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INVERSE PROBLEM FOR STOCHASTIC DIFFERENTIAL EQUATIONS ON HILBERT SPACES DRIVEN BY LÉVY PROCESSES

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Abstract. In this paper we consider inverse problem for a general class of nonlinear stochastic differential equations on Hilbert spaces whose generating operators (drift, diffusion and jump kernels) are unknown. We introduce a class of function spaces and put a suitable topology on such spaces and prove existence of optimal generating operators from these spaces. We present also necessary conditions of optimality including an algorithm and its convergence whereby one can construct the optimal generators (drift, diffusion and jump kernel).

1. Introduction

Most of the system dynamics found in physical and engineering sciences are developed on the basis of fundamental laws of science as currently understood by scientists and engineers. The basic parameters that determine the dynamics are often obtained from physical experiments which are not expected to guaranty absolute accuracy thereby presenting uncertainty in the model. In particular, in the study of biological, medical, management, economic and social sciences mathematical models are not so well developed and hence require empirical approach and mathematical analysis to develop such models.

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Dynamic systems arising from physical and applied sciences in general, are governed by either deterministic or stochastic ordinary or partial differential equations, or integral equations, or combinations thereof. Given the system dynamics, one is interested to find control policies from an admissible class to steer the system so as to meet certain objectives subject to various state and control constraints, see [2], [3], [6] and the references therein. These are the so called direct problems; in other words, the system dynamics is fully known and the primary objective is to find controls which can force the system to reach specified goals. In contrast, the inverse problem is concerned with the identification of the unknown system dynamics from available and possibly noisy data(see [1], [4], [5], [7], [8], [9], [10], [11], [12]).

In reference [1], inverse problems for infinite dimensional deterministic and stochastic systems, are considered with applications to partial differential equations and nonlinear filtering. Reference [4] presents techniques for identification of nonlinear systems given by input-output models popular in engineering sciences. In reference [12] the authors consider inverse problems for a class of partial differential equations from mathematical physics, in particular, a nonlinear heat equation with unknown system parameters or unknown initial or boundary data. In reference [9] the authors use the so called "collage theorem" based on Banach fixed point theorem to solve inverse problems for deterministic and random ordinary differential equations including mean field equations. In reference [10] the authors introduce a statistical treatment of inverse problems constrained by models with stochastic terms. In particular, the authors propose a technique along with the objective (score) functionals to determine the unknown space dependent coefficients for an elliptic partial differential equation (governing the spatial dynamics of subsurface flows) and a parameter inversion problem for power grid governed by ordinary differential-algebraic equations. Also they propose and discuss in details the merits and demerits of several alternative objective (or score) functionals. In reference [11] the author considers a linear filtering problem (arising in communication engineering) with an unknown linear filter for recovering signals embedded in additive noise. The problem is formulated as a min-max problem and some finite dimensional approximation techniques are used to treat the problem.

Recently we considered inverse problem for finite dimensional stochastic systems [5] governed by Itô differential equations identifying the infinitesimal generators controlling drift and diffusion only. In this paper we consider a much more general inverse problem for infinite dimensional stochastic systems of the form

$$dx = Axdt + F(x)dt + B(x)dW(t) + \int_{V_\delta} C(x,\xi)q(d\xi \times dt),$$
$$x(0) = x_0, \tag{1.1}$$

where the operator A is a known unbounded linear operator generating a C_0-semigroup of bounded linear operators on a Hilbert space, while the nonlinear operators $\{F, B, C\}$ are unknown. Our objective is to identify the unknown operators. Towards this goal we introduce the objective functional

$$J(F, B, C) = \mathbf{E}\{\int_0^T \ell(t, x(t))dt + \Phi(x(T))\}, \tag{1.2}$$

where $x = x(F, B, C)(t)$, $t \in I$, is the mild solution of equation (1.1). The integrand ℓ and the function Φ may also depend on available data not explicitly shown. The problem is to find a triple (F, B, C) from a suitable topological space of nonlinear maps or operators, to be introduced shortly, such that J attains its minimum. The results of this paper are applicable to both finite and infinite dimensional nonlinear stochastic differential equations including semilinear stochastic partial differential equations. These results are also applicable to control theory to determine optimal feedback control operators.

In order to consider the problem as stated above we need certain notations and terminologies. Let $\{H, U\}$ denote a pair of real separable Hilbert spaces with H denoting the state space, U the state space of Brownian motion, and $\mathcal{L}(U, H)$ the space of bounded linear operators from U to H. Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ be a complete filtered probability space where $\{\mathcal{F}_t, t \geq 0\}$ is an increasing family of sub-sigma algebras of the sigma algebra $\mathcal{F}$, continuous from the right and having limits from the left. Let $\{W(t), t \geq 0\}$ be an $\mathcal{F}_t$-adapted U valued Brownian motion with incremental covariance operator Q, a symmetric positive nuclear operator in $\mathcal{L}(U)$. Let $\{e_i, i \in N\}$ be a complete ortho-normal basis of U given by the eigen vectors of Q, $Qe_i = \lambda_i e_i$, with $\lambda_i \geq 0$ being the corresponding eigen values. Thus the Brownian motion W can be expressed as $W(t) = \sum (W(t), e_i)e_i$ where $\{(W(t), e_i) \equiv \beta_i(t), i \in N\}$ is a family of mutually independent real valued Brownian motions with mean zero and variance $\mathbf{E}(W(t), e_i)^2 = \mathbf{E}\beta_i(t)^2 = t(Qe_i, e_i) = t\lambda_i$. Since Q is nuclear, $\sum (Qe_i, e_i) = \sum \lambda_i < \infty$. To consider the jump process, we let V denote a Polish space (complete separable metric space) and $\mathcal{B}(V)$ denote the Borel algebra of subsets of V and $p(d\xi \times dt)$ denote a random measure defined on the sigma algebra of subsets of the set $V_\delta \times I$ where $I = [0, T]$ is the time interval and $V_\delta \equiv V \setminus B_\delta$ with B_δ denoting the open ball in V of radius $\delta > 0$ and centered at the origin. Throughout the rest of the paper it is assumed, without further notice, that for each $t \in I$, and $\Delta \subset V_\delta$, the measure process

$p(\Delta \times [0,t))$ is $\mathcal{F}_t$ adapted. The measure p is said to be a Poisson random measure (or a counting measure) on the measurable space $(V_\delta \times I, \mathcal{B}(V_\delta) \times \mathcal{B}(I))$ if for each Borel set $S \subset V_\delta$ and each time interval $\Gamma \subset I$, the probability that there are exactly n jumps of sizes (or with range) confined in the set S is given by

$$P\{p(S \times \Gamma) = n\} = \frac{(\pi(S)\lambda(\Gamma))^n}{n!} \exp - \{\pi(S)\lambda(\Gamma)\}$$

where λ denotes the Lebesgue measure on I and π (a positive measure) denotes the Lèvy (jump) measure on V_δ. The term $\pi(S)$ (the Lèvy measure of the set S) denotes the mean rate of jumps of sizes confined in the set S. The measure π can be chosen according to the specific needs of applications. Define the random measure

$$q(S \times \Gamma) \equiv p(S \times \Gamma) - \pi(S)\lambda(\Gamma)$$

with mean zero and variance $\pi(S)\lambda(\Gamma)$. The process $q(d\xi \times dt)$ is called the compensated Poisson random measure.

Throughout rest of the paper we assume without further notice that the initial state, the Brownian motion, and the Poisson random measure are stochastically independent.

2. Admissible class of drift-diffusion-jump triples

To consider the inverse problem as stated above it is necessary to give a precise characterization of the admissible set of drift-diffusion-jump triples denoted by $\mathcal{P}_{ad}$. Let $\{\alpha, K\}$ be any pair of positive numbers and let $\mathcal{F}_{\alpha,K}$, and $\mathcal{B}_{\alpha,K}$ denote the class of functions (operators) given by

$$\mathcal{F}_{\alpha,K} \equiv \Big\{F : H \to H \Big| \ \parallel F(0) \parallel_H \leq \alpha \text{ and } \\ \parallel F(x_1) - F(x_2) \parallel_H \leq K \parallel x_1 - x_2 \parallel_H \ \ \forall\, x_1, x_2 \in H\Big\} \quad (2.1)$$

and

$$\mathcal{B}_{\alpha,K} \equiv \Big\{B : H \to \mathcal{L}(U,H) \Big| \ \parallel B(0) \parallel_{\mathcal{L}(U,H)} \leq \alpha \text{ and } \\ \parallel B(x_1) - B(x_2) \parallel_{\mathcal{L}(U,H)} \leq K \parallel x_1 - x_2 \parallel_H \ \ \forall\, x_1, x_2 \in H\Big\}. \quad (2.2)$$

