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RUDNICK AND SOUNDARARAJAN'S THEOREM FOR FUNCTION FIELDS IN EVEN CHARACTERISTIC

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Abstract. In this paper we prove an even characteristic analogue of the result of Andrade on lower bounds for moment of quadratic Dirichlet L-functions in odd characteristic. We establish lower bounds for the moments of Dirichlet L-functions of characters defined by Hasse symbols in even characteristic.

1. Introduction

It is a fundamental problem in analytic number theory to estimate moments of central values of L-functions in families. In [3, 4], Rudnick and Soundararajan obtained a result on lower bounds for moments of central values of L-functions over the family of quadratic Dirichlet characters. More precisely, they showed that for every even natural number k, one has

$$\sum_{|d|\leq X}^{\flat} L(\tfrac{1}{2}, \chi_d)^k \gg_k X(\log X)^{k(k+1)/2},$$

where $\flat$ indicates that the sum is taken over fundamental discriminants d, and χ_d is the Dirichlet character associated to the quadratic extension $K/\mathbb{Q}$ of discriminant d. In [1], Andrade established a function field analogue of the result of Rudnick and Soundararajan in odd characteristic. Let $\mathbb{F}_q[T]$ be the polynomial ring over a finite field $\mathbb{F}_q$, where q is odd, and $\mathcal{H}_n$ denote the set of monic square-free polynomials in $\mathbb{F}_q[T]$

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of degree n. Andrade proved that for every even natural number k and $n = 2g+1$ or $n = 2g+2$, one has

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n} L(\tfrac{1}{2},\chi_D)^k \gg_k (\log_q |D|)^{k(k+1)/2},$$

where $|D| = q^{\deg(D)}$ and $L(s,\chi_D)$ is the Dirichlet L-function attached to a quadratic character χ_D. The aim of this paper is to give an even characteristic analogue of the result of Andrade.

Let us fix some basic notations. Let $\mathrm{k} = \mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[T]$. From now on q is assumed to be even and $q > 2$ for simplicity. We denote by $\mathbb{A}^+$ the set of monic polynomials in $\mathbb{A}$ and by $\mathbb{P}$ the set of monic irreducible polynomials in $\mathbb{A}$. Let $\mathbb{A}_n = \{f \in \mathbb{A} : \deg(f) = n\}$, and $\mathbb{A}_n^+ = \mathbb{A}^+ \cap \mathbb{A}_n$ for any positive integer n. The zeta function $\zeta_{\mathbb{A}}(s)$ of $\mathbb{A}$ is defined to be the following infinite series:

$$\zeta_{\mathbb{A}}(s) = \sum_{f\in\mathbb{A}^+}\frac{1}{|f|^s} = \prod_{P\in\mathbb{P}}\left(1-\frac{1}{|P|^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

It is well known that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$. For $f \in \mathbb{A}^+$, let $\Phi(f) = |(\mathbb{A}/f\mathbb{A})^\times|$.

1.1. Quadratic function field in even characteristic

In this subsection, we recall some basic facts on quadratic function field in even characteristic. For more details, we refer to [2, §2.2, §2.3]. Any separable quadratic extension of k is of the form $K_u = \mathrm{k}(x_u)$, where x_u is a zero of $X^2 + X + u = 0$ for some $u \in \mathrm{k}$. Fix an element $\xi \in \mathbb{F}_q \setminus \wp(\mathbb{F}_q)$, where $\wp : \mathrm{k} \to \mathrm{k}$ is the additive homomorphism defined by $\wp(x) = x^2 + x$. We say that $u \in \mathrm{k}$ is normalized if it is of the form

$$u = \sum_{i=1}^{m}\sum_{j=1}^{e_i}\frac{A_{ij}}{P_i^{2j-1}} + \sum_{\ell=1}^{n}\alpha_\ell T^{2\ell-1} + \alpha,$$

where $P_i \in \mathbb{P}$ are distinct, $A_{ij} \in \mathbb{A}$ with $\deg(A_{ij}) < \deg(P_i)$, $A_{ie_i} \neq 0$, $\alpha \in \{0, \xi\}$, $\alpha_\ell \in \mathbb{F}_q$ and $\alpha_n \neq 0$ for $n > 0$. Let $u \in \mathrm{k}$ be normalized one. The infinite prime $(1/T)$ of k splits, is inert or ramified in K_u according as $n = 0$ and $\alpha = 0$, $n = 0$ and $\alpha = \xi$, or $n > 0$. Then the field K_u is called real, inert imaginary, or ramified imaginary, respectively. The discriminant D_u of K_u is given by

