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CHARACTERIZATIONS ON ORBITAL INVERSE LIMIT SYSTEMS

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Abstract. In this article, we investigate minimality, transitivity and mixing property for a shift map on the orbital inverse limit systems.

1. Introduction

In this article, we study dynamical properties on the orbital inverse limit systems induced from two cross bonding maps. Actually the systems is a generalization of the inverse limit systems which is one of important subjects in dynamical systems, see [1], [2] and [5].

In the orbital inverse limit systems, horizontal directions express inverse limit systems and vertical directions mean orbits based on horizontal axes. In [3], the authors proved expansiveness of the shift maps on the orbital inverse limit spaces. However the two bonding maps in [3] move differently on two directions from the bonding maps in this article.

Now we first propose the construction of the systems. Let X be a compact metric space with metric d_X. We consider a countable product space of X,

$$X^{\mathbb{Z}} := \{(x_i)_{i\in\mathbb{Z}} \mid x_i \in X \text{ for } i \in \mathbb{Z}\}.$$

For points $(x_i)_{i\in\mathbb{Z}}$, $(y_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$, we define a compatible metric d_∞ on $X^{\mathbb{Z}}$ given by

$$d_\infty((x_i)_{i\in\mathbb{Z}}, (y_i)_{i\in\mathbb{Z}}) := \sum_{i\in\mathbb{Z}} \frac{d_X(x_i, y_i)}{2^{|i|}}.$$

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Let g be a homeomorphism from X to itself. So we also consider a subspace $\mathbf{X}^g$ of $X^{\mathbb{Z}}$ defined by

$$\mathbf{X}^g := \{(x_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}} \mid x_i = g(x_{i+1}) \text{ for } i \in \mathbb{Z}\}.$$

We say it an *orbital space* for g, and its element $(x_i)_{i\in\mathbb{Z}}$ is called a g-*orbit* of x where $x = x_0$. It is known that $\mathbf{X}^g$ is a compact metric space with the metric d_∞. For $k \in \mathbb{Z}$, let $p^k : X^{\mathbb{Z}} \to X$ be a natural projection given by $p^k((x_i)_{i\in\mathbb{Z}}) = x_k$. For each $k \in \mathbb{Z}$, we denote $p_k := p^k|_{\mathbf{X}^g} : \mathbf{X}^g \to X$ as the restriction of p^k to $\mathbf{X}^g$.

Next we consider a countable product space $\mathbf{X} := \left(X^{\mathbb{Z}}\right)^{\mathbb{N}}$ of $X^{\mathbb{Z}}$ and denote an element $(x_{ij})_{ij}$ of $\mathbf{X}$ as follows:

$$(x_{ij})_{ij} := \left(\begin{pmatrix} \vdots \\ x_{(-i)0} \\ \vdots \\ x_{00} \\ \vdots \\ x_{i0} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ x_{(-i)1} \\ \vdots \\ x_{01} \\ \vdots \\ x_{i1} \\ \vdots \end{pmatrix}, \cdots, \begin{pmatrix} \vdots \\ x_{(-i)j} \\ \vdots \\ x_{0j} \\ \vdots \\ x_{ij} \\ \vdots \end{pmatrix}, \cdots \right)$$

where $x_{ij} \in X$ for $i \in \mathbb{Z}$, $j \in \mathbb{N} = \{0, 1, 2, \dots\}$. That is, each element of X has a type of matrix. On the space $\mathbf{X}$, we define a metric $\tilde{d}$ given by

$$\tilde{d}((x_{ij})_{ij}, (y_{ij})_{ij}) := \sum_{\substack{i\in\mathbb{Z} \\ j\in\mathbb{N}}} \frac{d_X(x_{ij}, y_{ij})}{2^{|i|} \cdot 3^j}$$

for $(x_{ij})_{ij}$, $(y_{ij})_{ij} \in \mathbf{X}$.

A subspace $(\mathbf{X}^g)^{\mathbb{N}}$ of the product space $\mathbf{X}$ is denoted by

$$(\mathbf{X}^g)^{\mathbb{N}} := \{(x_{ij})_{ij} \in \mathbf{X} \mid x_{ij} = g(x_{(i+1)j}) \ \text{ for } \ i \in \mathbb{Z} \ \text{ and } \ j \in \mathbb{N}\},$$

so it is also a compact metric space. Now we give a shift map on the product space $(\mathbf{X}^g)^{\mathbb{N}}$. Let ς be a (left) shift map on $(\mathbf{X}^g)^{\mathbb{N}}$ given as follows:

