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EQUIVARIANT SEMIALGEBRAIC EMBEDDINGS

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Abstract. Let G be a semialgebraic group not necessarily compact. Let M be a proper semialgebraic G-set whose orbit space has a semialgebraic structure. In this paper, we prove the embeddability of M into a G-representation space when G is linear.

1. Introduction

In compact topological or smooth transformation group theory, Mostow [9] and Palais [10] used the slice theorem to establish the embedding of a G-space into a G-representation space. Palais extended the slice theorem and the G-embedding theorem to proper (topological or smooth) actions of noncompact groups. For semialgebraic transformation groups we consider semialgebraic groups G acting semialgebraically on semialgebraic sets M, i. e., the action map $\theta\colon G \times M \to M$ is semialgebraic. Note that a semialgebraic set is a subset of some $\mathbb{R}^n$ defined by finite number of polynomial equations and inequalities, and a semialgebraic map between semialgebraic sets is a map whose graph is a semialgebraic set. See Section 2 for some basic material for semialgebraic actions.

When G is compact, the semialgebraic slice theorem and the semialgebraic G-embedding theorem are also established in [4] and [16]. The purpose of this paper is to establish semialgebraic G-embedding theorem for proper semialgebraic G-sets. Using the slice theorem(Theorem 2.6) and following the general scheme of the topological embedding theorem

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by Palais we have the following embedding theorem which is a restatement of Theorem 4.4.

Embedding Theorem. *Let G be a semialgebraic linear group, and let M be a proper semialgebraic G-set whose orbit space has a semialgebraic structure. Then M can be equivariantly and semialgebraically embedded in a G-representation space.*

Here the linearity of G is not only sufficient but also necessary, see Remark 4.5. Note that there are semialgebraic groups which are not semialgebraically isomorphic to a semialgebraic linear group, see [8]. The embedding theorem is proved in Section 4 together with some applications.

2. Semialgebraic actions

In this section we study semialgebraic actions of semialgebraic groups on semialgebraic sets.

The class of *semialgebraic sets* in $\mathbb{R}^n$ is the smallest collection of subsets containing all subsets of the form $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for a real valued polynomial $p(x) = p(x_1, \ldots, x_n)$, which is stable under finite union, finite intersection and complement. A map $f\colon M \to N$ between semialgebraic sets M ($\subset \mathbb{R}^m$) and N ($\subset \mathbb{R}^n$) is called a *semialgebraic map* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. From now on we impose "Euclidian topology" on semialgebraic sets and mainly consider continuous semialgebraic maps. For the general theory of semialgebraic sets and semialgebraic maps, we refer the reader to [1, 6].

The definition of a semialgebraic group is given obviously, i.e., a semialgebraic set $G \subset \mathbb{R}^n$ is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion are semialgebraic. A semialgebraic homomorphism between two semialgebraic groups is a semialgebraic map that is a group homomorphism. If H is a subgroup and semialgebraic subset, then H is called a *semialgebraic subgroup*.

By a *semialgebraic transformation group* we mean a triple (G, M, θ), where G is a semialgebraic group, M is a semialgebraic set, and $\theta\colon G \times M \to M$ is a continuous semialgebraic map such that

(1) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in M$

(2) $\theta(e, x) = x$ for all $x \in M$, where e is the identity of G.

In this case M is called a *semialgebraic G-set*, and θ is called the *action map*. As usual we shortly write gx for $\theta(g, x)$. A *semialgebraic G-subset* of a semialgebraic G-set M is a G-invariant semialgebraic subset of M. A continuous semialgebraic map $f\colon M \to N$ between semialgebraic G-sets is called a *semialgebraic G-map* if it is G-equivariant, i.e., $f(gx) = gf(x)$ for all $g \in G$ and $x \in M$.

A continuous semialgebraic map $f\colon M \to N$ is called *semialgebraically proper* if $f^{-1}(C)$ is compact for every compact semialgebraic subset C of N. Since C should be semialgebraic in the definition, this notion is weaker than the condition that f is topologically proper. But we know that f is semialgebraically proper if and only if it is topologically proper, see [13, 15].