$$D_u = \begin{cases} \prod_{i=1}^{m} P_i^{2e_i} & \text{if } n = 0, \\ \prod_{i=1}^{m} P_i^{2e_i}\cdot(1/T)^{2n} & \text{if } n > 0, \end{cases}$$

and the genus g_u of K_u is given by

$$g_u = \frac{1}{2}\deg(D_u) - 1.$$

For $M \in \mathbb{A}^+$, write $r(M) = \prod_{P|M} P$ and $t(M) = M \cdot r(M)$. For $P \in \mathbb{P}$, let ν_P be the normalized valuation at P. Let $\mathcal{B}$ be the set of non-constant monic polynomials M such that $\nu_P(M)$ is zero or odd for any $P \in \mathbb{P}$, and $\mathcal{B}_n = \{M \in \mathcal{B} : \deg(t(M)) = 2n\}$. The map $\mathcal{B}_n \to \mathbb{A}_n^+$ defined by $M \mapsto \tilde{M} = \sqrt{M}$ is a bijection with the inverse $N \mapsto N^* = \frac{N^2}{r(N)}$. Hence, $|\mathcal{B}_n| = |\mathbb{A}_n^+| = q^n$. Let $\mathcal{F}$ be the set of rational functions $\frac{D}{M} \in \mathrm{k}$ with $D \in \mathbb{A}, M \in \mathcal{B}$, $\gcd(D, M) = 1$ and $\deg(D) < \deg(M)$ which can be written as

$$\frac{D}{M} = \sum_{P|M} \sum_{i=1}^{\ell_P} \frac{A_{P,i}}{P^{2i-1}},$$

where $\deg(A_{P,i}) < \deg(P)$ for any $P|M$ and $1 \le i \le \ell_P = \frac{1}{2}(\nu_P(M)+1)$. Under the correspondence $u \mapsto K_u$, $\mathcal{F}$ corresponds to the set of all real separable quadratic extensions K_u of k. For $M \in \mathcal{B}$, let $\mathcal{F}_M$ be the set of rational functions $u \in \mathcal{F}$ whose denominator is M. Then $\mathcal{F}$ is the disjoint union of $\mathcal{F}_M$ with $M \in \mathcal{B}$. For $u \in \mathcal{F}_M$, the discriminant D_u and the genus g_u of K_u are $D_u = t(M)$ and $g_u = \frac{1}{2}\deg(t(M)) - 1$. For $n \ge 1$, let $\mathcal{F}_n$ be the union of $\mathcal{F}_M$ with $M \in \mathcal{B}_n$. Then $\mathcal{F}_n$ corresponds to the set of all real separable quadratic extensions K_u of k with genus $n-1$. For $M \in \mathcal{B}_n$, there are $\Phi(\tilde{M})$ D's such that $\frac{D}{M} \in \mathcal{F}_n$, so that $|\mathcal{F}_M| = \Phi(\tilde{M})$ and

$$|\mathcal{F}_n| = \sum_{M \in \mathcal{B}_n} \Phi(\tilde{M}) = \sum_{\tilde{M} \in \mathbb{A}_n^+} \Phi(\tilde{M}) = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}.$$

For a positive integer s, let $\mathcal{G}_s$ be the set of polynomials $F(T) \in \mathbb{A}$ of the form

