : [2, K+1]_{\mathbb{Z}} \to \mathbb{R}$ is a function.

2. Methods for solving problem

A smooth version is a primary tool for identifying whether there are admit three solutions to the problem (1.1). Our next result is the implications of the existence result of a local minimum [18, Theorem 2.1], which is inspired by [18].

Definition 2.1. Suppose χ is a Banach space. $I : \chi \to \mathbb{R}$ is said to be coercive on χ if

$$\lim_{\|u\| \longrightarrow \infty} I(u) = +\infty$$

and the functional I is Gâteaux continuously differentiable. I is called fulfills the Palais-Smale in the brief PS-condition if for each sequence $\{u_n\} \subset \chi$ such that $I(u_n)$ is bounded and $I'(u_n)$ convergent to zero in χ^* has a convergent subsequence in χ that is mean

(1) $I(u_n)$ is bounded,
(2) for each $\{x_n\}$ with $I'(u_n) \longrightarrow 0$ in χ^* it has a subsequence in χ.

Theorem 2.2. ([19, Theorem 2.7]). *Assume χ being a real finite-dimensional Banach space, $\Phi, \Psi : \chi \to \mathbb{R}$ the functionals being coercive with*

$$\inf_{\chi} \Phi = \Phi(0) = \Psi(0) = 0.$$

Suppose that there exist $r > 0$ and $\overline{v} \in \chi$ with $r < \Phi(\overline{v})$ such that

(P_1) $\frac{\sup_{\Phi(v)\leq r}\Psi(v)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(P_2) *for $\lambda \in \Lambda_r := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(v)\leq r}\Psi(v)}\right)$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then the functional $\Phi - \lambda\Psi$ has at least three different critical points in χ for each $\lambda \in \Lambda_r$.

3. The basics notations and auxiliary result

In order to provide the problem (1.1) with a variational formulation, let's begin by introducing

$$\chi := \{u : [0, K+2]_{\mathbb{Z}} \to \mathbb{R} : u[0] = u[1] = u[K+1] = u[K+2] = 0\},$$

equipped with the norm

$$\|u\| = \left(\sum_{i=2}^{K+2}(|\Delta^2 u(i-2)|^p + |\Delta u(i-1)|^p + q(i)|u(i)|^p\right)^{\frac{1}{p}}.$$

Consider the K-dimentional Banach space concerning the norm

$$\chi_1 = \|u\|_p = \left\{\sum_{i=2}^{K+2}|u(i)|^p\right\}^{\frac{1}{p}}.$$

By using the notation in [7] and taking χ_1 into account we find the following formula

$$\begin{aligned}
\|u\| &= \left(\sum_{i=2}^{K+2}(|\Delta^2 u(i-2))|^p + |\Delta u(i-1)|^p + q(i)|u(i)|^p\right)^{\frac{1}{p}} \\
&\leq \left\{\sum_{i=2}^{K+2}(|\Delta^2 u(i-2))|^p\right\}^{\frac{1}{p}} + \left\{\sum_{i=2}^{K+2}(|\Delta u(i-1))|^p\right\}^{\frac{1}{p}} \\
&\quad + \left\{\sum_{i=2}^{K+2}q(i)\right\}^{\frac{1}{p}}\left\{\sum_{i=2}^{K+2}(|u(i))|^p\right\}^{\frac{1}{p}} \\
&\leq (K+2)\|u\|^p + (K+2)\|u\|^p \\
&\quad + \left\{\sum_{i=2}^{K+2}q(i)\right\}^{\frac{1}{p}}\sum_{i=2}^{K+2}|u(i)|^p, \qquad (3.1)
\end{aligned}$$

it implies that

$$\|u\| \leq \left\{ 2(K+2)\|u\|^p + \left\{ \sum_{i=2}^{K+2} q(i) \right\}^{\frac{1}{p}} \|u\|^p \right\}$$
$$\leq \left(2(K+2) + \left\{ \sum_{i=2}^{K+2} q(i) \right\}^{\frac{1}{p}} \right) \|u\|^p. \tag{3.2}$$