Clearly, these are Lipschitz maps whose values at zero vector do not exceed the number α, and the Lipschitz coefficients do not exceed K. The larger the parameters $\{\alpha, K\}$ are, the larger are these classes $\{\mathcal{F}_{\alpha,K}, \mathcal{B}_{\alpha,K}\}$. Let $BC(V_\delta)$ denote the space of bounded continuous real valued functions defined on the set $V_\delta \subset V$, and $L_2(\pi) = L_2(V_\delta, \pi)$ denote the class of real valued Borel measurable functions defined on V_δ which are square integrable with respect to the Lévy measure π. Let (a,b) be a pair of nonnegative Borel measurable real valued functions defined on V_δ so that $a,b \in BC(V_\delta) \cap L_2^+(\pi)$. We introduce the class of jump kernels $\mathcal{C}_{a,b}$ as follows:

$$\begin{aligned}\mathcal{C}_{a,b} \equiv \Big\{ & C : H \times V_\delta \longrightarrow H \text{ continuous} \Big| \\ & \| C(x,\xi) - C(y,\xi) \|_H \leq a(\xi) \| x - y \|_H, \ \ \forall\, x,y \in H, \xi \in V_\delta \\ & \text{and } \ \| C(0,\xi) \|_H \leq b(\xi), \ \ \xi \in V_\delta \Big\}. \end{aligned} \tag{2.3}$$

Using the above three classes we introduce the set of admissible drift-diffusion-jump triples given by the Cartesian product $\mathcal{P}_{ad} \equiv \mathcal{F}_{\alpha,K} \times \mathcal{B}_{\alpha,K} \times \mathcal{C}_{a,b}$. Using the notations of Willard [13] we denote the domain space $H \times V_\delta$ by X, and the range space $H_w \times \mathcal{L}_{wo}(U,H) \times H_w$ by Y where H_w is the Hilbert space H furnished with the weak topology τ_w, and $\mathcal{L}_{wo}(U,H)$ is the space of bounded linear operators $\mathcal{L}(U,H)$ endowed with the weak operator topology τ_{wo}. The range space Y is then endowed with the product topology $\mathcal{T}_w \equiv \tau_w \times \tau_{wo} \times \tau_w$. We consider the function space $Y^X = \mathcal{F}_p(X,Y)$ which has the natural Tychonoff product topology. Note that the set $\mathcal{P}_{ad}$ is a subset of the function space Y^X and it is given the topology of point wise convergence [10] denoted by τ_p. For each $(x,\xi) \in X \equiv H \times V_\delta$, let $\Pi_{x,\xi}$ denote the projection map given by

$$\Pi_{x,\xi}(Y^X) = \{(F(x), B(x), C(x,\xi)) \mid (F,B,C) \in Y^X\}.$$

This is simply the evaluation map. We are concerned with the set given by

$$\Pi_{x,\xi}(\mathcal{P}_{ad}) = \{(F(x), B(x), C(x,\xi)) \mid (F,B,C) \in \mathcal{P}_{ad}\}.$$

It is clear that for each $(x,\xi) \in X = H \times V_\delta$, the closure of the (x,ξ)-projection of $\mathcal{F}_{\alpha,K}$ denoted by $\Pi_{x,\xi}(\mathcal{F}_{\alpha,K}) = \Pi_x(\mathcal{F}_{\alpha,K}) = \{F(x), F \in \mathcal{F}_{\alpha,K}\}$, is a closed bounded convex subset of H and hence weakly (or τ_w) compact. Similarly, the closure of each (x,ξ)-projection of $\mathcal{B}_{\alpha,K}$, given by $\Pi_{x,\xi}(\mathcal{B}_{\alpha,K}) = \Pi_x(\mathcal{B}_{\alpha,K})$, is a closed bounded convex subset of $\mathcal{L}(U,H)$ and hence compact in the weak operator topology (τ_{wo}). For each $(x,\xi) \in X$, the closure of the (x,ξ)-projection of $\mathcal{C}_{a,b}$, given by $\Pi_{x,\xi}(\mathcal{C}_{a,b}) = \{C(x,\xi) : C \in \mathcal{C}_{a,b}\}$, is a closed bounded convex subset of H and hence weakly (or τ_w) compact.

Theorem 2.1. *The set $\mathcal{P}_{ad}$, a subset of the function space Y^X, is compact in the point wise topology τ_p.*

Proof. The Hilbert space H endowed with the weak topology τ_w is a Hausdorff topological space, and the space $\mathcal{L}(U,H)$ furnished with the weak operator topology τ_{wo} is also a Hausdorff space. The Cartesian product of Hausdorff spaces is Hausdorff. Thus the space Y furnished with the product topology

$\mathcal{T}_w$ is Hausdorff. It follows from the preceding discussions that the set $\mathcal{P}_{ad}$ is point wise closed and that for each $(x, \xi) \in X$ the closure of $\Pi_{x,\xi}(\mathcal{P}_{ad})$ is $\mathcal{T}_w$ compact. Hence it follows from Willard [13, Theorem 42.3], that the set $\mathcal{P}_{ad}$ is compact in point wise topology τ_p. □

3. Existence of optimal drift-diffusion-jump triples

Consider the system (1.1) with $(F, B, C) \in \mathcal{P}_{ad}$ and the objective functional (1.2). First, we present a result on existence, uniqueness, and regularity properties of solutions of equation (1.1). For this we introduce the following spaces of random processes. Throughout the rest of the paper, we let $I \equiv [0, T]$ denote the closed bounded interval and $B_\infty(I, H)$ the Banach space of H valued bounded measurable functions endowed with the sup-norm topology. In the study of stochastic differential equations subject to both Wiener process and Poisson random process (or Lévy process) we expect the solution trajectories to have discontinuities of no more than that of the first kind. In order to include such processes we may introduce the space $B^a_\infty(I, H)$ consisting of $\mathcal{F}_t$-adapted H valued random processes having finite second moments. Here we introduce the norm topology given by

$$\| x \| \equiv \sup\{\big(\mathbf{E} \parallel x(t) \parallel_H^2\big)^{1/2}, \ \ t \in I\}.$$

With respect to this norm topology, $B^a_\infty(I, H)$ is a Banach space.

Theorem 3.1. *Consider the system* (1.1) *and suppose A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$, on H with $S(t), t > 0$, being compact. Then, for each initial state $x_0 \in L_2(\mathcal{F}_0, H)$ (having finite second moment) and each drift-diffusion-jump triple $(F, B, C) \in \mathcal{P}_{ad}$, the stochastic differential equation* (1.1) *has a unique mild solution $x \in B^a_\infty(I, H)$.*

Proof. The proof is fairly standard. We present a brief outline. Before we begin we note that, for proof of existence of solution, compactness of the semigroup $S(t), t > 0$, is not required. We need it for the next theorem. Recall that by a mild solution of equation (1.1) we mean the solution of the corresponding stochastic integral equation,

$$\begin{aligned} x(t) \ &= \ S(t)x_0 + \int_0^t S(t-s)F(x(s))ds + \int_0^t S(t-s)B(x(s))dW(s) \\ &\quad + \int_0^t \int_{V_\delta} S(t-s)C(x(s), \xi)q(d\xi \times ds), \ \ t \in I. \end{aligned} \tag{3.1}$$

The proof is based on Banach fixed point theorem. We present a brief outline. Define the operator Λ given by

$$\begin{aligned}(\Lambda x)(t) &= S(t)x_0 + \int_0^t S(t-s)F(x(s))ds + \int_0^t S(t-s)B(x(s))dW(s)\\ &\quad + \int_0^t \int_{V_\delta} S(t-s)C(x(s),\xi)q(d\xi \times ds), \quad t \in I. \end{aligned} \tag{3.2}$$

We show that $\Lambda : B_\infty^a(I,H) \longrightarrow B_\infty^a(I,H)$. Computing the expected value of the norm square of the process $(\Lambda x)(t)$ and using Fubini's theorem and the properties of Itô integrals we obtain

$$\begin{aligned}\mathbf{E} \parallel (\Lambda x)(t) \parallel_H^2 &\leq 4M^2\mathbf{E} \parallel x_0 \parallel_H^2 + 4tM^2 \int_0^t \mathbf{E} \parallel F(x(s)) \parallel_H^2 ds\\ &\quad + 4M^2 \int_0^t \mathbf{E} \parallel B(x(s))Q^{1/2} \parallel_{\mathcal{L}(U,H)}^2 ds\\ &\quad + 4M^2 \int_0^t \mathbf{E}\Big(\int_{V_\delta} \parallel C(x(s),\xi) \parallel_H^2 \pi(d\xi)\Big)ds, \quad t \in I.\end{aligned} \tag{3.3}$$

By virtue of the property (2.3) of $C \in \mathcal{C}_{a,b}$, one can readily verify that

$$\begin{aligned}\int_{V_\delta} \parallel C(x(s),\xi) \parallel_H^2 \pi(d\xi) &\leq \big(2\int_{V_\delta} a^2(\xi)\pi(d\xi)\big) \parallel x(s)) \parallel_H^2\\ &\quad + \big(2\int_{V_\delta} b^2(\xi)\pi(d\xi)\big).\end{aligned} \tag{3.4}$$