$$F(T) = \alpha + \sum_{i=1}^{s} \alpha_i T^{2i-1},$$

where $\alpha \in \{0, \xi\}, \alpha_i \in \mathbb{F}_q$ and $\alpha_s \neq 0$. For any two subsets U, V of k, write $U + V = \{u + v : u \in U, v \in V\}$. Let $\mathcal{I} = (\mathcal{F} \cup \{0\}) + \mathcal{G}$, where $\mathcal{G} = \bigcup_{s \ge 1} \mathcal{G}_s$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{I}$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k. For $w \in \mathcal{F}_M + \mathcal{G}_s$, the discriminant D_w and the genus g_w of K_w are $D_w = t(M) \cdot (1/T)^{2s}$ and $g_w = \frac{1}{2}\deg(t(M)) + s - 1$. Let $\mathcal{F}_0 = \{0\}$. For any $r \ge 0$ and $s \ge 1$, let $\mathcal{I}_{(r,s)} = \mathcal{F}_r + \mathcal{G}_s$. If $w \in \mathcal{I}_{(r,s)}$, the genus g_w of K_w is $r + s - 1$. For $n \ge 1$, let $\mathcal{I}_n$ be the union of all $\mathcal{I}_{(r,s)}$, where (r, s) runs over all pairs of non-negative integers such that $s > 0$ and $r + s = n$. Then $\mathcal{I}_n$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k with genus $n-1$. Since $|\mathcal{G}_s| = 2\zeta_{\mathbb{A}}(2)^{-1}q^s$ for $s \geq 1$, we have

$$|\mathcal{I}_n| = \sum_{s=1}^{n} |\mathcal{F}_{n-s}||\mathcal{G}_s| = 2\zeta_{\mathbb{A}}(2)^{-1}q^{2n-1}.$$

1.2. Hasse symbol and L-functions

For any $u \in k$ whose denominator is not divisible by $P \in \mathbb{P}$, the Hasse symbol $[u, P)$ with values in $\mathbb{F}_2$ is defined by

$$[u, P) = \begin{cases} 0 & \text{if } X^2 + X \equiv u \bmod P \text{ is solvable in } \mathbb{A}, \\ 1 & \text{otherwise.} \end{cases}$$

For $N \in \mathbb{A}$ prime to the denominator of u, if $N = sgn(N)\prod_{i=1}^{s} P_i^{e_i}$, where $sgn(N)$ is the leading coefficient of N and $P_i \in \mathbb{P}$ are distinct and $e_i \geq 1$, the symbol $[u, N)$ is defined to be $\sum_{i=1}^{s} e_i[u, P_i)$.

For $u \in k$ and $0 \neq N \in \mathbb{A}$, the quadratic symbol $\{\frac{u}{N}\}$ is defined as follows:

$$\left\{\frac{u}{N}\right\} = \begin{cases} (-1)^{[u,N)} & \text{if } N \text{ is prime to the denominator of } u, \\ 0 & \text{otherwise.} \end{cases}$$

This symbol is clearly additive in its first variable, and multiplicative in the second variable.

For the field K_u, we associate a character χ_u on $\mathbb{A}^+$ which is defined by $\chi_u(f) = \{\frac{u}{f}\}$, and let $L(s, \chi_u)$ be the L-function associated to the character χ_u: for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 1$,

$$L(s, \chi_u) = \sum_{f \in \mathbb{A}^+} \frac{\chi_u(f)}{|f|^s} = \prod_{P \in \mathbb{P}} \left(1 - \frac{\chi_u(P)}{|P|^s}\right)^{-1}.$$

It is known that $L(s, \chi_u)$ is a polynomial in q^{-s} of degree $2g_u + \frac{1}{2}(1 + (-1)^{\varepsilon(u)})$, where $\varepsilon(u) = 1$ if K_u is ramified imaginary and $\varepsilon(u) = 0$ otherwise.

1.3. Results

The main result of this paper is the following theorem.

THEOREM 1.1. *For any even natural number k we have*

$$\frac{1}{|\mathcal{I}_{g+1}|}\sum_{u\in\mathcal{I}_{g+1}} L(\tfrac{1}{2},\chi_u)^k \gg_k g^{k(k+1)/2}.$$

REMARK 1.2. Comparing to the odd characteristic case, $\mathcal{I}_{g+1}$ corresponds to $\mathcal{H}_{2g+1}$ (more precisely, $\mathcal{H}_{2g+1}\cup\gamma\mathcal{H}_{2g+1}$, where γ is a generator of $\mathbb{F}_q^{\times}$), and $\mathcal{F}_{g+1}$ corresponds to $\mathcal{H}_{2g+2}$. Even though, for simplicity, we restrict ourselves to $\mathcal{I}_{g+1}$ in this paper, we also can prove that for any even natural number k,

$$\frac{1}{|\mathcal{F}_{g+1}|}\sum_{u\in\mathcal{F}_{g+1}} L(\tfrac{1}{2},\chi_u)^k \gg_k g^{k(k+1)/2}.$$

2. Preliminaries

In this section we present some auxiliary lemmas that will be used in the proof of the main theorem.

