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AN EFFICIENT THIRD ORDER MANN-LIKE FIXED POINT SCHEME

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Abstract. In this paper, we introduce a Mann-like three step iteration method and show that it can be used to approximate the fixed point of a weak contraction mapping. Furthermore, we prove that this scheme is equivalent to the Mann iterative scheme. A comparison is made with the other third order iterative methods. Results are presented in a table to support our conclusion.

1. Introduction

Let X be a Banach space, and C be a nonempty, closed and convex subset of X. Let T be a mapping from a set C to itself. An element $x^\star$ of C is called a fixed point of T if $Tx^\star = x^\star$. The iterative approximation of a fixed point is crucial in fixed point theory and has dominated this field to a large extent. Many iterative methods have been proposed and studied. Firstly Mann iteration [11] was proposed in 1953 and proved useful when Picard's iteration failed. In fact Mann iteration exploits the convexity of the underlying

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space. Subsequently many other third order methods arose as detailed below, and were compared to each other for their speed of convergence. Most of these methods are in fact nested operations by the map T. We have found in our investigations that the third order methods are rather robust. Here we advocate a Mann-like iterative process that compares favourably with other third order schemes. To our surprise this method has not been proposed before. Our method is a third order polynomial in the operator T that uses a convex combination of terms.

We first summarize some existing third order methods. In what follows $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ subject to some restrictions.

$$\begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases} \tag{1.1}$$

equivalently

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n T((1-\gamma_n)x_n + \gamma_n T x_n)), \tag{1.2}$$

proposed in 2000 by Noor [12], called the Noor scheme and denoted by NOO here.

$$\begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases} \tag{1.3}$$

equivalently

$$\begin{aligned} x_{n+1} = {} & (1-\alpha_n)((1-\beta_n)((1-\gamma_n)x_n + \gamma_n T x_n) \\ & + \beta_n T((1-\gamma_n)x_n + \gamma_n T x_n)) \\ & + \alpha_n T((1-\beta_n)((1-\gamma_n)x_n + \gamma_n T x_n) + \beta_n T((1-\gamma_n)x_n + \gamma_n T x_n)), \end{aligned} \tag{1.4}$$

proposed by Phuengrattana and Suanti [14] in 2011 called the SP iteration.

$$\begin{cases} x_1 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases} \tag{1.5}$$

equivalently

$$x_{n+1} = (1-\alpha_n)((1-\beta_n)Tx_n + \beta_n T((1-\gamma_n)x_n + \gamma_n Tx_n)) \\ + \alpha_n T((1-\beta_n)Tx_n + \beta_n T((1-\gamma_n)x_n + \gamma_n Tx_n)), \tag{1.6}$$

proposed by Chugh et al. [7] in 2012 called the CR iteration.

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n \end{cases} \tag{1.7}$$

equivalently

$$x_{n+1} = T((1-\alpha_n)Tx_n + \alpha_n T((1-\beta_n)x_n + \beta_n Tx_n)), \tag{1.8}$$

proposed by Gursoy and Karakaya [8] in 2014 called the Picard-S iterative process denoted by PS.

$$\begin{cases} x_1 \in C, \\ z_n = Tx_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n \end{cases} \tag{1.9}$$

equivalently

$$x_{n+1} = T((1-\alpha_n)Tx_n + \alpha_n T^2 x_n)), \tag{1.10}$$

proposed by Karakaya etal [9] in 2017 called the Karakaya scheme denoted by KA.

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n \end{cases} \tag{1.11}$$

equivalently

$$x_{n+1} = T((1-\alpha_n)x_n + \alpha_n T((1-\beta_n)x_n + \beta_n Tx_n)), \tag{1.12}$$

proposed by Okeke [13] in 2019 called the Picard-Ishikawa iteration denoted by PIK.