$$\begin{aligned}
\varsigma((x_{ij})_{ij}) &= \varsigma\left(\begin{pmatrix} \vdots \\ x_{(-i)0} \\ \vdots \\ x_{00} \\ \vdots \\ x_{i0} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ x_{(-i)1} \\ \vdots \\ x_{01} \\ \vdots \\ x_{i1} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ x_{(-i)2} \\ \vdots \\ x_{02} \\ \vdots \\ x_{i2} \\ \vdots \end{pmatrix}, \cdots \right) \\
&= \left(\begin{pmatrix} \vdots \\ x_{(-i)1} \\ \vdots \\ x_{01} \\ \vdots \\ x_{i1} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ x_{(-i)2} \\ \vdots \\ x_{02} \\ \vdots \\ x_{i2} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ x_{(-i)3} \\ \vdots \\ x_{03} \\ \vdots \\ x_{i3} \\ \vdots \end{pmatrix}, \cdots \right) = ((x_{i(j+1)})_{ij})
\end{aligned}$$

for every $(x_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{N}}$. It is obvious that this shift mapping is a continuous surjection.

Let $f : X \to X$ be a continuous surjection satisfying a commuting property $f \circ g = g \circ f$ for the above homeomorphism g. We define a subspace $\mathbf{X}_f^g$ of $(\mathbf{X}^g)^{\mathbb{N}}$ as

$$\mathbf{X}_f^g := \{(x_{ij})_{ij} \in (\mathbf{X}^g)^{\mathbb{N}} \mid x_{0j} = f(x_{0(j+1)}) \text{ for } j \in \mathbb{N}\}.$$

So $\mathbf{X}_f^g$ is a closed subset of $(\mathbf{X}^g)^{\mathbb{N}}$ and thus it is compact in $(\mathbf{X}^g)^{\mathbb{N}}$. See [4]. We sometimes write down $\varprojlim\{\mathbf{X}^g, \mathbf{F}^g\}$ instead of $\mathbf{X}_f^g$ and call the space $\mathbf{X}_f^g$ the *orbital inverse limit space* induced by f with respect to g. In the systems, the space X is called a *factor space* and the function f is called the *horizontal bonding function* and the function g is called the *vertical bonding function*. For more details, see [4].

Throughout this paper, we let that (X, d_X) is a compact metric space and that $f : X \to X$ is a continuous surjection and $g : X \to X$ is a homeomorphism with satisfying the commutative condition $f \circ g = g \circ f$.

2. Transitivity for orbital inverse limit systems

In this section, we deal with several dynamical properties on the orbital inverse limit systems. For the shift mapping ς, we consider a restriction $\sigma = \varsigma|_{\mathbf{X}_f^g} : \mathbf{X}_f^g \to \mathbf{X}_f^g$ of ς to $\mathbf{X}_f^g$. So we also obtain the inverse function, denoted $\sigma_{f,g} := \sigma^{-1} : \mathbf{X}_f^g \to \mathbf{X}_f^g$. Since

$$\begin{aligned}\sigma_{f,g}((x_{ij})_{ij}) &= \sigma_{f,g}\left(\begin{pmatrix}\vdots\\ x_{(-i)0}\\ \vdots\\ x_{00}\\ \vdots\\ x_{i0}\\ \vdots\end{pmatrix}, \begin{pmatrix}\vdots\\ x_{(-i)1}\\ \vdots\\ x_{01}\\ \vdots\\ x_{i1}\\ \vdots\end{pmatrix}, \begin{pmatrix}\vdots\\ x_{(-i)2}\\ \vdots\\ x_{02}\\ \vdots\\ x_{i2}\\ \vdots\end{pmatrix}, \cdots\right)\\ &= \left(\begin{pmatrix}\vdots\\ f(x_{(-i)0})\\ \vdots\\ f(x_{00})\\ \vdots\\ f(x_{i0})\\ \vdots\end{pmatrix}, \begin{pmatrix}\vdots\\ x_{(-i)0}\\ \vdots\\ x_{00}\\ \vdots\\ x_{i0}\\ \vdots\end{pmatrix}, \begin{pmatrix}\vdots\\ x_{(-i)1}\\ \vdots\\ x_{01}\\ \vdots\\ x_{i1}\\ \vdots\end{pmatrix}, \cdots\right)\end{aligned}$$

for $(x_{ij})_{ij} \in \mathbf{X}_f^g$, we have $\sigma_{f,g}((x_{ij})_{ij}) = (f(x_{ij}))_{ij}$ for all $(x_{ij})_{ij} \in \mathbf{X}_f^g$.