Let $\mathbb{A}^+_{\le x} = \{f \in \mathbb{A}^+ : \deg(f) \le x\}$ for any $x > 0$. The following lemma is an even characteristic analogue of [1, Lemma 3.1], which is an "Approximate" functional equations of $L(s,\chi_u)$.

LEMMA 2.1. *Let $u \in \mathcal{I}_{g+1}$. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \ge \frac{1}{2}$, we have*

$$L(s,\chi_u) = \sum_{f\in\mathbb{A}^+_{\le g}} \frac{\chi_u(f)}{|f|^s} + q^{(1-2s)g} \sum_{f\in\mathbb{A}^+_{\le g-1}} \frac{\chi_u(f)}{|f|^{1-s}}.$$

Proof. It follows immediately from Lemma 3.1 in [2] since $g_u = g$ for $u \in \mathcal{I}_{g+1}$. □

The following lemma quoted from [2, Lemma 3.3] that is needed in proof of Proposition 2.3.

LEMMA 2.2. *Let $L \in \mathbb{A}^+$. Given any $\epsilon > 0$, we have*

$$\sum_{\substack{f\in\mathbb{A}^+_n \\ (f,L)=1}} \Phi(f) = \frac{q^{2n}}{\zeta_{\mathbb{A}}(2)} \prod_{P|L}\left(1+\frac{1}{|P|}\right)^{-1} + O\left(q^{n(1+\epsilon)}\right).$$

We give the following orthogonality relations for sums over $\mathcal{I}_n$.

PROPOSITION 2.3. *Let $f \in \mathbb{A}^+$.*

1. *If f is not a square in $\mathbb{A}$, then*

$$\sum_{u\in\mathcal{I}_n} \chi_u(f) \ll 2^{\frac{\deg(f)}{2}} n\sqrt{|\mathcal{I}_n|}.$$

2. *If f is a square in $\mathbb{A}$, then*

$$\sum_{u\in\mathcal{I}_n} \chi_u(f) = |\mathcal{I}_n| \prod_{P|f} \left(1+\frac{1}{|P|}\right)^{-1} + O\left(|\mathcal{I}_n|^{\frac{1}{2}(1+\epsilon)}\right)$$

for any $\epsilon > 0$.

Proof. The case of f being not a square in $\mathbb{A}$ follows immediately from Proposition 3.20 in [2] since $|\mathcal{I}_n| = 2\zeta_{\mathbb{A}}(2)^{-1}q^{2n-1}$. Consider the case that f is a square in $\mathbb{A}$. Since $\mathcal{I}_n$ is the disjoint union of the $\mathcal{I}_{(r,n-r)}$'s for $0 \le r \le n-1$, we can write

$$\sum_{u\in\mathcal{I}_n} \chi_u(f) = \sum_{r=0}^{n-1} \sum_{u\in\mathcal{I}_{(r,n-r)}} \chi_u(f). \tag{2.1}$$

Note that $\mathcal{I}_{(0,n)} = \mathcal{G}_n$ and $|\mathcal{G}_n| = 2\zeta_{\mathbb{A}}(2)^{-1}q^n$. Then we have

$$\sum_{u\in\mathcal{I}_{(0,n)}} \chi_u(f) = |\mathcal{G}_n| \ll |\mathcal{I}_n|^{\frac{1}{2}(1+\epsilon)}. \tag{2.2}$$