Now we define semialgebraically proper actions as follows. Let G be a semialgebraic group not necessarily compact. A semialgebraic action of G on M is *proper* if the augmented action map

$$\vartheta_*\colon G \times M \to M \times M, \quad (g, x) \mapsto (gx, x)$$

is (semialgebraically) proper. In this case M is called a *proper semialgebraic G-set*. Note that when G is compact, every semialgebraic G-set is proper.

We summarize some results about semialgebraic actions. For more details, see [7, 14].

Proposition 2.1 ([17, 19]). (1) *Every semialgebraic group has a Lie group structure, and hence locally compact.*
(2) *Every semialgebraic subgroup of a semialgebraic group is closed.*

Proposition 2.2. *Let M be a proper semialgebraic G-set and let $x \in M$, then*
(1) *the isotropy subgroup $G_x = \{g \in G \mid g(x) = x\}$ is compact and semialgebraic,*
(2) *the orbit $G(x) = \{gx \in M \mid g \in G\}$ is a closed semialgebraic subset of M,*
(3) *the evaluation map $\theta_x\colon G \to M$, $g \mapsto gx$, is proper,*
(4) *the fixed point set $M^G = \{x \in M \mid gx = x \text{ for all } g \in G\}$ is closed semialgebraic subset of M.*

Working in semialgebraic category requires a lot of nontrivial efforts to establish some of the properties which are easy or well-known in topological or smooth category. One of such properties is the existence of semialgebraic structure on the orbit space of a semialgebraic G-set. Namely, it is not quite obvious whether the orbit space M/G of a semialgebraic G-set M has a semialgebraic structure such that the orbit map is semialgebraic. A *semialgebraic structure* of M/G is a semialgebraic set Y together with a semialgebraic map $f\colon M \to Y$ which is topologically quotient map of M of G. In this case we can substitute M/G and the orbit map $\pi\colon M \to M/G$ with Y and f respectively. Brumfiel [3] and Scheiderer [20] gave us a partially positive answer of this question as follows.

Proposition 2.3 ([20]). *Let G be a semialgebraic group and M a proper semialgebraic G-set which is locally compact. Then the orbit space M/G has a semialgebraic structure.*

Note that when G is compact, M/G has a semialgebraic structure even if M is not locally compact(see [3]).

As a specific example of a semialgebraic proper action, we consider the following situation: let G be a semialgebraic group and H a semialgebraic subgroup of G. Then G can be seen as a semialgebraic set where H acts by the right multiplication on G. So G can be seen as a semialgebraic H-set.

Proposition 2.4 ([14]). *Let G be a semialgebraic group and H a semialgebraic subgroup of G. Then the H-space G is a proper semialgebraic H-set, and thus G/H has a semialgebraic structure such that the quotient map $G \to G/H$ is semialgebraic.*

Semialgebraic transformation groups have some nice properties which are not enjoyed by general topological or smooth transformation groups. The following theorem is one of such nice properties in semialgebraic category.

Theorem 2.5 ([14]). *Every proper semialgebraic G-set has finitely many orbit types.*

Let G be a semialgebraic group. Let M be a semialgebraic G-set and H a semialgebraic subgroup of G. A semialgebraic subset S of M will be called an *semialgebraic H-slice* if GS is an open semialgebraic subset of M and there exists a semialgebraic G-map $f\colon GS \to G/H$ such that $f^{-1}(eH) = S$. For $x \in M$ a *semialgebraic slice* at x means a semialgebraic G_x-slice S in M such that $x \in S$. We call GS a *semialgebraic G-tube* about $G(x)$.

Theorem 2.6 ([7]). *Let G be a semialgebraic group, and let M be a proper semialgebraic G-set whose orbit space has a semialgebraic structure. Then there exists a semialgebraic slice at every point of M. Moreover M can be covered by finitely many semialgebraic G-tubes.*

3. Semialgebraic linear groups

In this section we discuss some properties of semialgebraic linear groups.