We hereby propose the following method:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n^{(0)} x_n + \alpha_n^{(1)} Tx_n + \alpha_n^{(2)} T^2 x_n + \alpha_n^{(3)} T^3 x_n, \end{cases} \tag{1.13}$$

where $\{\alpha_n^{(i)}\}_{n=1}^{\infty} \subset (0,1),\ i = 0,1,2,3$ satisfying $\sum_{i=0}^{3} \alpha_n^{(i)} = 1$ and denote it as the NEW iteration.

2. Some results

Lemma 2.1. ([15]) *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative sequences satisfying the condition*

$$a_{n+1} \leq (1-\mu_n)a_n + b_n, \tag{2.1}$$

where $\mu_n \in (0,1)$ for all $n \geq n_0$, $\sum_{n=1}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.2. ([4]) The self-map $T : C \to C$ is called a weak-contraction if there exist $\delta \in (0,1)$ and $L_1 \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L_1\|y - Tx\|.$$

Many iterative methods have been proposed and studied for the weak-contractive mappings [3, 10].

Definition 2.3. ([6]) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real convergent sequences with limits a and b respectively. Then $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$ if

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0. \tag{2.2}$$

Definition 2.4. ([5]) Let $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be two fixed point iterative processes, both converging to fixed point $x^\star$ of a given operator T. Suppose that the error estimates

$$\begin{aligned} \|u_n - x^\star\| &\leq a_n, \\ \|x_n - x^\star\| &\leq b_n, \end{aligned} \tag{2.3}$$

for all $n \in N$ are available, where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive numbers converging to 0. If $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$, then $\{u_n\}_{n=1}^{\infty}$ converges faster than $\{x_n\}_{n=1}^{\infty}$ to $x^\star$.

Remark 2.5. Let $T : x \to \frac{x}{5}$, $x \in [-2,2]$, choose $x_1 = 1$ and consider the Picard iteration $x_{n+1} = Tx_n$. It is easily verified that

$$x_n = \left(\frac{1}{5}\right)^n \leq \left(\frac{4}{5}\right)^n \tag{2.4}$$

$$= b_n. \tag{2.5}$$

Also consider the Mann iteration

$$u_{n+1} = \alpha u_n + (1-\alpha)Tu_n \tag{2.6}$$

with $u_n = 1$ and $\alpha = \frac{1}{2}$. Then $u_{n+1} = \frac{3}{5}u_n$ which implies that

$$\begin{aligned} u_n = \left(\frac{3}{5}\right)^n &\le \left(\frac{3}{5}\right)^n \\ &= a_n. \end{aligned} \tag{2.7}$$

Now by Definition 2.4 $\{a_n\}_{n=1}^{\infty}$ converges to zero faster than $\{b_n\}_{n=1}^{\infty}$, so we should expect $\{u_n\}_{n=1}^{\infty}$ converge to zero faster than $\{x_n\}_{n=1}^{\infty}$, but this is clearly false as per Definition 2.3. The shortcoming in Definition 2.4 is that it should refer to the least upper bounds. However for an arbitrary operator T which may be nonlinear, it may be very difficult or indeed impossible to find such a bound. Unfortunately Definition 2.4 has been used to claim that some methods are faster than others. Also it is almost impossible to deduce expressions for $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ and hence apply Definition 2.3. For this reason we avoid an analysis of the speed of convergence (see, [1, 2]) and claiming that one method is superior to the other.

Theorem 2.6. ([4]) *Let X be a Banach space and $T : X \to X$ be a weak-contraction. Then T has a fixed point in X, that is,*

$$F(T) := \{x \in X : Tx = x\} \neq \emptyset. \tag{2.8}$$

Theorem 2.7. ([4]) *Let X be a Banach space and $T : X \to X$ be a weak-contraction for which there exist $\delta \in (0, 1)$ and some $L \ge 0$ such that*

$$\|Tx - Ty\| \le \delta\|x - y\| + L\|x - Tx\|. \tag{2.9}$$

Then T has a unique fixed point.