Using the growth and Lipschitz properties (2.1)-(2.3) of the triple $(F,B,C) \in \mathcal{P}_{ad}$, in the inequality (3.3) one can verify that

$$\mathbf{E} \parallel (\Lambda x)(t) \parallel_H^2 \leq C_1(t) + C_2(t)\sup\big\{\mathbf{E} \parallel x(s) \parallel_H^2, \ 0 \leq s \leq t\big\}, \tag{3.5}$$

where

$$C_1(t) = 4M^2\mathbf{E} \parallel x_0 \parallel_H^2 + 8tM^2\big(t\alpha^2 + t \parallel b \parallel_{L_2(\pi)}^2 + TrQ\big)$$

and

$$C_2(t) = 8tM^2\big(K^2[t+TrQ] + \parallel a \parallel_{L_2(\pi)}^2\big), \ t \in I.$$

Hence, for $x \in B_\infty^a(I,H)$, we have

$$\begin{aligned}&\sup\{\mathbf{E} \parallel (\Lambda x)(t) \parallel_H^2, \ t \in I\}\\ &\leq C_1(T) + C_2(T)\sup\big\{\mathbf{E} \parallel x(s) \parallel_H^2, \ 0 \leq s \leq T\big\}.\end{aligned} \tag{3.6}$$

Thus it follows from the above inequality that $\Lambda x \in B_\infty^a(I,H)$ whenever $x \in B_\infty^a(I,H)$ proving that Λ maps $B_\infty^a(I,H)$ to itself. Following similar steps, one can verify that

$$\mathbf{E} \parallel (\Lambda x)(t) - (\Lambda y)(t) \parallel_H^2 \leq \tilde{\beta}(t) \int_0^t \mathbf{E} \parallel x(s) - y(s) \parallel_H^2 ds, \ \ t \in I, \qquad (3.7)$$

where

$$\tilde{\beta}(t) \equiv 4M^2\big(tK^2 + TrQ + \parallel a \parallel_{L_2(\pi)}^2\big), \ \ t \in I.$$

Define

$$\rho_t^2(x,y) \equiv \sup\{\mathbf{E} \parallel x(s) - y(s) \parallel_H^2, \ \ 0 \leq s \leq t\}$$

and

$$\beta \equiv \tilde{\beta}(T) = 4M^2\big(TK^2 + TrQ + \parallel a \parallel_{L_2(\pi)}^2\big),$$

and denote $\rho_t^2(x,y)$ by $\varrho_t(x,y)$ for all $t \in I$. Using these notations in the above inequality we find that

$$\varrho_t(\Lambda x, \Lambda y) \leq \beta \int_0^t \varrho_s(x,y) ds, \ \ t \in I. \qquad (3.8)$$

Let Λ^m denote the m-fold composition of the operator Λ. For $m = 2$, it follows from the above expression that

$$\begin{aligned} \varrho_t(\Lambda^2 x, \Lambda^2 y) &\leq \beta \int_0^t \varrho_s(\Lambda x, \Lambda y) ds \leq \beta^2 \int_0^t s \varrho_s(x,y) ds \\ &\leq \beta^2 (t^2/2!) \varrho_t(x,y), \ \ t \in I. \qquad (3.9) \end{aligned}$$

Repeating this iterative process m times one finds that

$$\varrho_t(\Lambda^m x, \Lambda^m y) \leq \beta^m (t^m/m!) \varrho_t(x,y), \ \ t \in I,$$

and hence $\rho_T(\Lambda^m x, \Lambda^m y) \leq \sqrt{\beta^m (T^m/m!)} \rho_T(x,y)$. In terms of the norm of the Banach space $B_\infty^a(I,H)$, this inequality is equivalent to the following inequality

$$\parallel \Lambda^m x - \Lambda^m y \parallel_{B_\infty^a(I,H)} \leq \gamma_m \parallel x - y \parallel_{B_\infty^a(I,H)},$$

where $\gamma_m \equiv \sqrt{\beta^m (T^m/m!)}$. It is clear that for $m_0 \in N$ sufficiently large, $0 < \gamma_{m_0} < 1$. Thus Λ^{m_0} is a contraction and hence it follows from Banach fixed point theorem that it has a unique fixed point $x^o \in B_\infty^a(I,H)$. This implies that the operator Λ itself has x^o as the unique fixed point. This completes the outline of our proof. □

As indicated in the introduction, our objective is to solve the inverse problem. The problem is to find a drift-diffusion-jump triple $(F^o, B^o, C^o) \in \mathcal{P}_{ad}$ for system (1.1) that minimizes the functional (1.2). The question of existence of an optimal triple is crucial. Before we consider this problem, we prove the continuity of the map $(F,B,C) \longrightarrow x(F,B,C)$ representing the (mild) solution of the stochastic differential equation (1.1) corresponding to the triple (F,B,C). We present this in the following theorem.

Theorem 3.2. *Consider the system* (1.1) *with the admissible set of drift-diffusion-jump triples* $\mathcal{P}_{ad}$ *and suppose the assumptions of Theorem* 3.1 *hold. Then the solution map* $(F, B, C) \longrightarrow x(F, B, C)$ *is continuous with respect to the* τ_p *topology on* $\mathcal{P}_{ad}$ *and the norm topology on the space* $B_\infty^a(I, H)$.

Proof. Let $(F^k, B^k, C^k) \in \mathcal{P}_{ad}$ be a generalized sequence such that it converges to $(F^o, B^o, C^o) \in \mathcal{P}_{ad}$ in the τ_p topology. Let $x^k \in B_\infty^a(I, H)$ be the mild solution of equation (1.1) corresponding to the triple (F^k, B^k, C^k), and $x^o \in B_\infty^a(I, H)$ the mild solution corresponding to the triple (F^o, B^o, C^o). We show that $x^k \xrightarrow{s} x^o$ in the Banach space $B_\infty^a(I, H)$ as the sequence

$$(F^k, B^k, C^k) \xrightarrow{\tau_p} (F^o, B^o, C^o).$$

Clearly, the pair (x^k, x^o) satisfies the following stochastic integral equations

$$\begin{aligned} x^k(t) = S(t)x_0 &+ \int_0^t S(t-s)F^k(x^k(s))ds + \int_0^t S(t-s)B^k(x^k(s))dW(s) \\ &+ \int_0^t \int_{V_\delta} S(t-s)C^k(x^k(s), \xi)q(d\xi \times ds), \quad t \in I, \end{aligned} \tag{3.10}$$

$$\begin{aligned} x^o(t) = S(t)x_0 &+ \int_0^t S(t-s)F^o(x^o(s))ds + \int_0^t S(t-s)B^o(x^o(s))dW(s) \\ &+ \int_0^t \int_{V_\delta} S(t-s)C^o(x^o(s), \xi)q(d\xi \times ds), \quad t \in I. \end{aligned} \tag{3.11}$$

Subtracting equation (3.11) from equation (3.10) term by term we obtain the following identity

$$\begin{aligned} x^k(t) - x^o(t) = &\int_0^t S(t-s)[F^k(x^k(s)) - F^k(x^o(s))]ds \\ &+ \int_0^t S(t-s)[F^k(x^o(s)) - F^o(x^o(s))]ds \\ &+ \int_0^t S(t-s)[B^k(x^k(s)) - B^k(x^o(s))]dW(s) \\ &+ \int_0^t S(t-s)[B^k(x^o(s)) - B^o(x^o(s))]dW(s) \\ &+ \int_0^t \int_{V_\delta} S(t-s)[C^k(x^k(s), \xi) - C^k(x^o(s), \xi)]q(d\xi \times ds) \\ &+ \int_0^t \int_{V_\delta} S(t-s)[C^k(x^o(s), \xi) - C^o(x^o(s), \xi)]q(d\xi \times ds), \quad t \in I. \end{aligned} \tag{3.12}$$

Computing the expected value of the norm square of the fifth term on the right hand side of the above expression using the Lipschitz property of the elements of the set $\mathcal{C}_{a,b}$ and the properties of the compensated Poisson random measure q and Fubini's theorem, we obtain

$$\begin{aligned}
&\mathbf{E} \parallel \int_0^t \int_{V_\delta} S(t-s)[C^k(x^k(s),\xi) - C^k(x^o(s),\xi)]q(d\xi \times ds) \parallel_H^2 \\
&= \mathbf{E} \int_0^t \int_{V_\delta} \parallel S(t-s)[C^k(x^k(s),\xi) - C^k(x^o(s),\xi)] \parallel_H^2 \pi(d\xi)ds \\
&\leq M^2 \mathbf{E} \int_0^t \int_{V_\delta} \parallel C^k(x^k(s),\xi) - C^k(x^o(s),\xi) \parallel_H^2 \pi(d\xi)ds \\
&\leq M^2 \parallel a \parallel_{L_2(\pi)}^2 \int_0^t \mathbf{E} \parallel x^k(s) - x^o(s) \parallel_H^2 ds, \ \ t \in I.
\end{aligned}$$