We say a semialgebraic group G is *linear* if it has a faithful real semialgebraic representation, i.e., G is semialgebraically isomorphic to a semialgebraic subgroup of a general linear group $GL_n(\mathbb{R})$ for some n.

Note that a closed subgroup of $GL_n(\mathbb{R})$ needs not be semialgebraic even if it is connected. For example the following one-parameter subgroup

$$\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{\sqrt{2}t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

is closed but is not a semialgebraic subgroup of $GL_2(\mathbb{R})$. However the following proposition shows that a compact subgroup of $GL_n(\mathbb{R})$ is semialgebraic.

Proposition 3.1. *If H is a compact subgroup of $GL_n(\mathbb{R})$, then H is an algebraic subgroup of $GL_n(\mathbb{R})$. Hence if H is a compact subgroup of a semialgebraic linear group G, then H is a semialgebraic subgroup of G.*

Proof. See Remark 4.7 of [18]. □

For later use, we give the following two propositions.

Proposition 3.2 (c.f., [12, Section 3.1]). *Let G be a semialgebraic linear group, and let H be a compact (hence semialgebraic) subgroup of G. For a semialgebraic H-module U, there exists a semialgebraic G-module V such that $\operatorname{res}_H V$ contains U as an H-submodule. Here $\operatorname{res}_H V$ denotes the restriction of the G-module V to H.*

Proof. Since G is a linear group, there is a faithful representation $\varphi\colon G \to GL_n(\mathbb{R})$. Let $R(G)$ be the $\mathbb{R}$-algebra of real valued functions on G, generated by the matrix entries $a_{ij}\colon G \to \mathbb{R}$ for $\varphi\colon G \to GL_n(\mathbb{R})$, $\varphi(g) = (a_{ij}(g))$. Note that $R(G)$ has a G-module structure as follows: for $f \in R(G)$ and $g, h \in G$, $g \cdot f(h) = f(g^{-1}h)$. $R(H)$ is defined similarly. Since H is compact, $R(H)$ is the $\mathbb{R}$-algebra of all representative functions on H, see [2, Proposition III. (4.3)]. Let $R(G)|_H$ be the restrictions of the functions in $R(G)$ to H, i.e.,

$$R(G)|_H = \{f|_H\colon H \to \mathbb{R} \mid f \in R(G)\}.$$

Since $R(G)$ is generated by the matrix entries a_{ij} we can see that every element $f \in R(G)$ generates a finite dimensional G-invariant subspace of the vector space $C^0(G,\mathbb{R}) = \{f\colon G \to \mathbb{R} \mid f \text{ is a continuous function}\}$. Therefore a fortiori every element $f|_H \in R(G)|_H$ generates a finite dimensional H-invariant subspace of $C^0(H,\mathbb{R})$, i.e., $f|_H$ is a representative function on H. This shows that $R(G)|_H \subset R(H)$. Since φ is a faithful representation, $R(G)$ separates points of G, so a fortiori $R(G)|_H$ separates points of H. By the Stone-Weierstrass theorem $R(G)|_H$ is dense in the space $C^0(H,\mathbb{R})$. Since $R(G)|_H$ is closed in $R(H)$ by III.1.4 of [2], $R(G)|_H = R(H)$.

Without loss of generality, we may assume that U is an irreducible H-module. Then there exists an H-invariant submodule of $R(H)$ which is isomorphic to U (see [2, III.1.5]), and we identify this submodule with U. Let $\{f_1,\dots,f_k\}$ be a basis for U and choose $h_i \in p^{-1}(f_i)$ for $i = 1,\dots,k$ where $p\colon R(G) \to R(H)$ is the restriction map. Let V be the G-invariant subspace of $R(G)$ generated by $\{h_1,\dots,h_k\}$. Then since every element of $R(G)$ generates a finite dimensional G-invariant subspace of $R(G)$, V is a finite dimensional G-module. Moreover from the construction $\operatorname{res}_H V$ contains U as an irreducible H-submodule. Therefore it is enough to show that V is a semialgebraic G-module. Let W be the G-module defined by the faithful semialgebraic representation $\varphi\colon G \to GL_n(\mathbb{R})$. Since φ is semialgebraic, W is a semialgebraic G-module.