For $M \in \mathcal{B}_r$ with $1 \le r \le n-1$, let $\mathcal{I}_M = \mathcal{F}_M + \mathcal{G}_{n-r}$. Then $\mathcal{I}_{(r,n-r)}$ is the disjoint union of the $\mathcal{I}_M$'s, where M runs over $\mathcal{B}_r$. For $u \in \mathcal{I}_M$ with $M \in \mathcal{B}_r$, we have $\chi_u(f) = 1$ if $(M,f) = 1$ and $\chi_u(f) = 0$ otherwise. Thus, we have

$$\sum_{u\in\mathcal{I}_{(r,n-r)}} \chi_u(f) = \sum_{\substack{M\in\mathcal{B}_r\\(M,f)=1}} \sum_{u\in\mathcal{I}_M} 1 = 2\zeta_{\mathbb{A}}(2)^{-1}q^{n-r} \sum_{\substack{M\in\mathcal{B}_r\\(M,f)=1}} \Phi(\tilde{M}) \tag{2.3}$$

since $|\mathcal{I}_M| = |\mathcal{F}_M||\mathcal{G}_{n-r}| = 2\zeta_{\mathbb{A}}(2)^{-1}q^{n-r}\Phi(\tilde{M})$ for $M \in \mathcal{B}_r$. The map $\mathcal{B}_n \to \mathbb{A}_n^+$ defined by $M \mapsto \tilde{M}$ is a bijection and $(M,f) = 1$ if and only if $(\tilde{M},f) = 1$. Thus we have

$$\sum_{\substack{M\in\mathcal{B}_r\\(M,f)=1}} \Phi(\tilde{M}) = \sum_{\substack{\tilde{M}\in\mathbb{A}_r^+\\(\tilde{M},f)=1}} \Phi(\tilde{M})$$

$$= \frac{q^{2r}}{\zeta_{\mathbb{A}}(2)} \prod_{P|f} \left(1+\frac{1}{|P|}\right)^{-1} + O\left(q^{r(1+\epsilon)}\right), \tag{2.4}$$

where the second equality follows from Lemma 2.2. We insert (2.4) into (2.3) to get

$$\sum_{u\in\mathcal{I}_{(r,n-r)}}\chi_u(f)=2\zeta_{\mathbb{A}}(2)^{-2}q^{n+r}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}+O\left(q^{n+r\epsilon}\right). \tag{2.5}$$

Since $|\mathcal{I}_n|=2\zeta_{\mathbb{A}}(2)^{-1}q^{2n-1}$, by inserting (2.2) and (2.5) into (2.1) and arranging the terms, we complete the proof. □

For $f\in\mathbb{A}^+$, let $d_k(f)$ represent the number of ways to write f as a product of k factors. We have the following asymptotic formula whose proof can be found in [1, §4.1].

LEMMA 2.4. *We have*

$$\sum_{f\in\mathbb{A}^+_{\le z}}\frac{d_k(f^2)}{|f|}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\sim C(k)z^{k(k+1)/2}$$

for some positive constant $C(k)$ explicitly given in [1, (4.26)].

3. Proof of Theorem 1.1

Let k be a given even natural number, and set $x=\frac{2(2g)}{15k}$. For $u\in\mathcal{I}_{g+1}$, define

$$A(u)=\sum_{f\in\mathbb{A}^+_{\le x}}\frac{\chi_u(f)}{\sqrt{|f|}}.$$

We use Triangle inequality and Holder's inequality to obtain that

$$\begin{aligned}\left|\sum_{u\in\mathcal{I}_{g+1}}L(\tfrac12,\chi_u)A(u)^{k-1}\right| &\le \sum_{u\in\mathcal{I}_{g+1}}|L(\tfrac12,\chi_u)||A(u)|^{k-1}\\ &\le\left(\sum_{u\in\mathcal{I}_{g+1}}L(\tfrac12,\chi_u)^k\right)^{\frac1k}\left(\sum_{u\in\mathcal{I}_{g+1}}A(u)^k\right)^{\frac{k-1}{k}}.\end{aligned}$$

It follows that

$$\sum_{u\in\mathcal{I}_{g+1}}L(\tfrac12,\chi_u)^k\ge\frac{\mathcal{S}_1^k}{\mathcal{S}_2^{k-1}}, \tag{3.1}$$

where

$$\mathcal{S}_1=\sum_{u\in\mathcal{I}_{g+1}}L(\tfrac12,\chi_u)A(u)^{k-1}\quad\text{and}\quad\mathcal{S}_2=\sum_{u\in\mathcal{I}_{g+1}}A(u)^k.$$