Using this estimate and computing the expected value of the norm square of the process $[x^k(t) - x^o(t)]$ given by equation (3.12) and following similar procedure, we obtain the following inequality

$$\begin{aligned}
&\mathbf{E} \parallel x^k(t) - x^o(t) \parallel_H^2 \\
&\leq \gamma \int_0^t \mathbf{E} \parallel x^k(s) - x^o(s) \parallel^2 ds \\
&\quad + 8T \int_0^t \mathbf{E} \parallel S(t-s)[F^k(x^o(s)) - F^o(x^o(s))] \parallel_H^2 ds \\
&\quad + 8 \int_0^t \mathbf{E} \parallel S(t-s)(B^k(x^o(s)) - B^o(x^o(s)))Q^{1/2} \parallel_{\mathcal{L}(U,H)}^2 ds \\
&\quad + 8 \int_0^t \int_{V_\delta} \mathbf{E} \parallel S(t-s)[C^k(x^o(s),\xi) - C^o(x^o(s),\xi)] \parallel_H^2 \pi(d\xi)ds, \ t \in I,
\end{aligned} \tag{3.13}$$

where $\gamma \equiv 8M^2 \big[K^2(1+TrQ) + \parallel a \parallel_{L_2(\pi)}^2\big]$. Define

$$e_1^k(t) \equiv 8T \int_0^t \mathbf{E} \parallel S(t-s)[F^k(x^o(s)) - F^o(x^o(s))] \parallel_H^2 ds, \ \ t \in I, \tag{3.14}$$

$$\begin{aligned}
e_2^k(t) &\equiv 8 \int_0^t \mathbf{E} \parallel S(t-s)[B^k(x^o(s)) - B^o(x^o(s))]Q^{1/2} \parallel_{\mathcal{L}(U,H)}^2 ds \\
&= 8 \int_0^t \sum \lambda_i \mathbf{E} \parallel S(t-s)[B^k(x^o(s)) - B^o(x^o(s))]e_i \parallel_H^2 ds, \ \ t \in I,
\end{aligned} \tag{3.15}$$

$$e_3^k(t) \equiv 8 \int_0^t \int_{V_\delta} \mathbf{E} \parallel S(t-s)[C^k(x^o(s),\xi) - C^o(x^o(s),\xi)] \parallel_H^2 \pi(d\xi)ds, \ \ t \in I. \tag{3.16}$$

and

$$\varphi^k(t) \equiv \mathbf{E} \parallel x^k(t) - x^o(t) \parallel_H^2, \;\; t \in I, \tag{3.17}$$

$$e^k(t) \equiv e_1^k(t) + e_2^k(t) + e_3^k(t), \;\; t \in I. \tag{3.18}$$

Using the expressions (3.17) and (3.18) in the inequality (3.13) we obtain the following inequality

$$\varphi^k(t) \le \gamma \int_0^t \varphi^k(s)ds + e^k(t), \;\; t \in I. \tag{3.19}$$

It follows from Grönwall inequality applied to the expression (3.19) that

$$\varphi^k(t) \le e^k(t) + \gamma \int_0^t \{\exp \gamma(t-s)\} e^k(s)ds, \;\; t \in I. \tag{3.20}$$

Considering the expression (3.14), and recalling that $F^k \xrightarrow{\tau_w} F^o$ point wise in H and the semigroup $S(t), t > 0$, is compact, it is easy to verify that the integrand (within the norm symbol) of the expression for $e_1^k(t)$ converges to zero strongly in H for almost all $s \in [0,t)$, P-almost surely and for every $t \in I$. Further, using the growth and Lipschitz properties of the elements of the set $\mathcal{F}_{\alpha,K}$, one can verify that the H-norm of the integrand is dominated by an integrable random process. Thus it follows from Lebesgue dominated convergence theorem that $e_1^k(t)$ converges to zero for each $t \in I$. Considering the expression (3.15) for $e_2^k(t)$, we note that $B^k \xrightarrow{\tau_{wo}} B^o$ (in the weak operator topology of $\mathcal{L}(U,H)$) point wise, and the semigroup $S(t), t > 0$, is compact. Thus each component of the integrand within the norm symbol for the process $e_2^k(t)$ converges strongly to zero in H for almost all $s \in [0,t)$ P-a.s and each $t \in I$. Recall that the operator $Q(\in \mathcal{L}(U))$ is positive nuclear and hence the sum $\sum \lambda_i < \infty$. Further, note that all of the components of the integrand are dominated by a single integrable random process and thus, by dominated convergence theorem, the integral of the sum converges to zero for each $t \in I$. Hence, $e_2^k(t) \longrightarrow 0$ for each $t \in I$. Considering the third component $e_3^k(t)$ for any $t \in I$, we recall that $C^k \xrightarrow{\tau_w} C^o$ in H point wise on X. Thus again, by virtue of compactness of the semigroup, we conclude that the norm of the integrand converges strongly in H. Since $x^o \in B_\infty^a(I,H)$ it follows from the properties (i) and (ii) of the set $\mathcal{C}_{a,b}$ that the integrand is dominated by a square integrable random process. Hence, again by dominated convergence theorem, it follows from the expression (3.16) that $e_3^k(t) \longrightarrow 0$ for each $t \in I$. Since $e^k(t)$, given by the expression (3.18), is uniformly bounded on I and converges to zero for each $t \in I$, it follows from Lebesgue bounded convergence theorem that the integral in the expression (3.20) converges to zero. Thus it follows from the inequality (3.20) that $\varphi^k(t) \to 0$ uniformly on I. This proves the continuity of the map $(F,B,C) \longrightarrow x(F,B,C)$ from $\mathcal{P}_{ad}$ to $B^a_\infty(I,H)$ in their respective topologies. This completes the proof. $\square$

Remark 3.3. For proof of convergence in the strong (norm) topology, we have used in the above theorem the compactness property of the semigroup $S(t)$ for $t>0$. This is easily verified by following the same technique as seen in [3, Theorem 4.4, p.3180]. Since similar technique applies to $\{e_1^k, e_2^k\}$ also, we demonstrate this for the component $\{e_3^k\}$ only. For any $t \in (0,T]$ and for any $0<\varepsilon<t$, we rewrite the expression (3.16) as follows:

$$\begin{aligned}
e_3^k(t) &\equiv 8\int_{[0,t]\times V_\delta} \mathbf{E} \parallel S(t-s)[C^k(x^o(s),\xi) - C^o(x^o(s),\xi)] \parallel_H^2 \pi(d\xi)ds \\
&= 8\int_{[0,t-\varepsilon]\times V_\delta} \mathbf{E}||S(\varepsilon)\big\{S(t-\varepsilon-s)[C^k(x^o(s),\xi) - C^o(x^o(s),\xi)]\big\}||_H^2 \pi(d\xi)ds \\
&\quad + 8\int_{t-\varepsilon}^t \int_{V_\delta} \mathbf{E} \parallel S(t-s)[C^k(x^o(s),\xi) - C^o(x^o(s),\xi)] \parallel_H^2 \pi(d\xi)ds, \quad t \in I.
\end{aligned}$$

Since $C^k \xrightarrow{\tau_w} C^o$ in H point wise, the term within the curly bracket converges to zero weakly in H for almost all $s \in I$, P-a.s and the operator $S(\varepsilon)$ is compact, it follows from the arguments given in the proof that the first integral on the righthand side converges to zero as $k \to \infty$. Using the growth properties of $\mathcal{C}_{a,b}$ and recalling that $x^o \in B^a_\infty(I,H)$, one can easily verify that the second term satisfies the following inequality,

$$\begin{aligned}
&8\int_{t-\varepsilon}^t \int_{V_\delta} \mathbf{E} \parallel S(t-s)[C^k(x^o(s),\xi) - C^o(x^o(s),\xi)] \parallel_H^2 \pi(d\xi)ds \\
&\leq 32M^2\Big(\parallel b \parallel_{L_2(\pi)}^2 + \parallel a \parallel_{L_2(\pi)}^2 \parallel x^o \parallel_{B_\infty(I,H)}^2\Big)\varepsilon.
\end{aligned}$$

Thus our conclusion stating that $e_3^k(t)$ converges to zero point wise on I is verified.

Now we are prepared to prove existence of optimal drift-diffusion-jump triple.

Theorem 3.4. *Consider the system* (1.1) *with the objective functional* (1.2) *and admissible set of drift-diffusion-jump triples $\mathcal{P}_{ad}$. Suppose the assumptions of Theorem* 3.2 *hold and that ℓ is a real valued Borel measurable function on $I\times H$ and lower semi-continuous in the second variable, and Φ is also a Borel measurable real valued function and lower semi-continuous on H satisfying the* *following growth properties:*

$$|\ell(t,x)| \leq \alpha_1(t) + \alpha_2 \parallel x \parallel_H^2, \tag{3.21}$$

$$|\Phi(x)| \leq \alpha_3 + \alpha_4 \parallel x \parallel^2 \tag{3.22}$$

for $\alpha_1 \in L_1^+(I)$*, and* $\alpha_2, \alpha_3, \alpha_4 > 0$*. Then there exists an optimal triple* $(F^o, B^o, C^o) \in \mathcal{P}_{ad}$ *that minimizes the cost functional* (1.2).