Note that the elements of V are consist of sum of products of $\{a_{ij}\}$. Consider a semialgebraic G-module

$$T = \bigoplus_{n=0}^{k}((W^* \otimes W) \otimes \cdots \otimes (W^* \otimes W))$$

for some k. Mapping $e_i^* \otimes e_j \mapsto a_{ij}$ defines a G-map from $\Psi\colon T \to R(G)$. By taking sufficiently large k we can assume V is contained in $\Psi(T)$. Then, since $\Psi(T)$ is semialgebraic, V is semialgebraic. □

The following proposition is the semialgebraic analogue of the result in Section 3.2 of [12], and the proof of it is simply the verbatim semialgebraic translation of the topological proof in the cited reference. Therefore we may skip the proof of the following proposition.

Proposition 3.3. *Let G be a semialgebraic linear group, and let H be a compact subgroup of G. Then there exists a semialgebraic G-module V and a point $v \in V$ such that the isotropy subgroup G_v at v is equal to H.*

4. Equivariant semialgebraic embeddings

In this section we prove the semialgebraic embedding of proper semialgebraic G-sets into a finite dimensional semialgebraic G-representation space. The general scheme of the proof follows the idea of Palais embedding theorem of proper topological G-spaces in [12]. When G is a compact semialgebraic linear group, the semialgebraic embedding theorem is proved in [4] and [16] in two different ways. Here we extend the method in [4] to a semialgebraic linear group which is not necessarily compact.

LEMMA 4.1. *Let G be a semialgebraic group and M a proper semialgebraic G-set. If $M - M^G$ admits an equivariant semialgebraic embedding in some semialgebraic representation space of G then so does M.*

Proof. If G is noncompact, then M^G is empty, there is nothing to prove. So we assume G is compact. Moreover, in this case, we can assume that the representation is orthogonal. Let M be a semialgebraic subset of $\mathbb{R}^n$ and M/G a semialgebraic subset of $\mathbb{R}^k$. We define a semialgebraic map $h\colon M/G \to \mathbb{R}$ by $h(z) = \operatorname{dist}(z, M^G/G) = \inf\{\|z-y\| \mid y \in M^G/G\}$. Then the composition map $\tilde{h} = h \circ \pi\colon M \to \mathbb{R}$ is semialgebraic and G-invariant. Let $f\colon M - M^G \to \Omega$ be a semialgebraic G-embedding.

Moreover, we can assume $\|f(x)\| = 1$ for all $x \in M - M^G$: let v be a non-zero real number. Clearly the map $\psi\colon \Omega \to \Omega \oplus \mathbb{R}$ defined by $\psi(x) = (x, v)$ is a semialgebraic G-embedding. Define $\varphi\colon \Omega \to \Omega \oplus \mathbb{R}$ by $\varphi(x) = \psi(x)/\|\psi(x)\|$, then $\varphi \circ f$ is the desired semialgebraic G-embedding.

So we assume $\|f(x)\| = 1$ for all $x \in M - M^G$ and define $\tilde{f}\colon M \to \Omega$ by

$$\tilde{f}(x) = \begin{cases} \tilde{h}(x)f(x) & \text{if } x \in M - M^G \\ 0 & \text{if } x \in M^G. \end{cases}$$

That $\tilde{f}$ is clearly a semialgebraic G-map. Now we define $\phi\colon M \to \mathbb{R}^k \oplus \Omega$ by $\phi(x) = (\pi(x), \tilde{f}(x))$ where $\mathbb{R}^k$ denote k-dimensional trivial real G-representation space. Then ϕ can be shown to be continuous (see [11, p.22]). Hence ϕ is a semialgebraic G-embedding. □

LEMMA 4.2. *Let G be a semialgebraic group and let M be a proper semialgebraic G-set whose orbit space has a semialgebraic structure. Let $\{U_1, \ldots, U_k\}$ be a covering of M by open semialgebraic G-subsets of*