For $l \in \mathbb{N}$, we denote a natural projection with respect to horizontal direction of $(\mathbf{X}^g)^{\mathbb{N}}$ as $\mathbf{p}^l : (\mathbf{X}^g)^{\mathbb{N}} \to \mathbf{X}^g$. Then we have that $\mathbf{p}^l((x_{ij})_{ij}) := Orb^g(x_{0l})$ for each $l \in \mathbb{N}$. We denote $\mathbf{p}_l$ a restriction of $\mathbf{p}^l$ to $\mathbf{X}_f^g$. Let us define a continuous surjection $\mathbf{F}^g : \mathbf{X}^g \to \mathbf{X}^g$ given by

$$\mathbf{F}^g({}^{\mathrm{T}}(y_j)_j) = {}^{\mathrm{T}}(f(y_j))_j \quad \text{for} \quad {}^{\mathrm{T}}(y_j)_j \in \mathbf{X}^g.$$

Thus we get that the restriction $\mathbf{p}_l$ holds the commutativity as follows:

$$\begin{array}{ccc} \mathbf{X}_f^g & \xrightarrow{\sigma_{f,g}} & \mathbf{X}_f^g \\ {\scriptstyle \mathbf{p}_l}\downarrow & & \downarrow{\scriptstyle \mathbf{p}_l} \\ \mathbf{X}^g & \xrightarrow[\mathbf{F}^g]{} & \mathbf{X}^g \end{array}$$

that is, $\mathbf{p}_l \circ \sigma_{f,g} = \mathbf{F}^g \circ \mathbf{p}_l$.

The continuous mapping f from X to itself is said to be *minimal* if for any $x \in X$, an orbit of x for f is dense in X. A subset M is *minimal* in X with respect to f if M is a closed invariant subset of X and the restriction $f|_M : M \to M$ is minimal.

THEOREM 2.1. *Let $\mathbf{X}_f^g$ be an orbital inverse limit space induced by f with respect to g and let $\sigma_{f,g} : \mathbf{X}_f^g \to \mathbf{X}_f^g$ be a shift map. Let $\mathbf{M}$ be a minimal set in $\mathbf{X}_f^g$ with respect to $\sigma_{f,g}$, then for every $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\pi_{ij}(\mathbf{M})$ is also minimal in X with respect to f. Here π_{ij} is a (i,j)-th projection map from $\mathbf{X}_f^g$.*

Proof. Fixed $k \in \mathbb{Z}$ and $l \in \mathbb{N}$, since π_{kl} is a closed map and $\mathbf{M}$ is closed in $\mathbf{X}_f^g$, $\pi_{kl}(\mathbf{M})$ is closed in X. Since

$$f(\pi_{kl}(\mathbf{M})) = \pi_{kl}(\sigma_{f,g}(\mathbf{M})) = \pi_{kl}(\mathbf{M}),$$

$\pi_{kl}(\mathbf{M})$ is invariant under f. Now we prove that $\pi_{kl}(\mathbf{M}) = \overline{O_f^+(x)}$ for all $x \in \pi_{kl}(\mathbf{M})$. Let $x \in \pi_{kl}(\mathbf{M})$. It is enough to show that $\pi_{kl}(\mathbf{M}) \subseteq \overline{O_f^+(x)}$. Here, $O_f^+(x) := \{f^n(x) \mid n \in \mathbb{N}\}$. We choose $\mathbf{x} \in \mathbf{M}$ such that $\pi_{kl}(\mathbf{x}) = x$. Then $\overline{O_{\sigma_{f,g}}(\mathbf{x})} = \mathbf{M}$ because $\mathbf{M}$ is minimal. Here, $O_{\sigma_{f,g}}(\mathbf{x}) := \{\sigma_{f,g}^n(\mathbf{x}) \mid n \in \mathbb{Z}\}$. For any $y \in \pi_{kl}(\mathbf{M})$, take $\mathbf{y} \in \mathbf{M}$ such that $\pi_{kl}(\mathbf{y}) = y$. Then we can take a subsequence n_t of positive integers such that $\sigma_{f,g}^{n_t}(\mathbf{x}) \to \mathbf{y}$ as $t \to \infty$. Therefore we obtain that

$$y = \pi_{kl}(\mathbf{y}) = \lim_{t\to\infty} \pi_{kl}(\sigma_{f,g}^{n_t}(\mathbf{x})) = \lim_{t\to\infty} f^{n_t}(\pi_{kl}(\mathbf{x})) = \lim_{t\to\infty} f^{n_t}(x).$$

Hence $y \in \overline{O_f^+(x)}$. □

Using Theorem 2.1, we obtain directly the next corollaries.