Proof. Since the set $\mathcal{P}_{ad}$ is compact in the point wise topology, it suffices to show that J is lower semicontinuous in this topology. Let $(F^k, B^k, C^k) \in \mathcal{P}_{ad}$ be a generalized sequence converging to $(F^o, B^o, C^o) \in \mathcal{P}_{ad}$ in the point wise topology. Let $(x^k, x^o) \in B_\infty^a(I,H)$ denote the corresponding mild solutions of equation (1.1). It follows from the continuity Theorem 3.2 that $x^k \xrightarrow{s} x^o$ in $B_\infty^a(I,H)$. Since ℓ is lower semicontinuous in the state variable it is clear that

$$\ell(t, x^o(t)) \leq \underline{\lim}_{k\to\infty} \ell(t, x^k(t)) \text{ for a.e } t \in I,\ P - a.s. \tag{3.23}$$

The elements of $\mathcal{P}_{ad}$ have at most linear growth and therefore the solutions $\{(x^k, x^o)\}$ are contained in a bounded subset of $B_\infty^a(I,H)$. Thus it follows from the growth property of ℓ as described by the inequality (3.21), that $\ell(t, x^k(t)), t \in I$, is dominated from bellow by an integrable random process. Hence by virtue of generalized Fatou's Lemma we conclude that

$$\mathbf{E}\int_0^T \ell(t, x^o(t))dt \leq \mathbf{E}\int_0^T \underline{\lim}_{k\to\infty} \ell(t, x^k(t))dt \leq \underline{\lim}_{k\to\infty} \mathbf{E}\int_0^T \ell(t, x^k(t))dt. \tag{3.24}$$

Since Φ is also lower semicontinuous on H and has the growth property (3.22), it follows from similar argument that

$$\mathbf{E}\Phi(x^o(T)) \leq \mathbf{E}\,\underline{\lim}_{k\to\infty} \Phi(x^k(T)) \leq \underline{\lim}_{k\to\infty} \mathbf{E}\Phi(x^k(T)). \tag{3.25}$$

Sum of lower semi continuous functionals is lower semi continuous. Thus by adding (3.24) and (3.25) we obtain

$$J(F^o, B^o, C^o) \leq \underline{\lim}_{k\to\infty} J(F^k, B^k, C^k)$$

proving lower semicontinuity of J on $\mathcal{P}_{ad}$ in the point wise topology τ_p. Hence it follows from compactness of the set $\mathcal{P}_{ad}$ that J attains its minimum on $\mathcal{P}_{ad}$. This completes the proof of existence of an optimal drift-diffusion-jump triple as stated. □

Remark 3.5. The results presented above also hold for drift-diffusion-jump triples which are functions of both time and space $\{F(t,x), B(t,x), C(t,x,\xi)\}$ under the assumption that the family of functions $\{\mathcal{F}_{\alpha,K}, \mathcal{B}_{\alpha,K}, \mathcal{C}_{a,b}\}$ satisfy the properties (2.1),(2.2) and (2.3) and that they are also uniformly Hölder continuous exponent $0 < \theta < 1$ in $t \in I$.

4. Necessary conditions of optimality

In this section we present the necessary conditions of optimality characterizing the optimal drift-diffusion-jump triple whereby one can determine the optimal triple from the admissible class $\mathcal{P}_{ad}$ and hence construct the stochastic dynamic model. We recall that $B^a_\infty(I,H) \subset L^a_\infty(I, L_2(\Omega, H))$ denotes the space of $\mathcal{F}_t$-adapted $L_2(\Omega, H)$ valued norm bounded measurable processes defined on I. Similarly, $B^a_\infty(I, \mathcal{L}(U,H)) \subset L^a_\infty(I, L_2(\Omega, \mathcal{L}(U,H)))$ denotes the space of $\mathcal{F}_t$-adapted $L_2(\Omega, \mathcal{L}(U,H))$ valued norm bounded measurable processes on I. For convenience of notation we use $\{DF, DB, DC\}$ to denote respectively the Gâteaux differentials (directional derivatives) of $\{F, B, C\}$ in the state variable $x \in H$. Throughout the rest of the paper we assume that the initial state x_0, the Wiener process W, and the compensated Poisson random measure q are mutually stochastically independent.

Theorem 4.1. *Consider the system given by equation* (1.1) *with* $(F, B, C) \in \mathcal{P}_{ad}$ *and the cost functional given by* (1.2). *Suppose the assumptions of Theorem* 3.4 *hold and that the elements of* $\mathcal{P}_{ad}$ *are once continuously Gâteaux differentiable in the state variable with the derivatives uniformly bounded. Then, in order for the triple* $(F^o, B^o, C^o) \in \mathcal{P}_{ad}$ *with the corresponding solution* $x^o \in B^a_\infty(I,H)$ *to be optimal, it is necessary that there exists a triple* $(\psi, \Xi, \varphi) \in B^a_\infty(I,H) \times L^a_\infty(I, \mathcal{L}(U,H)) \times L^a_\infty(I \times V, H)$ *satisfying the inequality* (4.1) *and the stochastic adjoint and state differential equations* (4.2)-(4.3) *as presented below:*

$$
\begin{aligned}
&\mathbf{E}\int_I < F(x^o) - F^o(x^o), \psi >_H dt + \mathbf{E}\int_I Tr[(B(x^o) - B^o(x^o))\; Q\; \Xi^*]dt \\
&\quad + \mathbf{E}\int_I \int_{V_\delta} < C(x^o,\xi) - C^o(x^o,\xi), \varphi >_H \pi(d\xi)dt \geq 0 \\
&\geq 0,\ \forall\ (F,B,C) \in \mathcal{P}_{ad},
\end{aligned}
\tag{4.1}
$$

where $\Xi(t) \equiv -DB^o(x^o(t);\psi(t))$ *and* $\varphi(t,\xi) \equiv -(DC^o(x^o(t),\xi))^*\psi(t)$, $(t,\xi) \in I \times V_\delta$, *with* $\{DB^o, DC^o\}$ *denoting the Gâteaux derivatives of* $\{B^o, C^o\}$ *with respect to the state variable evaluated at* x^o. *The function* ψ *denotes the solution of the following adjoint equation,*

$$
\begin{aligned}
-d\psi = {} & A^*\psi dt + (DF^o(x^o(t)))^*\psi dt + V_1(x^o(t))\psi dt + V_2(x^o(t))\psi dt \\
& + \ell_x(t, x^o(t))dt + DB^o(x^o(t);\psi(t))dW \\
& + \int_{V_\delta} (DC^o(x^o,\xi))^*\psi q(d\xi \times dt)
\end{aligned}
\tag{4.2}
$$

for $t \in I$, $\psi(T) = \Phi_x(x^o(T))$, *where* $V_1(x^o(t))$, $t \in I$, *is a non-positive symmetric* $\mathcal{L}(H)$ *(bounded linear operators in* H*) valued random process following from the bilinear form*

$$-Tr\big(DB^o(x^o;y)Q(DB^o(x^o;\psi))^*\big) \equiv < V_1(x^o)y, \psi >;$$

and $V_2(x^o(t))$, $t \in I$, *is another non-positive symmetric* $\mathcal{L}(H)$ *valued random process given by the bilinear form*

$$-\int_{V_\delta} < DC^o(x^o,\xi)y, (DC^o(x^o,\xi))^*\psi > \equiv < V_2(x^o(t))y, \psi >$$

with $x^o \in B^a_\infty(I,H)$ *being the solution of the sate equation*

$$dx^o = Ax^o dt + F^o(x^o)dt + B^o(x^o)dW + \int_{V_\delta} C^o(x^o,\xi)q(d\xi \times dt) \tag{4.3}$$

for $t \in I$, $x^o(0) = x_0$.

Proof. Let $(F^o, B^o, C^o) \in \mathcal{P}_{ad}$ be the optimal drift-diffusion-jump triple with the corresponding (mild) solution of equation (1.1) denoted by $x^o \in B^a_\infty(I,H)$. Let $(F,B,C) \in \mathcal{P}_{ad}$ be an arbitrary element and $\varepsilon \in [0,1]$. Define the triple $(F^\varepsilon, B^\varepsilon, C^\varepsilon)$ as follows

$$F^\varepsilon \equiv F^o + \varepsilon(F - F^o),\; B^\varepsilon = B^o + \varepsilon(B - B^o),\; C^\varepsilon = C^o + \varepsilon(C - C^o),\; \varepsilon \in [0,1].$$

It follows from convexity of the set $\mathcal{P}_{ad}$ that $(F^\varepsilon, B^\varepsilon, C^\varepsilon) \in \mathcal{P}_{ad}$, and by virtue of optimality of the triple (F^o, B^o, C^o), we have

$$J(F^\varepsilon, B^\varepsilon, C^\varepsilon) \geq J(F^o, B^o, C^o), \quad \forall\, \varepsilon \in [0,1]. \tag{4.4}$$