3.1. Estimating $\mathcal{S}_2$

In this subsection we follow the arguments in [1, §4.1] to estimate $\mathcal{S}_2$. We have

$$\mathcal{S}_2 = \sum_{f_1,\dots,f_k\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|f_1|\cdots|f_k|}} \sum_{u\in\mathcal{I}_{g+1}} \chi_u(f_1\cdots f_k). \tag{3.2}$$

We use Proposition 2.3 to obtain that

$$\begin{aligned}\mathcal{S}_2 = |\mathcal{I}_{g+1}| &\sum_{\substack{f_1,\dots,f_k\in\mathbb{A}^+_{\le x}\\ f_1\cdots f_k=\square}} \frac{1}{\sqrt{|f_1|\cdots|f_k|}} \prod_{P|f_1\cdots f_k}\left(1+\frac{1}{|P|}\right)^{-1}\\
&+ \sum_{\substack{f_1,\dots,f_k\in\mathbb{A}^+_{\le x}\\ f_1\cdots f_k=\square}} \frac{1}{\sqrt{|f_1|\cdots|f_k|}} O\left(|\mathcal{I}_{g+1}|^{\frac{1}{2}(1+\epsilon)}\right)\\
&+ \sum_{\substack{f_1,\dots,f_k\in\mathbb{A}^+_{\le x}\\ f_1\cdots f_k\neq\square}} \frac{1}{\sqrt{|f_1|\cdots|f_k|}} O\left(2^{\frac{\deg(f_1\cdots f_k)}{2}}(g+1)\sqrt{|\mathcal{I}_{g+1}|}\right).\end{aligned}$$

Since $x = \frac{2(2g)}{15k}$, the second term above can be estimated as

$$\begin{aligned}&\le |\mathcal{I}_{g+1}|^{\frac{1}{2}(1+\epsilon)} \sum_{f_1\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|f_1|}}\cdots \sum_{f_k\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|f_k|}}\\
&\ll |\mathcal{I}_{g+1}|^{\frac{1}{2}(1+\epsilon)} q^{\frac{kx}{2}} = |\mathcal{I}_{g+1}|^{\frac{17}{30}+\frac{1}{2}\epsilon}\end{aligned}$$

and the last term above can be estimated as

$$\begin{aligned}&\le (g+1)\sqrt{|\mathcal{I}_{g+1}|} \sum_{f_1\in\mathbb{A}^+_{\le x}} \frac{2^{\frac{\deg(f_1)}{2}}}{\sqrt{|f_1|}}\cdots \sum_{f_k\in\mathbb{A}^+_{\le x}} \frac{2^{\frac{\deg(f_k)}{2}}}{\sqrt{|f_k|}}\\
&\ll (g+1)(2q)^{\frac{kx}{2}}\sqrt{|\mathcal{I}_{g+1}|} \ll |\mathcal{I}_{g+1}|^{\frac{2}{3}}.\end{aligned}$$

Hence, we get

$$\begin{aligned}\mathcal{S}_2 = |\mathcal{I}_{g+1}| &\sum_{\substack{f_1,\dots,f_k\in\mathbb{A}^+_{\le x}\\ f_1\cdots f_k=\square}} \frac{1}{\sqrt{|f_1|\cdots|f_k|}} \prod_{P|f_1\cdots f_k}\left(1+\frac{1}{|P|}\right)^{-1}\\
&+ O\left(|\mathcal{I}_{g+1}|^{\frac{2}{3}}\right).\end{aligned} \tag{3.3}$$

For $f \in \mathbb{A}^+$, put

$$\alpha_f = \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1}.$$

Writing $f_1 \cdots f_k = m^2$, we see that

$$\sum_{m \in \mathbb{A}^+_{\le x/2}} \frac{d_k(m^2)}{|m|} \alpha_m \le \sum_{\substack{f_1, \dots, f_k \in \mathbb{A}^+_{\le x} \\ f_1 \cdots f_k = \Box}} \frac{1}{\sqrt{|f_1| \cdots |f_k|}} \alpha_{f_1 \cdots f_k}$$
$$\le \sum_{m \in \mathbb{A}^+_{\le kx/2}} \frac{d_k(m^2)}{|m|} \alpha_m.$$