M. If each U_i admits a semialgebraic G-embedding in a semialgebraic G-representation space Ω_i then so does M.

Proof. Let $\pi\colon M \to M/G$ be the semialgebraic orbit map. Let $U_i^* = \pi(U_i)$ and let $h_1^*, \dots, h_k^*\colon M/G \to [0,1]$ be a semialgebraic partition of unity subordinate to $U_1^*, \dots, U_k^*$ (see [5, Theorem 1.6]). Define a semialgebraic G-invariant map $h_i\colon M \to [0,1]$ by $h_i = h_i^* \circ \pi$.

Let $\phi_i\colon U_i \to \Omega_i$ be semialgebraic G-embeddings. Now we define continuous semialgebraic G-maps $\varphi_i\colon M \to \Omega_i$ by

$$\varphi_i(x) = \begin{cases} h_i(x)\phi_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \notin U_i. \end{cases}$$

Let $\mathbb{R}^k$ denote k-dimensional trivial real G-representation space. Then the map $\varphi_0\colon M \to \mathbb{R}^k$ defined by $\varphi_0(x) = (h_1(x), \dots, h_k(x))$ is a semialgebraic G-invariant map. The map

$$\varphi\colon M \to \mathbb{R}^k \oplus \Omega_1 \cdots \oplus \Omega_k,\ x \mapsto (\varphi_0(x), \varphi_1(x), \dots, \varphi_k(x))$$

is a G-embedding(see [11] or [12] for the detail). Hence φ is a desired semialgebraic G-embedding. □

Lemma 4.3. *Let G be a semialgebraic linear group and H a compact semialgebraic subgroup of G. If Ω is a semialgebraic H-representation space then there exists a semialgebraic H-embedding of Ω onto a semialgebraic H-slice in some semialgebraic G-representation space Ξ.*

Proof. By Proposition 3.2, there is a semialgebraic G-representation space Ω' which includes Ω as an H-invariant linear subspace. By Proposition 3.3, there exist a semialgebraic G-representation space Ξ' and a point $u_0 (\neq 0)$ of Ξ' such that $G_{u_0} = H$. Set $\Xi = \Xi' \oplus \Omega'$. Then Ξ is a semialgebraic G-representation space. Clearly the map $\varphi\colon \Omega \hookrightarrow \Xi = \Xi' \oplus \Omega'$ defined by $\varphi(v) = (u_0, v)$ is a semialgebraic H-embedding. We claim that the image $S = \varphi(\Omega)$ is an H-slice in GS. To construct a continuous semialgebraic G-map $f\colon GS \to G(u_0)$ with $f^{-1}(u_0) = S$, consider the projection map $\Xi = \Xi' \oplus \Omega' \to \Xi'$ which is obviously G-equivariant. Take f as the restriction on GS of the projection map, then its image is clearly $G(u_0)$. Moreover if $g \notin H$ and $(u_0, v) \in S$ then $g(u_0, v) = (gu_0, gv) \notin S$ because $g \notin H = G_{u_0}$. This leads to the equality $f^{-1}(u_0) = S$, now the proof is complete. □

We now prove the embedding theorem for proper semialgebraic actions.

Theorem 4.4 (Embedding Theorem). *Let G be a semialgebraic linear group, and let M be a proper semialgebraic G-set whose orbit space M/G has a semialgebraic structure. Then M can be equivariantly and semialgebraically embedded in a finite dimensional semialgebraic G-representation space.*