Let $x^\varepsilon \in B^a_\infty(I,H)$ denote the (mild) solution of equation (1.1) corresponding to the triple $(F^\varepsilon, B^\varepsilon, C^\varepsilon)$. Clearly, the processes $\{x^\varepsilon, x^o\}$ satisfy respectively the following stochastic integral equations,

$$\begin{aligned} x^\varepsilon(t) = S(t)x_0 &+ \int_0^t S(t-s)F^\varepsilon(x^\varepsilon(s))ds + \int_0^t S(t-s)B^\varepsilon(x^\varepsilon(s))dW(s) \\ &+ \int_0^t \int_{V_\delta} S(t-s)C^\varepsilon(x^\varepsilon(s),\xi)q(d\xi \times ds), \quad t \in I, \end{aligned} \tag{4.5}$$

$$\begin{aligned} x^o(t) = S(t)x_0 &+ \int_0^t S(t-s)F^o(x^o(s))ds + \int_0^t S(t-s)B^o(x^o(s))dW(s) \\ &+ \int_0^t \int_{V_\delta} S(t-s)C^o(x^o(s),\xi)q(d\xi \times ds), \quad t \in I \end{aligned} \tag{4.6}$$

and they are elements of $B^a_\infty(I,H)$. Clearly $(F^\varepsilon, B^\varepsilon, C^\varepsilon) \xrightarrow{\tau_p} (F^o, B^o, C^o)$ and hence it follows from Theorem 3.2 that $x^\varepsilon \xrightarrow{s} x^o$ in $B^a_\infty(I,H)$. As justified below, the process y given by the following limit

$$y(t) = \lim_{\varepsilon \downarrow 0} (1/\varepsilon)(x^\varepsilon(t) - x^o(t)), \quad t \in I,$$

exists and belongs to $B^a_\infty(I, H)$. Subtracting equation (4.6) from equation (4.5) term by term and dividing by ε and letting $\varepsilon \downarrow 0$ one can easily verify that y satisfies the following stochastic integral equation,

$$\begin{aligned} y(t) = &\int_0^t S(t-s) DF^o(x^o(s)) y(s) ds \\ &+ \int_0^t S(t-s)[F(x^o(s)) - F^o(x^o(s))] ds \\ &+ \int_0^t S(t-s) DB^o(x^o(s); y(s)) dW(s) \\ &+ \int_0^t S(t-s)[B(x^o(s)) - B^o(x^o(s))] dW(s) \\ &+ \int_0^t \int_{V_\delta} S(t-s) DC^o(x^o(s), \xi) y(s) q(d\xi \times ds) \\ &+ \int_0^t \int_{V_\delta} S(t-s)[C(x^o(s), \xi) - C^o(x^o(s), \xi)] q(d\xi \times ds), \quad t \in I, \end{aligned} \tag{4.7}$$

where for any $x, z \in H$, $DF^o(x)z$ denotes the Gâteaux differential of F^o evaluated at x in the direction z with $DF^o(x) \in \mathcal{L}(H)$, and $DB^o(x; z)$ denotes the Gâteaux differential of B^o evaluated at x in the direction z with $DB^o(x, \cdot) \in \mathcal{L}(H, \mathcal{L}(U, H))$, and similarly, for any $\xi \in V_\delta$, $DC^o(x, \xi))z$ denotes the Gâteaux differential of C^o evaluated at $x \in H$ in the direction $z \in H$ with $DC^o(x, \xi) \in \mathcal{L}(H)$. By assumption these Gâteaux derivatives are uniformly bounded in $x \in H$. It follows from the integral equation (4.7) that y is the mild solution (if one exists) of the following linear stochastic differential equation on H,

$$\begin{aligned} dy &= Aydt + DF^o(x^o(t)) y(t) dt + DB^o(x^o(t); y(t)) dW(t) \\ &\quad + \int_{V_\delta} DC^o(x^o(t), \xi)) y(t) q(d\xi \times dt) + dM_t^{F,B,C}, \\ y(0) &= 0, \quad t \in I, \end{aligned} \tag{4.8}$$

driven by the (semimartingale) process $M^{F,B,C} \equiv \{M_t^{F,B,C}, t \in I\}$ which is given by

$$\begin{aligned} dM_t^{F,B,C} &= [F(x^o(t)) - F^o(x^o(t))] dt + [B(x^o(t)) - B^o(x^o(t))] dW(t) \\ &\quad + \int_{V_\delta} [C(x^o(t), \xi) - C^o(x^o(t), \xi)] q(d\xi \times dt), \quad t \in I. \end{aligned} \tag{4.9}$$

Let $\mathcal{S}M_2$ denote the Hilbert space of norm square integrable H valued $\mathcal{F}_t$-adapted semi martingales $\{M_t, t \geq 0\}$ starting from the origin, that is $M_0 = 0$. Since $(F,B,C),(F^o,B^o,C^o) \in \mathcal{P}_{ad}$ and $x^o \in B_\infty^a(I,H)$, it is straightforward to verify that the drift (vector) $[F(x^o(\cdot)) - F^o(x^o(\cdot))]$, the diffusion (operator) $[B(x^o(\cdot)) - B^o(x^o(\cdot))]$, and the jump kernel $[C(x^o(\cdot),\xi)) - C^o(x^o(\cdot),\xi)], \xi \in V_\delta$, are norm square integrable $\mathcal{F}_t$-adapted random processes. Hence $M^{F,B,C} \in \mathcal{S}M_2$ and therefore, as a special case, it follows from Theorem 3.1 that equation (4.8) has a unique mild solution $y \in B_\infty^a(I,H)$. Thus $M^{F,B,C} \longrightarrow y$ is a bounded linear map, denoted by Υ, from the Hilbert space $\mathcal{S}M_2$ to the Banach space $B_\infty^a(I,H)$ and hence continuous. We denote this by $y = \Upsilon(M^{F,B,C})$. Using the inequality (4.4) and dividing the following expression,

$$J(F^\varepsilon, B^\varepsilon, C^\varepsilon) - J(F^o,B^o,C^o) \geq 0, \ \forall \ \varepsilon \in [0,1],$$

by ε and letting $\varepsilon \downarrow 0$ we obtain the Gâteaux differential of J at $(F^o,B^o,C^o) \in \mathcal{P}_{ad}$ in the direction $(F-F^o, B-B^o, C-C^o)$ satisfying the following inequality,

$$\begin{aligned} &dJ((F^o,B^o,C^o),(F-F^o,B-B^o,C-C^o)) \\ &= \mathbf{E}\left\{\int_0^T < \ell_x(t,x^o(t)), y(t) >_H dt + < \Phi_x(x^o(T)), y(T) >_H \right\} \geq 0 \\ &\text{for all } (F,B,C) \in \mathcal{P}_{ad}. \end{aligned} \tag{4.10}$$

For notational convenience, we introduce the following linear functional:

$$L(y) \equiv \mathbf{E}\left\{\int_0^T < \ell_x(t,x^o(t)), y(t) >_H dt + < \Phi_x(x^o(T)), y(T) >_H \right\}. \tag{4.11}$$

Since $\ell_x(\cdot, x^o(\cdot)) \in L_1^a(I, L_2(\Omega,H))$, $y \in B_\infty^a(I,H) \subset L_\infty^a(I, L_2(\Omega,H))$, and $\Phi_x(x^o(T)) \in L_2(\Omega, \mathcal{F}_T, H)$ and $y(T) \in L_2(\Omega,\mathcal{F}_T,H)$, we conclude that $y \longrightarrow L(y)$ is a continuous linear functional on $B_\infty^a(I,H)$. Hence it follows from the above analysis that the functional $\tilde{L}$, given by the composition map

$$M^{F,B,C} \longrightarrow y \longrightarrow L(y) = (L \circ \Upsilon)(M^{F,B,C}) \equiv \tilde{L}(M^{F,B,C}), \tag{4.12}$$

is a continuous linear functional on the Hilbert space of semi martingales $\mathcal{S}M_2$. Thus it follows from representation of Hilbert space valued semimartingales and Riesz representation theorem for Hilbert spaces that there exists a triple

$$\{\psi, \Xi, \varphi\} \in L_2^a(I, L_2(\Omega,H)) \times L_2^a(I, L_2(\Omega, \mathcal{L}(U,H))) \times L_2^a(I, L_2(\pi, L_2(\Omega,H)))$$

such that

$$\tilde{L}(M^{F,B,C}) = \mathbf{E}\int_0^T < F(x^o(s)) - F^o(x^o(s)), \psi(s) >_H ds$$
$$+\mathbf{E}\int_0^T Tr\{(B(x^o(s)) - B^o(x^o(s)))Q\Xi^*(s)\}ds \tag{4.13}$$
$$+\mathbf{E}\int_0^T \int_{V_\delta} < C(x^o(s),\xi) - C^o(x^o(s),\xi), \varphi(s,\xi) >_H \pi(d\xi)ds.$$