It follows from Lemma 2.4 that

$$\sum_{m \in \mathbb{A}^+_{\le x/2}} \frac{d_k(m^2)}{|m|} \alpha_m \sim C(k) \left(\tfrac{2g}{15k}\right)^{\frac{k(k+1)}{2}}, \tag{3.4}$$

and

$$\sum_{m \in \mathbb{A}^+_{\le kx/2}} \frac{d_k(m^2)}{|m|} \alpha_m \sim C(k) \left(\tfrac{2g}{15}\right)^{\frac{k(k+1)}{2}}. \tag{3.5}$$

From (3.3) with (3.4) and (3.5), we can conclude that

$$\mathcal{S}_2 \ll |\mathcal{I}_{g+1}| \, g^{k(k+1)/2}. \tag{3.6}$$

3.2. Estimating $\mathcal{S}_1$

In this subsection we follow the arguments in [1, §4.2] to estimate $\mathcal{S}_1$ and give a proof of the main theorem. Using Lemma 2.1 with $s = \frac{1}{2}$, we have that

$$L(\tfrac{1}{2}, \chi_u) = \sum_{f \in \mathbb{A}^+_{\le g}} \frac{\chi_u(f)}{\sqrt{|f|}} + \sum_{f \in \mathbb{A}^+_{\le g-1}} \frac{\chi_u(f)}{\sqrt{|f|}}.$$

Since

$$A(u)^{k-1} = \sum_{f_1, \dots, f_{k-1} \in \mathbb{A}^+_{\le x}} \frac{\chi_u(f_1 \cdots f_{k-1})}{\sqrt{|f_1| \cdots |f_{k-1}|}},$$

we can write $\mathcal{S}_1 = \mathcal{S}_{1;g} + \mathcal{S}_{1;g-1}$, where

$$\mathcal{S}_{1;\ell} = \sum_{\substack{f \in \mathbb{A}^+_{\leq \ell} \\ f_1,\ldots,f_{k-1} \in \mathbb{A}^+_{\leq x}}} \frac{1}{\sqrt{|f||f_1|\cdots|f_{k-1}|}} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(ff_1\cdots f_{k-1})$$

for $\ell \in \{g, g-1\}$. Write $\mathcal{S}_{1;\ell} = (\mathcal{S}_{1;\ell})_{\square} + (\mathcal{S}_{1;\ell})_{\neq\square}$, where

$$(\mathcal{S}_{1;\ell})_{\square} = \sum_{\substack{f \in \mathbb{A}^+_{\leq \ell} \\ f_1,\ldots,f_{k-1} \in \mathbb{A}^+_{\leq x} \\ ff_1\cdots f_{k-1} = \square}} \frac{1}{\sqrt{|f||f_1|\cdots|f_{k-1}|}} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(ff_1\cdots f_{k-1})$$

and

$$(\mathcal{S}_{1;\ell})_{\neq\square} = \sum_{\substack{f \in \mathbb{A}^+_{\leq \ell} \\ f_1,\ldots,f_{k-1} \in \mathbb{A}^+_{\leq x} \\ ff_1\cdots f_{k-1} \neq \square}} \frac{1}{\sqrt{|f||f_1|\cdots|f_{k-1}|}} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(ff_1\cdots f_{k-1}).$$

We use Proposition 2.3 (1) to obtain that

$$\begin{aligned}(\mathcal{S}_{1;\ell})_{\neq\square} &\ll (g+1)\sqrt{|\mathcal{I}_{g+1}|} \sum_{f \in \mathbb{A}^+_{\leq \ell}} \frac{2^{\frac{\deg(f)}{2}}}{\sqrt{|f|}} \sum_{f_1 \in \mathbb{A}^+_{\leq x}} \frac{2^{\frac{\deg(f_1)}{2}}}{\sqrt{|f_1|}} \cdots \sum_{f_{k-1} \in \mathbb{A}^+_{\leq x}} \frac{2^{\frac{\deg(f_{k-1})}{2}}}{\sqrt{|f_{k-1}|}} \\ &\ll (g+1)(2q)^{\frac{\ell + x(k-1)}{2}} \sqrt{|\mathcal{I}_{g+1}|} \ll |\mathcal{I}_{g+1}|^{\frac{59}{60}}.\end{aligned}$$