Define the functional requirements $\Phi, \Psi : \chi \longrightarrow \mathbb{R}$

$$\Phi(u) = \|u\| - \sum_{i=2}^{K+2} H(u(i)) \tag{3.3}$$

and

$$\Psi(u) = \sum_{i=2}^{K} (F(i, u(i))) + \frac{\mu}{\lambda} \sum_{i=2}^{K} (G(i, u(i))), \tag{3.4}$$

where $F, G : [2, K]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$, $H : \mathbb{R} \to \mathbb{R}$ is shown bellow, accordingly

$$F(i,x) := \int_0^x \; f(i,s)\,ds, \;\forall\; (i,x) \in [2,K]_{\mathbb{Z}} \times \mathbb{R},$$
$$G(i,x) := \int_0^x \; g(i,s)\,ds, \;\forall\; (i,x) \in [2,K]_{\mathbb{Z}} \times \mathbb{R},$$
$$H(x) := \int_0^x \; h(s)\,ds, \;\forall\, x \in \mathbb{R}.$$

Let

$$I_\lambda = \Phi(u) - \lambda \Psi(u). \tag{3.5}$$

Then a simple calculation ensures that I_λ is of class C^1 on χ with

$$\begin{aligned}\Phi'(u)(v) = &\sum_{i=0}^{K+2} [\phi_p(\Delta^2 u(i-2)\Delta^2 v(i-2)) + \phi_p(\Delta u(i-1)\Delta v(i-1)) \\ &+ q(i)|u(i)|^{p-2}u(i)v(i)] - \sum_{i=2}^{K} h(u(i))v(i) \\ = &- \sum_{i=2}^{K} \Delta^2(\phi_p(\Delta^2 u(i-2))\Delta^4 v(i) - \Delta(\phi_p \Delta u(i-1)\Delta^2 v(i)) \\ &- q(i)|u(i)|^{p-2}u(i)v(i) + h(u(i))v(i)\end{aligned} \tag{3.6}$$

and

$$\Psi'(u)(v) = \sum_{i=2}^{K} [f(i, u(i)) + \frac{\mu}{\lambda} g(i, u(i))]v(i)] \tag{3.7}$$

for anyone at all $u, v \in \chi$, the weak solutions to the equation (1.1) are obviously the most essential feature of I_λ. The norm $|\cdot|$ is clearly the same as the norm $||\cdot||_\chi$. Given that $(\chi, ||\cdot||)$, it is compactly embedded in $C([2,K]_{\mathbb{Z}}, \mathbb{R})$,

$$\|u\|_\infty \leq \frac{(K+2)^{\frac{p-1}{p}}}{2}\|u\|, \quad \forall\ u \in \chi, \tag{3.8}$$

where $\|u\|_\infty = \max_{i\in[2,K]_{\mathbb{Z}}} |u(k)|$.

($\Re_1$) Assume that the functions $f, g : [2,K]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

($\Re_2$) Assuming $h : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous Lipschitz mapping and strictly monotone of order $p-1$ fulfilling Lipschitzian fixed L with $0 < L < \frac{1}{2}$, that is,

$$|h(i_1) - h(i_2)| \leq L|i_1 - i_2|^{p-1} \quad \forall\ i_1, i_2 \in \mathbb{R},\ \ p > 1$$

and $h(0) = 0$.

In the following lemmas, we first give some essential characteristics of the functionals Φ and Ψ.

Lemma 3.1. *The functional I_λ in* (3.5) *is differentiable.*

Proof. From (3.6) and (3.7) the functionals (3.3) and (3.4) are differentiable, and since I_λ given in the formula of (3.5) so, we have obtained the proof. □

Lemma 3.2. Φ *is a coercive operator.*

Proof. From the relation (3.3) and the formula in (3.2) we get the following result:

$$\begin{aligned}
\Phi(u) &= \|u\| - \sum_{i=2}^{K+2} H(u(i)) \\
&\leq \left\{(2(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}}\right\}\|u\|^p - \sum_{i=2}^{K+2}\int_0^{u(k)} h(k)dk \\
&\leq \left\{(2(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}}\right\}\|u\|^p - \frac{1}{p}L\|u\|^p \\
&= \|u\|^p\left\{(2(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L\right\}.
\end{aligned}$$

Hence, we have $\Phi(u) \longrightarrow +\infty$ as $\|u\|^p \longrightarrow +\infty$, so, we obtained that the functional Φ is corecive. □

4. There are at least three nontrivial solutions

Using the conditions of Theorem 2.2, we prove that the problem (1.1) has at least three nontrivial weak solutions.