Hence, it follows from (4.10),(4.11),(4.12) and (4.13)that

$$dJ((F^o,B^o,C^o);(F-F^o,B-B^o,C-C^o))$$
$$= \mathbf{E}\int_0^T < F(x^o(s)) - F^o(x^o(s)), \psi(s) > ds$$
$$+\mathbf{E}\int_0^T Tr[(B(x^o(s)) - B^o(x^o(s)))\ Q\ \Xi^*(s)]ds \tag{4.14}$$
$$+\mathbf{E}\int_0^T \int_{V_\delta} < C(x^o(s),\xi) - C^o(x^o(s),\xi), \varphi(s,\xi) > \pi(d\xi)ds$$
$$\geq 0$$

for all $(F,B,C) \in \mathcal{P}_{ad}$. This proves the necessary condition (4.1). We show that the triple (ψ, Ξ, φ) is given by the solution of the adjoint equation (4.2). Since $y \in B_\infty^a(I,H) \subset L_2^a(I, L_2(\Omega,H))$ and $\psi \in L_2^a(I, L_2(\Omega,H))$, the scalar product $< y, \psi >$ is well defined for almost all $t \in I$, $P - a.s.$ Computing the Itô differential of this scalar product we have

$$d < y,\psi > = < dy,\psi > + < y, d\psi > + << dy, d\psi >>, \tag{4.15}$$

where the third component on the right hand side of the above equation denotes the quadratic variation term. Using the stochastic variational equation (4.8) in the first term on the right hand side of the above expression we obtain

$$< dy,\psi > + < y, d\psi >$$
$$= < Aydt + DF^o(x^o)ydt + DB^o(x^o;y)dW, \psi >$$
$$+ < \int_{V_\delta} DC^o(x^o,\xi)yq(d\xi \times dt), \psi >$$
$$+ < dM^{F,B,C}, \psi > + < y, d\psi >$$
$$= < y, d\psi + A^*\psi dt + (DF^o(x^o))^*\psi dt + DB^o(x^o;\psi)dW >$$
$$+ < y, \int_{V_\delta} (DC^o(x^o,\xi))^*\psi q(d\xi \times dt) > + < \psi, dM^{F,B,C} > . \tag{4.16}$$

In order to consider the quadratic variation term in equation (4.15), let us note that the variational equation for y given by (4.8)-(4.9) contains (the sum of) four martingale terms as follows,

$$DB^o(x^o;y)dW + [B(x^o) - B^o(x^o)]dW$$
$$+ \int_{V_\delta} DC^o(x^o,\xi)yq(d\xi \times dt) + \int_{V_\delta} [C(x^o,\xi) - C^o(x^o,\xi)]q(d\xi \times dt).$$

In contrast, it is clear from the expression (4.16) that the equation for ψ contains at most the sum of two martingale terms given by

$$-DB^o(x^o;\psi)dW - \int_{V_\delta} (DC^o(x^o,\xi))^*\psi q(d\xi \times dt).$$

Hence the quadratic variation term is given by

$$<< dy, d\psi >>=<< dy, d\psi >>_1 + << dy, d\psi >>_2$$

where the first term corresponds to the Wiener martingale and the second term corresponds to the martingale generated by the Lévy (poisson) jump process. Integrating the first term of the quadratic variation we obtain,

$$\begin{aligned}
&\mathbf{E}\int_I << dy, d\psi >>_1 \\
&= -\mathbf{E}\int_I << DB^o(x^o;y)dW + [B(x^o) - B^o(x^o)]dW, DB^o(x^o,\psi)dW >> \\
&= -\mathbf{E}\int_I Tr\{(DB^o(x^o;y))Q(DB^o(x^o;\psi))^*\}dt \\
&\quad - \mathbf{E}\int_I Tr\{[B(x^o) - B^o(x^o)]Q[DB^o(x^o;\psi)]^*\}dt \qquad (4.17)\\
&\equiv \mathbf{E}\int_I \big\{< y, V_1(x^o)\psi > - Tr\{[B(x^o) - B^o(x^o)]Q[DB^o(x^o;\psi)]^*\}\big\}dt.
\end{aligned}$$

Note that $V_1(x^o(t))$, $t \in I$, is a non-positive symmetric $\mathcal{L}(H)$ valued essentially norm bounded random process following from the first component of the above quadratic variation. Similarly, integrating the second quadratic variation we obtain,

$$\begin{aligned}
&\mathbf{E}\int_I << dy, d\psi >>_2 \\
&= -\mathbf{E}\int_I\int_{V_\delta} < DC^o(x^o,\xi)y, (DC^o(x^o,\xi))^*\psi > \pi(d\xi)dt \\
&\quad - \mathbf{E}\int_I\int_{V_\delta} < C(x^o,\xi) - C^o(x^o,\xi), (DC^o(x^o,\xi))^*\psi > \pi(d\xi)dt
\end{aligned}$$

$$\equiv \mathbf{E}\int_I < y, V_2(x^o(t))\psi > dt$$
$$- \mathbf{E}\int_I\int_{V_\delta} < C(x^o,\xi) - C^o(x^o,\xi), (DC^o(x^o,\xi))^*\psi > \pi(d\xi)dt, \tag{4.18}$$

where $V_2(x^o(t))$, $t \in I$, is a non-positive symmetric $\mathcal{L}(H)$ valued essentially norm bounded random process following from the first component of the above quadratic variation term. Integrating the expression (4.15) and substituting the expressions (4.16), (4.17) and (4.18) we arrive at the following expression

$$\begin{aligned}
&\mathbf{E}\int_I d < y, \psi > \\
&= \mathbf{E}\int_I < y, \big\{d\psi + A^*\psi dt + (DF^o(x^o))^*\psi dt + [V_1(x^o) + V_2(x^o(t))]\psi dt\big\} > \\
&\quad + \mathbf{E}\int_I < y, \big\{DB^o(x^o;\psi)dW + \int_{V_\delta}(DC^o(x^o,\xi))^*\psi q(d\xi,dt)\big\} > \\
&\quad + \mathbf{E}\int_I \big\{< \psi, dM^{F,B,C} > - Tr\{[B(x^o) - B^o(x^o)]Q(DB^o(x^o;\psi))^*\}dt\big\} \\
&\quad - \mathbf{E}\int_I\int_{V_\delta} < C(x^o,\xi) - C^o(x^o,\xi), (DC^o(x^o,\xi))^*\psi > \pi(d\xi)dt.
\end{aligned} \tag{4.19}$$

Then setting

$$\begin{aligned}
&d\psi + A^*\psi dt + (DF^o(x^o))^*\psi dt + [V_1(x^o) + V_2(x^o)]\psi dt \\
&\qquad + DB^o(x^o;\psi)dW + \int_{V_\delta}(DC^o(x^o,\xi))^*\psi q(d\xi,dt) \\
&= -\ell_x(t,x^o)dt, \quad t \in I,
\end{aligned} \tag{4.20}$$

in the expression (4.19) we obtain

$$\begin{aligned}
&\mathbf{E}\int_I d < y, \psi > \\
&= -\mathbf{E}\int_I < y, \ell_x(t,x^o)dt > + \mathbf{E}\int_I < \psi, dM^{a,b,c} > \\
&\quad - \mathbf{E}\int_I Tr\{[B(x^o) - B^o(x^o)]Q(DB^o(x^o;\psi))^*\}dt \\
&\quad - \mathbf{E}\int_I\int_{V_\delta} < C(x^o,\xi) - C^o(x^o,\xi), (DC^o(x^o,\xi))^*\psi > \pi(d\xi)dt.
\end{aligned} \tag{4.21}$$

Next, using the identity (4.9) (characterizing the semi martingale $M^{F,B,C}$) in the above expression and integrating we obtain

$$\mathbf{E} < y(T), \psi(T) > + \mathbf{E} \int_0^T < y(t), \ell_x(t, x^o(t)) > dt$$
$$= \mathbf{E} \int_0^T \Big\{ < \psi, [F(x^o) - F^o(x^o)] >$$
$$- Tr\big([B(x^o) - B^o(x^o)] Q [DB^o(x^o; \psi)]^*\big)$$
$$- \int_{V_\delta} < [C(x^o, \xi) - C^o(x^o, \xi)], (DC^o(x^o, \xi))^* \psi > \pi(d\xi) \Big\} dt$$
$$+ \mathbf{E} \int_0^T < \psi, [B(x^o(t)) - B^o(x^o(t))] dW(t) >$$
$$+ \mathbf{E} \int_0^T \int_{V_\delta} < \psi, [C(x^o, \xi) - C^o(x^o, \xi)] > q(d\xi \times dt). \tag{4.22}$$

Using stopping time argument one can verify that the last two stochastic integrals in equation (4.22) vanish. Hence, for $\psi(T) = \Phi_x(x^o(T))$, the identity (4.22) reduces to the following one,

$$\mathbf{E} < y(T), \Phi_x(x^o(T)) > + \mathbf{E} \int_0^T < y(t), \ell_x(t, x^o(t)) > dt$$
$$= \mathbf{E} \int_0^T \Big\{ < \psi, [F(x^o) - F^o(x^o)] > - Tr\big([B(x^o) - B^o(x^o)] Q [DB^o(x^o; \psi)]^*\big)$$
$$- \int_{V_\delta} < [C(x^o, \xi) - C^o(x^o, \xi)], [(DC^o(x^o, \xi))^* \psi] > \pi(d\xi) \Big\} dt. \tag{4.23}$$

Taking

$$\Xi(t) \equiv -DB^o(x^o(t); \psi(t)), t \in I,$$
$$\varphi(t, \xi) \equiv -(DC^o(x^o(t), \xi))^* \psi, (t, \xi) \in I \times V_\delta,$$

and using this in the above expression, it is easy to see that the right hand member equals $\tilde{L}(M^{F,B,C})$ as seen in equation (4.13), while the left hand member equals $L(y)$ as seen in the expression (4.11), thereby satisfying the required identity (4.12). As seen above in the expression (4.14), this gives the necessary condition (4.1). Thus equation (4.20) with the terminal condition $\psi(T) = \Phi_x(x^o(T))$ is a necessary condition. Hence equation (4.2), being identical to equation (4.20) with the terminal condition as stated above, is a necessary condition giving the adjoint equation. Necessary condition (4.3) needs no proof since it is the system equation (1.1) corresponding to the optimal drift-diffusion-jump triple (F^o, B^o, C^o) with x^o being the corresponding solution. This proves all the necessary conditions of optimality. □

Remark 4.2. Let us note that Remark 3.5 also holds for the necessary conditions of optimality given by the above theorem.