COROLLARY 2.2. *Let $\mathbf{X}^g$ be an orbital space for g and $\mathbf{F}^g : \mathbf{X}^g \to \mathbf{X}^g$ an orbital function for g with respect to f. Let $\mathbf{X}_f^g$ be an orbital inverse limit space induced by f with respect to g and let $\sigma_{f,g} : \mathbf{X}_f^g \to \mathbf{X}_f^g$ be a shift map. Let M be a closed subset of X such that it is invariant under f. If $\varprojlim\{Orb^g(M), \mathbf{F}^g\}$ is minimal in $\mathbf{X}_f^g$ with respect to $\sigma_{f,g}$, then M is also minimal in X with respect to f.*

Corollary 2.3. *Let $\mathbf{X}_f^g$ be an orbital inverse limit space induced by f with respect to g and let $\sigma_{f,g} : \mathbf{X}_f^g \to \mathbf{X}_f^g$ be a shift map. If $\sigma_{f,g}$ is minimal, then f is minimal.*

We define that a continuous function $f : X \to X$ is *(topologically) transitive* if for any nonempty open subsets U and V of X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. If X is compact, then it is equivalent to the fact that there is an element x of X such that the orbit of x for f is dense in X. The mapping f is said to be *(topologically) mixing* if for any nonempty open subsets U and V of X, there is a positive integer N such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. We define that f is *chain transitive* if for any $x, y \in X$ and for any $\varepsilon > 0$, there is a ε-chain in X of f from x to y, that is, a finite sequence $x = x_0, x_1, \cdots, x_{n-1}, x_n = y$ in X such that $d(f(x_i), x_{i+1})) < \varepsilon$ for every $i \in \{0, 1, \cdots, n-1\}$.

The next theorem shows that the notions of transitivity, mixing and chain transitivity for the original dynamical systems are equivalent to the notions of the corresponding properties for the orbital inverse limit systems induced from the original systems.

Theorem 2.4. *Let $\mathbf{X}_f^g$ be an orbital inverse limit space induced by f with respect to g and let $\sigma_{f,g} : \mathbf{X}_f^g \to \mathbf{X}_f^g$ be a shift map. Then the following properties hold.*

(1) *f is transitive if and only if $\sigma_{f,g}$ is transitive.*
(2) *f is mixing if and only if $\sigma_{f,g}$ is mixing.*
(3) *f is chain transitive if and only if $\sigma_{f,g}$ is chain transitive.*

Proof. **(1)** Suppose that f is transitive. Let $\mathbf{U}$ and $\mathbf{V}$ be nonempty open subsets of the product space $\mathbf{X}_f^g$, respectively. Then we choose $u, v \in X$, $k \in \mathbb{Z}$, $l \in \mathbb{N}$, and $\varepsilon > 0$ such that

$$\pi_{kl}^{-1}(B_\varepsilon(u)) \subseteq \mathbf{U} \quad \text{and} \quad \pi_{kl}^{-1}(B_\varepsilon(v)) \subseteq \mathbf{V}.$$

Since f is transitive, there is a positive integer n such that $f^n(B_\varepsilon(u)) \cap B_\varepsilon(v) \neq \emptyset$. Thus

$$\begin{aligned} \emptyset &\neq \pi_{kl}^{-1}(B_\varepsilon(u) \cap f^{-n}(B_\varepsilon(v))) \\ &= \pi_{kl}^{-1}(B_\varepsilon(u)) \cap \pi_{kl}^{-1}(f^{-n}(B_\varepsilon(v))) \\ &= \pi_{kl}^{-1}(B_\varepsilon(u)) \cap \sigma_{f,g}^{-n}(\pi_{kl}^{-1}(B_\varepsilon(v))) \\ &\subseteq \mathbf{U} \cap \sigma_{f,g}^{-n}(\mathbf{V}). \end{aligned}$$

Thus we have $\sigma_{f,g}^n(\mathbf{U}) \cap \mathbf{V} \neq \emptyset$, so $\sigma_{f,g}$ is transitive.

Conversely, we assume that $\sigma_{f,g}$ is transitive. Let U and V be nonempty open subsets of X. Then $\pi_{00}^{-1}(U)$ and $\pi_{00}^{-1}(V)$ are also nonempty open subsets of $\mathbf{X}_f^g$. Since $\sigma_{f,g}$ is transitive, we can choose a positive integer n such that

$$\sigma_{f,g}^n(\pi_{00}^{-1}(U)) \cap \pi_{00}^{-1}(V) \neq \emptyset.$$

Then we get

$$\begin{aligned} \emptyset &\neq \pi_{00}(\sigma_{f,g}^n(\pi_{00}^{-1}(U)) \cap \pi_{00}^{-1}(V)) \\ &\subseteq \pi_{00}(\sigma_{f,g}^n(\pi_{00}^{-1}(U))) \cap \pi_{00}(\pi_{00}^{-1}(V)) \\ &= f^n(\pi_{00}(\pi_{00}^{-1}(U))) \cap V \\ &= f^n(U) \cap V, \end{aligned}$$

so f is transitive.