For our convenience, set

$$G^c := \sum_{i=2}^{K} \max_{|\xi|\leq c} G(k,\xi) \ \text{ for all } c>0$$

and

$$G_d := \sum_{i=2}^{K} G(k,d) \ \text{ for all } d>0.$$

To define the best method for giving the solution to the first part of problem (1.1). Fix $c, d > 0$ in our first outcome, such that

($\Re_3$) Set the following:

$$\frac{(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)d^p}{\sum_{i=2}^{K} F(i,d)}$$

$$< \frac{2c^p(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}} \sum_{i=2}^{K} \max_{|\xi|\leq c} F(i,\xi)}.$$

($\Re_4$) Pick:

$$\lambda \in \Lambda = \Big(\frac{(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)d^p}{\sum_{i=2}^{K} F(i,d)},$$

$$\frac{2c^p(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}} \sum_{i=2}^{K} \max_{|\xi|\leq c} F(i,\xi)}\Big).$$

Moreover, for finding λ according to the parameter μ for the part of the function G we put the following formulas:

$$\delta_{\lambda,g}$$

$$=\min\Bigg\{\frac{2c^p(2(K+2)+\left\{\sum_{i=2}^{K+2}q(i)\right\}^{\frac{1}{p}}-\frac{1}{p}L)-\lambda p(K+2)^{\frac{p-1}{p}}\sum_{i=2}^{K}\max_{|\xi|\le c}F(i,\xi)}{(K+2)^{\frac{p-1}{p}}G^c},$$

$$\left|\frac{(2(K+2)+\left\{\sum_{i=2}^{K+2}q(i)\right\}^{\frac{1}{p}}-\frac{1}{p}L)d^p-\lambda p\sum_{i=2}^{K}F(i,d)}{\min\{0,G_d\}}\right|\Bigg\} \tag{4.1}$$

and

$$\overline{\delta_{\lambda,g}}=\min\left\{\delta_{\lambda,g}\ ,\ \frac{1}{\max\left\{0,(K+2)^{\frac{p-1}{p}}\limsup_{|\xi|\to+\infty}\frac{\sum_{i=2}^{K}G(k,\xi)}{|\xi|^p}\right\}}\right\}, \tag{4.2}$$

where $\frac{1}{0}=+\infty$ is used whenever this condition happens.

The following theorem holds if $(\Re_1)$, $(\Re_2)$ and $(\Re_3)$ are all true.

Theorem 4.1. *Assume that there are three positive constants c, p and d with*

$$2^{\frac{1}{p}}c<d(K+2)^{p-1}\ \text{and}\ d<c,$$

$(\Re_5)$

$$\frac{(K+2)^{\frac{p-1}{p}}\sum_{i=2}^{K}\max_{|\xi|\le c}F(i,\xi)}{2c^p}<\frac{\sum_{i=2}^{K}F(i,d)}{d^p},$$

$(\Re_6)$

$$\limsup_{|\xi|\to+\infty}\frac{\sum_{i=2}^{K}F(i,\xi)}{\xi^p}<\frac{\sum_{i=2}^{K}\max_{|\xi|\le c}F(i,\xi)}{2c^p}.$$

Furthermore, for any continuous function $g:[2,K]_{\mathbb{Z}}\times\mathbb{R}\to\mathbb{R}$ and $\lambda\in\Lambda$ and as a result,

$(\Re_7)$

$$\limsup_{|\xi|\to+\infty}\frac{\sum_{i=2}^{K}G(i,\xi)}{\xi^p}<+\infty.$$

Then there is $\delta_{\lambda,g}>0$ provided by (4.1) *such that the problem* (1.1) *has at least three nontrivial weak solutions for each $\mu\in\left(0\ ,\ \overline{\delta_{\lambda,g}}\right)$.*