5. Convergence of numerical algorithm

Here we present an algorithm whereby one can construct the optimal drift-diffusion-jump triple.

Proposition 5.1. *Suppose the assumptions of Theorem* 4.1 *hold. Then there exists (and one can construct) a sequence* $\{(F^k, B^k, C^k)\} \in \mathcal{P}_{ad}$ *along which the corresponding sequence of values of the cost functional* $\{J(F^k, B^k, C^k)\}$ *converges monotonically to a (possibly) local minimum.*

Proof. We will divide the proof into five steps.

(Step 1): Choose a triple $(F^1, B^1, C^1) \in \mathcal{P}_{ad}$ and consider the system equation (4.3) with (F^o, B^o, C^o) replaced by the triple (F^1, B^1, C^1) and let x^1 denote the corresponding solution.

(Step 2): Use the quadruple (F^1, B^1, C^1, x^1) in place of (F^o, B^o, C^o, x^o) in the adjoint equation (4.2) with $V_1(x^1(t))$ and $V_2(x^1(t))$ given by

$$
\begin{aligned}
&< V_1(x^1(t))\eta_1, \eta_2 > \\
&= -Tr\big\{DB^1(x^1; \eta_1)Q(DB^1(x^1; \eta_2))^*\big\}, \text{ for } \eta_1, \eta_2 \in H, \ \ t \in I,
\end{aligned}
$$

$$
\begin{aligned}
&< V_2(x^1(t))\eta_1, \eta_2 > \\
&= -\int_{V_\delta} < DC^1(x^1, \xi)\eta_1, (DC^1(x^1, \xi))^*\eta_2 > \pi(d\xi), \text{ for } \eta_1, \eta_2 \in H, \ \ t \in I,
\end{aligned}
$$

and solve this adjoint equation for ψ^1. Then define

$$
\begin{aligned}
&\Xi(t) = \Xi^1(t) \equiv -DB^1(x^1(t); \psi^1(t)), t \in I, \\
&\varphi(t, \xi) = \varphi^1(t, \xi) = -(DC^1(x^1(t), \xi))^*\psi^1(t), (t, \xi) \in I \times V_\delta.
\end{aligned}
$$

This step yields the septuple $(F^1, B^1, C^1, x^1, \psi^1, \Xi^1, \varphi^1)$.

(Step 3): At this step, replace the septuple $(F^o, B^o, C^o, x^o, \psi^o, \Xi^o, \varphi^o)$ by the septuple $(F^1, B^1, C^1, x^1, \psi^1, \Xi^1, \varphi^1)$ in the inequality (4.1) giving

$$
\begin{aligned}
&\mathbf{E}\int_0^T < F(x^1(t)) - F^1(x^1(t)), \psi^1(t) > dt \\
&\quad + \mathbf{E}\int_0^T Tr\big\{(B(x^1(t)) - B^1(x^1(t)))Q\Xi^1(t)^*\big\}dt \\
&\quad + \mathbf{E}\int_{I\times V_\delta} < [C(x^1(t), \xi) - C^1(x^1(t), \xi)], \varphi^1(t, \xi)\pi(d\xi)dt \\
&\geq 0, \ \forall \ (F, B, C) \in \mathcal{P}_{ad}. \qquad (5.1)
\end{aligned}
$$

If this inequality holds then the septuple $(F^1, B^1, C^1, x^1, \psi^1, \Xi^1, \varphi^1)$ is optimal. Since an arbitrary choice of the triple (F^1, B^1, C^1) is not expected to be optimal we must ignore this and proceed to the next step.

(Step 4): Here we choose a new triple (F^2, B^2, C^2) as follows

$$F^2 \equiv F^1 - \varepsilon\psi^1, B^2 \equiv B^1 - \varepsilon\Xi^1, C^2 = C^1 - \varepsilon\varphi^1, \tag{5.2}$$

where $\varepsilon > 0$ is chosen sufficiently small so that $(F^2, B^2, C^2) \in \mathcal{P}_{ad}$. Then the Gâteaux differential of the cost functional J evaluated at (F^1, B^1, C^1) in the direction $-(\psi^1, \Xi^1, \varphi^1)$ with step size $\varepsilon > 0$ is given by

$$\begin{aligned} dJ((F^1, B^1, C^1); -\varepsilon(\psi^1, \Xi^1, \varphi^1)) = &-\varepsilon\mathbf{E}\int_0^T \parallel \psi^1(t) \parallel^2 dt \\ &- \varepsilon\mathbf{E}\int_0^T Tr(\Xi^1(t)Q(\Xi^1(t))^*)dt \\ &- \varepsilon\mathbf{E}\int_{I\times V_\delta} \parallel \varphi^1(t,\xi) \parallel_H^2 \pi(d\xi)dt. \end{aligned} \tag{5.3}$$

For notational convenience, let us define

$$\begin{aligned} G(\psi^1, \Xi^1, \varphi^1) \equiv \mathbf{E}\int_0^T \parallel \psi^1(t) \parallel_H^2 dt + \mathbf{E}\int_0^T Tr(\Xi^1(t)Q(\Xi^1(t))^*)dt \\ + \mathbf{E}\int_{I\times V_\delta} \parallel \varphi^1(t,\xi) \parallel_H^2 \pi(d\xi)dt. \end{aligned} \tag{5.4}$$

Using Lagrange formula and the expressions (5.3) and (5.4), the cost functional evaluated at (F^2, B^2, C^2) can be written as follows:

$$J(F^2, B^2, C^2) = J(F^1, B^1, C^1) - \varepsilon G(\psi^1, \Xi^1, \varphi^1) + o(\varepsilon). \tag{5.5}$$

It is clear from the above expression that for $\varepsilon > 0$ sufficiently small

$$J(F^1, B^1, C^1) \geq J(F^2, B^2, C^2).$$

(Step 5): Returning to (step1) with the triple (F^2, B^2, C^2) and repeating the process, one generates the sequence $\{(F^k, B^k, C^k)\}_{k\geq 1}$ and the corresponding sequence of values of J given by $\{J(F^k, B^k, C^k)\}_{k\geq 1}$ that satisfies the following train of inequalities,

$$\begin{aligned} J(F^1, B^1, C^1) &\geq J(F^2, B^2, C^2) \\ &\vdots \\ &\geq J(F^k, B^k, C^k) \geq J(F^{k+1}, B^{k+1}, C^{k+1}) \cdots . \end{aligned}$$

Further, it follows from the assumptions of Theorem 3.4, in particular (3.21)-(3.22), and the growth properties (2.1)-(2.3) that

$$\inf\{J(F, B, C), (F, B, C) \in \mathcal{P}_{ad}\} > -\infty.$$

Thus the sequence $\{J(F^k, B^k, C^k)\}_{k\geq 1}$ is a monotone decreasing sequence bounded away from $-\infty$ and hence it converges possibly to a local minimum. This completes the proof. $\square$

Remark 5.2. It is interesting note that the admissible set $\mathcal{P}_{ad}$ can be easily expanded by increasing any of the parameters $\{\alpha, K\}$ and the functions $\{a, b \in L_2^+(\pi)\}$. For example, if $\{\alpha \leq \tilde{\alpha}, K \leq \tilde{K}\}$ and $\{a \leq \tilde{a}, b \leq \tilde{b} \in L_2^+(\pi)$ point wise $\pi - a.e$ on $V_\delta\}$, then $\mathcal{P}_{ad} \subseteq \tilde{\mathcal{P}}_{ad}$. Hence we expect the following inequality to hold,

$$\inf\{J(F, B, C), (F, B, C) \in \tilde{\mathcal{P}}_{ad}\} \leq \inf\{J(F, B, C), (F, B, C) \in \mathcal{P}_{ad}\},$$

thereby increasing the possibility of further improvement if desired.

Open Problem: It would be interesting to extend the results of this paper to general Banach spaces.

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