Theorem 2.8. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak-contraction map satisfying the additional condition (2.9). Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence (1.13) generated by a real sequences $\{\alpha_n^{(i)}\}_{n=1}^{\infty} \subset (0, 1)$, $i = 0, 1, 2, 3$ satisfying $\sum_{n=1}^{\infty} \alpha_n^{(3)} = \infty$ and $\sum_{i=0}^{3} \alpha_n^{(i)} = 1$. Then $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point $x^\star$ of T.*

Proof. The existence of a fixed point $x^\star$ is guaranteed by Theorem 2.6. The uniqueness follows from Theorem 2.7 as is shown by using (2.9). Suppose that $x^\star = Tx^\star$ and $x^{\star\star} = Tx^{\star\star}$ are two fixed points. Then

$$\|x^\star - x^{\star\star}\| \le \delta\|x^\star - x^{\star\star}\| + L\|x^\star - Tx^\star\|. \tag{2.10}$$

If $x^\star \neq x^{\star\star}$, then $\delta \geq 1$ is a contradiction which ensures uniqueness.
It follows from (2.9) that

$$\begin{aligned}\|T^k x_n - T^k x^\star\| &= \|TT^{k-1}x_n - TT^{k-1}x^\star\| \\ &\leq \delta\|T^{k-1}x_n - T^{k-1}x^\star\| \\ &\leq \delta^k\|x_n - x^\star\|.\end{aligned} \tag{2.11}$$

Hence

$$\begin{aligned}\|x_{n+1} - x^\star\| &= \|\alpha_n^{(0)}(x_n - x^\star) + \alpha_n^{(1)}(Tx_n - Tx^\star) \\ &\quad + \alpha_n^{(2)}(T^2x_n - T^2x^\star) + \alpha_n^{(3)}(T^3x_n - T^3x^\star)\| \\ &\leq \alpha_n^{(0)}\|x_n - x^\star\| + \alpha_n^{(1)}\delta\|x_n - x^\star\| + \alpha_n^{(2)}\delta^2\|x_n - x^\star\| \\ &\quad + \alpha_n^{(3)}\delta^3\|x_n - x^\star\| \\ &= \left(\alpha_n^{(0)} + \alpha_n^{(1)}\delta + \alpha_n^{(2)}\delta^2 + \alpha_n^{(3)}\delta^3\right)\|x_n - x^\star\| \\ &\leq \prod_{i=1}^{n}\left(\alpha_i^{(0)} + \alpha_i^{(1)}\delta + \alpha_i^{(2)}\delta^2 + \alpha_i^{(3)}\delta^3\right)\|x_1 - x^\star\| \\ &\leq \prod_{i=1}^{n}\left(\left(\alpha_i^{(0)} + \alpha_i^{(1)} + \alpha_i^{(2)}\right) + \alpha_i^{(3)}\delta^3\right)\|x_1 - x^\star\| \\ &= \prod_{i=1}^{n}\left[1 - \alpha_i^{(3)}\left(1 - \delta^3\right)\right]\|x_1 - x^\star\|.\end{aligned} \tag{2.12}$$

Using $1 - x \leq e^{-x}$ for $x \in (0,1)$ in (2.12) we simplify

$$\begin{aligned}\|x_{n+1} - x^\star\| &\leq \prod_{i=1}^{n} e^{-\alpha_i^{(3)}(1-\delta^3)}\|x_1 - x^\star\| \\ &= e^{-(1-\delta^3)\sum_{i=1}^{n}\alpha_i^{(3)}}\|x_1 - x^\star\|.\end{aligned} \tag{2.13}$$

Now since $\sum_{i=1}^{n}\alpha_i^{(3)} \to \infty$ as $n \to \infty$, it follows that $x_n \to x^\star$. □