Proof. Let us establish that the functionals Φ, Ψ meet the requirements in Theorem 2.2 in order to apply in our situation. The functionals $\Phi,\Psi:W\longrightarrow\mathbb{R}$ are provided by (3.3) and (3.4) for any $u\in W$. For each $\lambda\in\mathbb{R}$, we now set the functional $I_\lambda:=\Phi(u)-\lambda\Psi(u)$. Then the functional I_λ is differentiable as a result of Lemma 3.1 and the functional Φ is coercive as a conclusion of Lemma 3.2. Set

$$r = \frac{2c^p(2(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}}}$$

and

$$\overline{v(i)} = \begin{cases} d\ ,\ i \in [2,K]_{\mathbb{Z}}, \\ 0\ ,\ \text{otherwise.} \end{cases}$$

Clearly $\overline{v} \in W$ and $\Phi(\overline{v}) = \left(2(K+2) + \sum_{i=2}^{K+2} q(i) - L\right) d^p$. Since $2c^{\frac{1}{p}} < d(K+2)^{p-1}$, we can get that $r < \Phi(\overline{v})$. Taking into account, one has $\max_{t\in[2,K]_{\mathbb{Z}}} |u(k)| \le c$, and we can obtain

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u)}{r} &= \frac{\sup_{\Phi(u)\le r} \sum_{k=2}^{K} \left[F(k,u(k) + \frac{\mu}{\lambda} G(t,u(k)\right]}{\frac{2c^p(2((K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}}}} \\
&\le \frac{\sum_{i=2}^{K} \max_{|\xi|\le c} \left[F(i,\xi) + \frac{\mu}{\lambda} G(t,\xi)\right]}{\frac{2c^p(2*((K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}}}} \\
&\le \frac{\sum_{i=2}^{K} \max_{|\xi|\le c} F(i,\xi)}{\frac{2c^p(2*((K+2)+\left\{\sum_{i=2}^{K+2} q(k)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}}}} \\
&\quad + \frac{\mu}{\lambda} \frac{G^c}{\frac{(2c)^p(2((K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}}}}.
\end{aligned}$$

From this, if $G^c = 0$, we get

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u)}{r} &\le \frac{\sum_{i=2}^{K} \max_{|\xi|\le c} F(t,\xi)}{\frac{2c^p(2((K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}}}} \\
&\le \frac{(K+2)^{\frac{p-1}{p}} \sum_{i=2}^{K} \max_{|\xi|\le c} F(i,\xi)}{2c^p(2(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)} \\
&< \frac{1}{\lambda},
\end{aligned}$$

while, if $G^c > 0$, given that, it turned out to be correct $\mu < \delta_{\lambda,g}$.

Furthermore, one must

$$\frac{\Psi(\overline{v})}{\Phi(\overline{v})} = \frac{\sum_{i=2}^{K}(F(i,d) + \frac{\mu}{\lambda}G(t,d)}{\left(2*(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L\right) d^p}$$
$$= \frac{\sum_{i=2}^{K} F(i,d)}{\left(2*(K+2) + \sum_{i=2}^{K+2} q(i) - L\right) d^p}$$
$$+ \frac{\mu}{\lambda}\frac{G_d}{\left(2*(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L\right) d^p}.$$

Hence, if $G_d \geq 0$, it will be as follows:

$$\frac{\Psi(\overline{v})}{\Phi(\overline{v})} > \frac{1}{\lambda} \tag{4.3}$$

and if $G_d < 0$, also we have $\frac{\Psi(\overline{v})}{\Phi(\overline{v})} > \frac{1}{\lambda}$, since $\mu < \delta_{\lambda,g}$. From (P_1) of Theorem 2.2, we now establish the functional's coercivity for $\Phi - \lambda\Psi$.