Proof. By the induction argument, we can assume that the theorem is true for all proper semialgebraic subgroups of G. By Lemma 4.1 it suffices to show that the semialgebraic G-set $M - M^G$ admits a semialgebraic G-embedding in a semialgebraic G-representation space. By Theorem 2.6 there are a finite number of semialgebraic H_i-slices $S_1, \ldots, S_k$ of $M - M^G$ such that $GS_1, \ldots, GS_k$ cover $M - M^G$. Since each H_i is a strictly smaller compact subgroup of G, by the induction hypothesis, there is a semialgebraic H_i-embedding $\varphi_i \colon S_i \to \Omega_i$ in a semialgebraic H_i-representation space Ω_i. By Lemma 4.3, there exists a semialgebraic H_i-embedding ψ_i of Ω_i onto a semialgebraic H_i-slice in some semialgebraic G-representation space Ξ_i. Then the map $f_i \colon GS_i \to \Xi_i$, defined by $f_i(gs) = g\psi_i(\varphi_i(s))$, is a semialgebraic G-embedding. Since each GS_i is a G-invariant open semialgebraic subset in M, by Lemma 4.2, $M - M^G$ admits a semialgebraic G-embedding in a semialgebraic G-representation space. □

Remark 4.5. The linearity condition of the semialgebraic group is necessary as well as sufficient if the action is effective. Indeed, let G be a semialgebraic group and let M be equal to G viewed as a semialgebraic G-set with the left multiplication. If M has a semialgebraic embedding $f \colon M \to \mathbb{R}^n(\rho)$ for some semialgebraic representation space $\mathbb{R}^n(\rho)$ of G, it follows that G acts effectively on $\mathbb{R}^n(\rho)$, i.e. ρ is faithful, so that G is a semialgebraic linear group. Moreover there is a compact semialgebraic group which is not linear(see [16]).

Any locally compact semialgebraic set can be semialgebraically embedded in some $\mathbb{R}^n$ as a closed semialgebraic subset, see [5, 6]. Now we have the similar result for semialgebraic proper linear actions.

Corollary 4.6. *Let G be a semialgebraic linear group. Then every locally compact proper semialgebraic G-set can be equivariantly and semialgebraically embedded in some semialgebraic representation space Ω of G as a closed semialgebraic G-subset of Ω.*

Proof. Let M be a locally compact proper semialgebraic G-set. By Theorem 4.4, we can view M as a semialgebraic G-subset of a semialgebraic G-representation space Ω'.

Set $A = \overline{M} - M$ where $\overline{M}$ is the closure of M in Ω'. Since M is locally compact, A is a closed semialgebraic subset of Ω' (see [5, Proposition 3.3]). We may assume $A \neq \varnothing$ unless M is already closed. The map $f\colon \Omega' \to \mathbb{R}$, defined by, $f(x) = \mathrm{dist}(x, A)$, is semialgebraic. Define a semialgebraic embedding $\varphi\colon M \to \Omega' \oplus \mathbb{R}$ by $\varphi(x) = (x, 1/f(x))$. Since Ω' is a G-representation, φ is a G-map. Clearly, the image of φ is the closed semialgebraic set defined by

$$\{(x, y) \in \Omega' \oplus \mathbb{R} \mid x \in \overline{M},\ yf(x) = 1\}.$$

□

Corollary 4.7. *Let G be a semialgebraic linear group. Then every proper semialgebraic G-manifold can be equivariantly and semialgebraically embedded in some semialgebraic representation space Ω of G as a closed semialgebraic G-subset of Ω.*

Proof. It is immediate from Corollary 4.6 since every semialgebraic manifold is locally compact. □

Corollary 4.8. *Let G be a semialgebraic linear group. If M is a locally compact proper semialgebraic G-set, then there exists a semialgebraic one point G-compactification of M.*

Proof. By Corollary 4.6, we may assume that M is a closed semialgebraic G-subset of some semialgebraic representation space Ω of G. We may assume that $0 \notin M$ because otherwise we can replace M by $M \times \{1\} \subset \Omega \oplus \mathbb{R}$. Let $\tau\colon \Omega - \{0\} \to \Omega - \{0\}$ be the inversion through the unit sphere, $\tau(x) = x/\|x\|^2$. Clearly τ is a semialgebraic homeomorphism, and thus $\tau(M) \cup \{0\}$ is a semialgebraic set in Ω. From this we can see that $\tau(M) \cup \{0\}$ is the desired compact semialgebraic G-set. □

Remark 4.9. In the above corollary, if G is not compact, then the action of G on the one-point compactification is no more proper.

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