Theorem 2.9. *Let X be a Banach space, C be a nonempty, closed and convex subset of X and $T : C \to C$ be a weak-contraction map satisfying condition (2.9) with a fixed point $x^\star$. Let $\{u_n\}_{n=1}^{\infty}$ be the Mann iteration process with $u_1 \in C$ and $\{x_n\}_{n=1}^{\infty}$ be defined by (1.13) with $x_1 \in C$ with real sequences $\{\alpha_n^{(i)}\}_{n=1}^{\infty} \subset (0,1)$, $i = 0, 1, 2, 3$ satisfying $\sum_{n=1}^{\infty}\alpha_n^{(3)} = \infty$, $\sum_{i=0}^{3}\alpha_n^{(i)} = 1$ and $\sum_{n=1}^{\infty}\beta_n = \infty$. Then the following assertions are equivalent:*

(a) *Mann iteration converges to $x^\star$.*
(b) *The new iteration method (1.13) converges to $x^\star$.*

Proof. We write Mann iteration as $u_{n+1} = (1-\beta_n)u_n + \beta_n T u_n$ and first show that $(a) \implies (b)$.

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &= \|(1-\beta_n)u_n + \beta T u_n - \alpha_n^{(0)} x_n \\ &\quad - \alpha_n^{(1)} T x_n - \alpha_n^{(2)} T^2 x_n - \alpha_n^{(3)} T^3 x_n\| \\ &= \|(\alpha_n^{(0)} + \alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)})u_n + \beta_n(Tu_n - u_n) - \alpha_n^{(0)} x_n \\ &\quad - \alpha_n^{(1)} T x_n - \alpha_n^{(2)} T^2 x_n - \alpha_n^{(3)} T^3 x_n\| \\ &\leq \alpha_n^{(0)}\|u_n - x_n\| + \alpha_n^{(1)}\|u_n - Tx_n\| + \alpha_n^{(2)}\|u_n - T^2 x_n\| \\ &\quad + \alpha_n^{(3)}\|u_n - T^3 x_n\| + \beta_n\|Tu_n - u_n\|. \end{aligned} \tag{2.14}$$

Now for $k \geq 1$ we have

$$\begin{aligned} \|u_n - T^k x_n\| &= \|u_n - Tu_n + Tu_n - T^k x_n\| \\ &\leq \|u_n - Tu_n\| + \|Tu_n - T(T^{k-1}x_n)\| \\ &\leq \|u_n - Tu_n\| + \delta\|u_n - T^{k-1}x_n\| + L\|u_n - Tu_n\| \\ &= (1+L)\|u_n - Tu_n\| + \delta\|u_n - T^{k-1}x_n\|. \end{aligned} \tag{2.15}$$

It follows from (2.15) that

$$\|u_n - Tx_n\| \leq (1+L)\|u_n - Tx_n\| + \delta\|u_n - x_n\|, \tag{2.16}$$

$$\|u_n - T^2 x_n\| \leq (1+L)(1+\delta)\|u_n - Tu_n\| + \delta^2\|u_n - x_n\|, \tag{2.17}$$

$$\|u_n - T^3 x_n\| \leq (1+L)(1+\delta+\delta^2)\|u_n - Tu_n\| + \delta^3\|u_n - x_n\|. \tag{2.18}$$

Substitute (2.16)-(2.18) into (2.14) to obtain

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &\leq (\alpha_n^{(0)} + \alpha_n^{(1)}\delta + \alpha_n^{(2)}\delta^2 + \alpha_n^{(3)}\delta^3)\|u_n - x_n\| \\ &\quad + \Big[\alpha_n^{(1)}(1+L) + \alpha_n^{(2)}(1+L)(1+\delta) \\ &\quad + \alpha_n^{(3)}(1+L)(1+\delta+\delta^2) + \beta_n\Big]\|u_n - Tu_n\| \\ &\leq (\alpha_n^{(0)} + \alpha_n^{(1)}\delta + \alpha_n^{(2)}\delta^2 + \alpha_n^{(3)}\delta^3)\|u_n - x_n\| \\ &\quad + [(1+L) + 2(1+L) + 3(1+L) + 1]\|u_n - Tu_n\| \\ &= [1 - \alpha_n^{(3)}(1-\delta^3)]\|u_n - x_n\| + (6L+7)\|u_n - Tu_n\|. \end{aligned} \tag{2.19}$$