First, we start with the assumption that

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{i=2}^{K} F(i,\xi)}{|\xi|^p} > 0.$$

As a result, verify that

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{i=2}^{K} F(i,\xi)}{|\xi|^p} < \epsilon < \frac{\sum_{i=2}^{K} \max_{|\xi|\leq c} F(i,\xi)}{(2c)^p}.$$

A positive constant h_ϵ exists, obtained from $(\Re_6)$ as a result

$$\sum_{i=2}^{K} F(i,\xi) \leq \epsilon|\xi|^p + h_\epsilon \text{ for each } \xi \in \mathbb{R}.$$

And also because

$$\lambda < \frac{2c^p(2(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}} \sum_{i=2}^{K} \max_{|\xi|\leq c} F(i,\xi)},$$

it follows that

$$\lambda \sum_{i=2}^{K} F(i,u(i)) \leq \lambda\epsilon|\xi|^p + \lambda h_\epsilon$$

$$\leq \epsilon \frac{2c^p(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}} \sum_{i=2}^{K} \max_{|\xi|\leq c} F(k,\xi)} \|u\|^p$$

$$+ h_\epsilon \frac{2c^p(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{(K+2)^{\frac{p-1}{p}} \sum_{i=2}^{K} \max_{|\xi|\leq c} F(k,\xi)} \tag{4.4}$$

for each $u \in W$. Furthermore, if $\mu < \overline{\delta_{\lambda,g}}$ is taken into consideration, then it follows that

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{i=2}^{K} G(k,\xi)}{|\xi|^p} < \frac{2^p}{\mu(K+2)^{p-1}}.$$

Thus, for any $\xi \in \mathbb{R}$ and some constant $\tau_\mu > 0$, one has

$$\sum_{i=2}^{K} G(k,\xi) \leq \frac{2^p}{\mu(K+2)^{p-1}}|\xi|^p + \tau_\mu.$$

Hence, by using (3.8) for each $u \in W$, we get

$$\sum_{i=2}^{K} G(i,u(i)) \leq \frac{2^p}{\mu(K+2)^{p-1}}|u(i)|^p + \tau_\mu \leq \frac{\|u\|^p}{\mu} + \tau_\mu. \tag{4.5}$$

Now, from formula (3.3), (3.4) and considering the (3.5). Then, from (4.4) and (4.5) we have

$$\Phi(u) - \lambda\Psi(u) \leq \|u\|^p\Big(2*(K+2) + \left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L\Big)$$

$$+ \left(1 - \epsilon\frac{(2c)^p(2*(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}} - \frac{1}{p}L)}{\sum_{i=2}^{K} \max_{|\xi|\leq c} F(i,\xi)}\right)\|u\|^p$$

$$- h_\epsilon \frac{(2c)^p(2*(K+2)+\sum_{i=2}^{K} q(i) - L)}{(K+2)\sum_{i=2}^{K} \max_{|\xi|\leq c} F(i,\xi)} - \mu\tau_\mu.$$

If

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{i=2}^{K} F(i,\xi)}{|\xi|^p} \leq 0,$$

then for any $\xi \in \mathbb{R}$, there exists h_ϵ such that $\sum_{i=2}^{K} F(i,\xi) \le h_\epsilon$ and reasoning as previously, we obtain

$$\Phi(u) - \lambda\Psi(u) \le \|u\|^p \left(2 * (K+2) + \left\{ \sum_{i=2}^{K+2} q(k) \right\}^{\frac{1}{p}} - \frac{1}{p} L \right)$$
$$- h_\epsilon \frac{(2c)^p (2 * (K+2) + \left\{ \sum_{i=2}^{K+2} q(i) \right\}^{\frac{1}{p}} - \frac{1}{p} L)}{(K+2) \sum_{i=2}^{K} \max_{|\xi| \le c} F(i,\xi)} - \mu\tau_\mu.$$

Both instance results in $\Phi - \lambda\Psi$ coercivity. Thus, Theorem 2.2 (P_2) fulfills. Using the relationships (4.3) and (4.4), one may indeed

$$\lambda \in \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})} \ , \ \frac{r}{\sup_{\Phi(u) \le r} \Psi(u)} \right).$$

Finally, all of the requirements of Theorem 2.2 are met. Hence we have the desired result. □

We have shown the following instances in which the hypotheses of Theorem 4.1 are met, resulting in the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ for each $\lambda \in \Lambda$. There are at least three nontrivial solutions to the problem (1.1), because χ has at least three nontrivial critical points.