(2) This proof is similar to that of (1).

(3) We first assume that f is chain transitive. Let $\mathbf{x} = (x_{ij})_{ij}$, $\mathbf{y} = (y_{ij})_{ij} \in \mathbf{X}_f^g$. For any $\varepsilon > 0$ we choose a positive integer N satisfying $\frac{D}{2^N} < \frac{\varepsilon}{12}$ and $\frac{D}{3^N} < \frac{\varepsilon}{6}$ where $D := \operatorname{diam} X$. By the uniform continuity of f and g, there exists a positive real number δ such that for $x, y \in X$,

$$\sum_{\substack{-N \le i \le N \\ 0 \le j \le N}} \frac{d_X(g^i(f^j(x)), g^i(f^j(y)))}{2^{|i|} \cdot 3^j} < \frac{\epsilon}{4}.$$

if $d_X(x, y) < \delta$. Since f is chain transitive, there is a δ-chain in X of f from x_{0N} to y_{0N}, say

$$z_{0N}^0 = x_{0N}, z_{0N}^1, z_{0N}^2, \cdots, z_{0N}^{n-1}, z_{0N}^n = y_{0N}.$$

For each integer $k \in (0, n)$, put $z_{0(N-j)}^i = f^j(z_{0N}^k)$ if $0 \le j \le N$ and take $z_{i(N-j)}^k = f^{-1}(z_{i(N-j-1)}^k)$ if $j < 0$. For each $j \in \mathbb{N}$, we denote $z_{ij}^k = g^{-i}(z_{0j}^k)$ for all $i \in \mathbb{Z}$. For $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, let $z_{ij}^0 = x_{ij}$ and $z_{ij}^n = y_{ij}$. Then a finite sequence

$$(z_{ij}^0)_{ij} = \mathbf{x}, (z_{ij}^1)_{ij}, (z_{ij}^2)_{ij}, \cdots, (z_{ij}^{n-1})_{ij}, (z_{ij}^n)_{ij} = \mathbf{y}$$

becomes an ϵ-chain in $\mathbf{X}_f^g$ of $\sigma_{f,g}$ from $\mathbf{x}$ to $\mathbf{y}$. Indeed, we have that for $0 \le k < n$

$$\tilde{d}(\sigma_{f,g}((z_{ij}^k)_{ij}), (z_{ij}^{k+1})_{ij}) = \sum_{\substack{i \in \mathbb{Z} \\ j \in \mathbb{N}}} \frac{d_X(f(z_{ij}^k), z_{ij}^{k+1})}{2^{|i|} \cdot 3^j} < \varepsilon.$$

Thus $\sigma_{f,g}$ is chain transitive.

Conversely, suppose that $\sigma_{f,g}$ is chain transitive. Let x, $y \in X$. We take two points $(x_{ij})_{ij}$ and $(y_{ij})_{ij}$ of $\mathbf{X}_f^g$ with $x_{00} = x$ and $y_{00} = y$, respectively. Since $\sigma_{f,g}$ is chain transitive, for any $\varepsilon > 0$ we can choose an ε-chain in $\mathbf{X}_f^g$ of $\sigma_{f,g}$ from $(x_{ij})_{ij}$ to $(y_{ij})_{ij}$, that is,

$$(z_{ij}^0)_{ij} = (x_{ij})_{ij}, (z_{ij}^1)_{ij}, (z_{ij}^2)_{ij}, \cdots , (z_{ij}^{n-1})_{ij}, (z_{ij}^n)_{ij} = (y_{ij})_{ij}.$$

Then we have that for $0 \leq k < n$

$$d_X(f(z_{00}^k), z_{00}^{k+1}) \leq \tilde{d}(\sigma_{f,g}((z_{ij}^k)_{ij}), (z_{ij}^{k+1})_{ij}) < \varepsilon,$$

which means that the following finite sequence

$$z_{00}^0 = x, z_{00}^1, z_{00}^2, \cdots , z_{00}^{n-1}, z_{00}^n = y$$

is an ε-chain in X of f from x to y. This completes the proof.

□

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