As

$$\begin{aligned} \|u_n - Tu_n\| &\leq \|u_n - x^\star\| + \|Tx^\star - Tu_n\| \\ &\leq \|u_n - x^\star\| + \delta\|u_n - x^\star\| \\ &\leq 2\|u_n - x^\star\|. \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.19) finally yields

$$\|u_{n+1}-x_{n+1}\| \leq \left[1-\alpha_n^{(3)}(1-\delta^3)\right]\|u_n-x_n\| + (12L+14)\|u_n-x^\star\|. \quad (2.21)$$

Let $a_n = \|u_n - x_n\|$, $b_n = (12L+14)\|u_n - x^\star\|$ and $\mu_n = \alpha_n^{(3)}(1-\delta^3)$ and apply Lemma 2.1 to obtain $\|u_n - x_n\| \to 0$. Hence

$$\|x_n - x^\star\| \leq \|x_n - u_n\| + \|u_n - x^\star\|, \quad (2.22)$$

proving that $x_n \to x^\star$ since $u_n \to x^\star$.

We now show that $(b) \implies (a)$.

$$\begin{aligned} \|u_{n+1} - x^\star\| &\leq \|u_{n+1} - x^\star\| + \|x_n - x^\star\| \quad (2.23)\\ &= \|(1-\beta_n)u_n + \beta_n T u_n - (1-\beta_n)x^\star - \beta_n T x^\star\| + \|x_n - x^\star\| \\ &\leq (1-\beta_n)\|u_n - x^\star\| + \beta_n\|Tu_n - Tx^\star\| + \|x_n - x^\star\| \\ &\leq (1-\beta_n)\|u_n - x^\star\| + \beta_n\delta\|u_n - x^\star\| + \|x_n - x^\star\| \\ &= (1-\beta_n(1-\delta))\|u_n - x^\star\| + \|x_n - x^\star\|. \end{aligned}$$

Let $a_n = \|u_n - x^\star\|$, $b_n = \|x_n - x^\star\|$ and $\mu_n = \beta_n(1-\delta)$ and apply Lemma 1 to obtain $u_n \to x^\star$. This completes the proof. □

3. Examples

Example 3.1. Let $T : [0,6] \to [0,6]$ be defined by $Tx = \sqrt[3]{2x+4}$ with $x_0 = 5.0$. The exact solution is given by $x^\star = 2$.

Example 3.2. Let $T : [1,2] \to [1,2]$ be defined by $Tx = \frac{3}{4}(1+\frac{1}{x})$ with $x_0 = 1.0$. The exact solution is given by $x^\star = \sqrt{3}$.

Example 3.3. Let $T : [0,2] \to [0,2]$ be defined by $Tx = \frac{1}{1+x^2}$ with $x_0 = 0.5$. The exact solution is given by

$$x^\star = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{31}{108}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{31}{108}}}.$$

Example 3.4. Let $T : [0,2] \to [0,2]$ be defined by $Tx = \frac{x^2+9}{10}$ with $x_0 = 2.0$. The exact solution is given by $x^\star = 1$.

Example 3.5. Let $T : [0,2] \to [0,2]$ be defined by $Tx = \frac{-x^2+10}{9}$ with $x_0 = 2.0$. The exact solution is given by $x^\star = 1$.

Example 3.6. Let $T : [1.5,2] \to [1.5,2]$ be defined by $Tx = 2\sin x$ with $x_0 = 2.0$. The exact solution is given by $x^\star = 1.895494267033$ to twelve decimal digits.

Example 3.7. Let $T : [0, 0.5] \to [0, 0.5]$ be defined by $Tx = \frac{(1-x)^7}{10}$ with $x_0 = 0.5$. The exact solution is given by $x^\star = 0.063280205813$ to twelve decimal digits.