Example 4.2. Let $c = 3$, $d = 2$, $K = 11$, $i \in [2, 11]_{\mathbb{Z}}$, $q(i) = i^2 + 4 * i$, $L = 0.002$. Consider the following discrete fourth-order problem:

$$\begin{cases} -\Delta^2(\phi_4 \Delta^2 u(i-2)) - \Delta(\phi_4 \Delta u(i-1)) + (i^2 + 4 * i)\phi_4(u(i)) \\ \quad = \lambda f(i, u(i)) + \mu g(i, u(i)) + h(u(i), \ t \in [2, 8]_{\mathbb{Z}}, \\ u(0) = u(1) = u(12) = u(13) = 0 \end{cases} \tag{4.6}$$

for every $i \in [2, 11]_{\mathbb{Z}}$ and $u(t) \in \mathbb{R}$, where

$$f(i, u(i)) = i + \sin(u(i)) + 4, \quad h(u(i)) = \arcsin(0.002u(i)),$$
$$g(i, u(i)) = 2i^2 + \cos(u(i)),$$
$$F(i,\xi) = \int_0^\xi f(i, u(i))\, du = \int_0^\xi (i + \sin(u(i)) + 4) \ du = (i\xi 1 + \cos(\xi) + 4\xi - 1),$$
$$G(i,\xi) = \int_0^\xi g(i, u(i))\, du = \int_0^\xi (2i^2 + \cos(u(i))) \ du = 2i^2\xi + \sin(\xi),$$
$$\sum_{i=2}^{11} \max_{|\xi| \le 2} F(i,\xi) = 30, \quad \sum_{i=2}^{11} F(i,d) = 295.013,$$
$$\sum_{i=2}^{11} \max_{|\xi| \le 2} G(i,\xi) = 242.0348, \quad \sum_{i=2}^{11} G(i,d) = 3011.977,$$

$$\left(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}}-\frac{1}{p}L\right)=31.8626.$$

By using conditions of Theorem 4.1, we can get that $\lambda \in (1.728\ ,\ 25.1317)$,

$$\limsup_{|\xi|\to+\infty}\frac{\sum_{i=2}^{K} F(i,\xi)}{\xi^p}=0<\frac{\sum_{i=2}^{K}\max_{|\xi|\leq c} F(k,\xi)}{(2c)^p}=0.117,$$

$$\delta_{\lambda,g}=\frac{5161.7898-\lambda 205.0348}{1657.04891},\quad \overline{\delta_{\lambda,g}}=\delta_{\lambda,g},$$

so, for every $\mu \in \left(0\ ,\ \overline{\delta_{\lambda,g}}\right)$, the problem (4.6) has at least three nontrivial weak solutions.

Remark 4.3. This study discovered no asymptotic constraints on f and g are necessary in the results and only algebraic restrictions on f must be satisfied. Furthermore, one of the three solutions may be trivial in the findings of the preceding results since the values of $f(i,0)$ and $g(i,0)$ for $i \in [2,K]_{\mathbb{Z}}$ are not specified.

With $\mu = 0$, the following outcome is a particular instance of Theorem 4.1.

Remark 4.4. Suppose that all hypotheses in Theorem 4.1 are valid. Then for each

$$\lambda \in \Lambda = \Bigg(\frac{\left(2(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}}-\frac{1}{p}L\right)d^p}{\sum_{i=2}^{K} F(i,d)}\ ,\ \frac{(2c)^p\left(2*(K+2)+\left\{\sum_{i=2}^{K+2} q(i)\right\}^{\frac{1}{p}}-\frac{1}{p}L\right)}{(K+2)^{p-1}\sum_{i=2}^{K}\max_{|\xi|\leq c} F(i,\xi)}\Bigg),$$

the problem

$$\begin{cases} -\Delta^2(\phi_p(i)\Delta^2 u(i-2))-\Delta(\phi_p\Delta u(i-1))+q(i)\phi_p(u(i)) \\ \quad =\lambda f(i,u(i))+h(u(i)),\ i\in[2,K]_{\mathbb{Z}}, \\ u(0)=u(1)=u(K+1)=u(K+2)=0 \end{cases} \tag{4.7}$$

has three weak solutions in χ.

Example 4.5. Taking Example 4.2 into account and suppose that $\mu = 0$ we can find the existence of three solutions by using Remark 4.4.

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