We use Proposition 2.3 (2) to obtain that

$$\begin{aligned}(\mathcal{S}_{1;\ell})_{\square} = |\mathcal{I}_{g+1}| &\sum_{\substack{f \in \mathbb{A}^+_{\leq \ell} \\ f_1,\ldots,f_{k-1} \in \mathbb{A}^+_{\leq x} \\ ff_1\cdots f_{k-1} = \square}} \frac{1}{\sqrt{|f||f_1|\cdots|f_{k-1}|}} \prod_{P | ff_1\cdots f_{k-1}} \left(1 + \frac{1}{|P|}\right)^{-1} \\ &+ O\Bigg(|\mathcal{I}_{g+1}|^{\frac{1}{2}(1+\epsilon)} \sum_{\substack{f \in \mathbb{A}^+_{\leq \ell} \\ f_1,\ldots,f_{k-1} \in \mathbb{A}^+_{\leq x} \\ ff_1\cdots f_{k-1} = \square}} \frac{1}{\sqrt{|f||f_1|\cdots|f_{k-1}|}}\Bigg).\end{aligned}$$

The error term above is bounded by

$$\begin{aligned}&|\mathcal{I}_{g+1}|^{\frac{1}{2}(1+\epsilon)} \sum_{f\in\mathbb{A}^+_{\le\ell}} \frac{1}{\sqrt{|f|}} \sum_{f_1\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|f_1|}} \cdots \sum_{f_{k-1}\in\mathbb{A}^+_{\le x}} \frac{1}{\sqrt{|f_{k-1}|}} \\ &\quad \ll q^{\frac{g+x(k-1)}{2}} |\mathcal{I}_{g+1}|^{\frac{1}{2}(1+\epsilon)} \ll |\mathcal{I}_{g+1}|^{\frac{49}{60}+\frac{1}{2}\epsilon}.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathcal{S}_{1;\ell} = |\mathcal{I}_{g+1}| &\sum_{\substack{f\in\mathbb{A}^+_{\le\ell} \\ f_1,\ldots,f_{k-1}\in\mathbb{A}^+_{\le x} \\ ff_1\cdots f_{k-1}=\square}} \frac{1}{\sqrt{|f||f_1|\cdots|f_{k-1}|}} \prod_{P|ff_1\cdots f_{k-1}} \left(1+\frac{1}{|P|}\right)^{-1} \\ &+ O\left(|\mathcal{I}_{g+1}|^{\frac{59}{60}}\right).\end{aligned}$$

Write $f_1\cdots f_{k-1} = rh^2$, where $r,h \in \mathbb{A}^+$ and r is square-free. If $ff_1\cdots f_{k-1}$ is a square, then f is of the form rl^2 for some $l \in \mathbb{A}^+$. With this notation, the main term contribution is

$$|\mathcal{I}_{g+1}| \sum_{\substack{f_1,\ldots,f_{k-1}\in\mathbb{A}^+_{\le x} \\ f_1\cdots f_{k-1}=rh^2}} \frac{1}{|rh|} \sum_{l\in\mathbb{A}^+_{\le(\ell-\deg(r))/2}} \frac{1}{|l|}\alpha_{rhl}. \tag{3.7}$$

As in [1, (4.39)], we have

$$\sum_{l\in\mathbb{A}^+_{\le(\ell-\deg(r))/2}} \frac{1}{|l|}\alpha_{rhl} \sim C(r,h)\alpha_{rh} g$$

for some positive constant $C(r,h)$. Thus, (3.7) is

$$\gg g|\mathcal{I}_{g+1}| \sum_{\substack{r,h\in\mathbb{A}^+ \\ \deg(rh^2)\le x}} \frac{d_{k-1}(rh^2)}{|rh|}\alpha_{rh} \gg |\mathcal{I}_{g+1}|\, g^{k(k+1)/2},$$

where the last bound follows from Lemma 2.4. Hence, we obtain that

$$\mathcal{S}_{1;\ell} \gg |\mathcal{I}_{g+1}|\, g^{k(k+1)/2}$$

for $\ell \in \{g, g-1\}$. Therefore we can conclude that

$$\mathcal{S}_1 \gg |\mathcal{I}_{g+1}|\, g^{k(k+1)/2}. \tag{3.8}$$

Combining (3.6) and (3.8), we complete the proof of Theorem 1.1.

References

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