Example 3.8. The bitcoin elliptic curve $y^2 = x^3 + 7$ called **Secp256k1** has a fixed point in $[-1.75, -1.5]$. Define $T : [-1.75, -1.5] \to [-1.75, -1.5]$ by $Tx = \sqrt[3]{x^2 - 7}$ with $x_0 = -1.6$. The exact solution is given by

$$x^\star = \sqrt[3]{-\frac{187}{54} + \sqrt{\frac{1295}{108}}} - \sqrt[3]{\frac{187}{54} + \sqrt{\frac{1295}{108}}} + \frac{1}{3}.$$

The number of iterations to converge to within 10^{-12} of $x^\star$ is summarized in the tables 1-4, here X denotes non convergence after a maximum of 500 iterations. We have chosen constant sequences $\{\alpha_n^{(0)}\}_{n=1}^\infty = \{\alpha_n\}_{n=1}^\infty$, $\{\alpha_n^{(1)}\}_{n=1}^\infty = \{\beta_n\}_{n=1}^\infty$ and $\{\alpha_n^{(2)}\}_{n=1}^\infty = \{\gamma_n\}_{n=1}^\infty$ as parameters.

4. Tables

Ex	**NEW**	**PS**	**KK**	**NOO**	**SP**	**PIK**	**CR**
1	12	9	9	X	226	17	17
2	18	21	20	X	358	39	39
3	17	30	28	315	103	50	49
4	12	10	10	X	228	19	19
5	9	11	11	443	147	19	19
6	16	30	27	308	100	49	49
7	10	19	18	365	120	33	33
8	8	15	14	339	112	27	26

Table 1. $\alpha_n^{(0)} = 0.05,\ \alpha_n^{(1)} = 0.05,\ \alpha_n^{(2)} = 0.05$

Ex	**NEW**	**PS**	**KK**	**NOO**	**SP**	**PIK**	**CR**
1	15	9	9	326	111	17	17
2	22	21	20	X	177	37	37
3	13	30	25	157	50	43	41
4	15	10	10	327	113	18	18
5	12	11	10	218	72	19	18
6	13	29	25	153	49	43	41
7	6	19	17	180	58	30	29
8	8	15	14	167	54	25	24

Table 2. $\alpha_n^{(0)} = 0.1,\ \alpha_n^{(1)} = 0.1,\ \alpha_n^{(2)} = 0.1$

Ex	NEW	PS	KK	NOO	SP	PIK	CR
1	25	9	9	119	42	16	15
1	37	20	19	178	69	33	33
3	14	27	20	61	18	31	25
4	25	10	10	119	43	17	16
5	19	11	10	82	27	17	16
6	14	27	19	59	17	31	25
7	16	18	15	69	21	24	21
8	15	14	12	64	20	20	19

Table 3. $\alpha_n^{(0)} = 0.25,\ \alpha_n^{(1)} = 0.25,\ \alpha_n^{(2)} = 0.25$

Ex	NEW	PS	KK	NOO	SP	PIK	CR
1	16	9	9	320	53	17	16
2	25	21	20	474	86	37	36
3	7	29	25	165	22	44	37
4	16	10	10	320	54	18	18
5	12	11	10	223	33	19	18
6	7	29	25	161	22	44	36
7	10	18	17	188	27	30	27
8	9	15	14	174	25	25	23

Table 4. $\alpha_n^{(0)} = 0.1,\ \alpha_n^{(1)} = 0.2,\ \alpha_n^{(2)} = 0.3$

5. Conclusion

An examination of the number of iterations required for convergence shows that the NEW method is surprisingly quick. One can of course choose non constant sequences and other parameters to ensure that other methods are just as fast. However such an approach would be problem dependent. Our aim has been to illustrate that a simple non nested polynomial third order method is quite robust and fast compared to existing third order methods. We believe that we have achieved this for constant parameters and it is worthwhile investigating for non constant parameters.

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