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# RELATIVE $(p,q,t)L$-TH TYPE AND RELATIVE $(p,q,t)L$-TH WEAK TYPE ORIENTED GROWTH PROPERTIES OF WRONSKIAN

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions using relative $(p,q,t)L$-th order, relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type and that of Wronskian generated by one of the factors.

## 1. Introduction

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [7, 10, 15, 16]. We also use the standard notations and definitions of the theory of entire functions which are available in [14] and therefore we do not explain those in details. Let $f$ is an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to $f$ is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r}|f(z)|$. If $f$ is non-constant then it has the following property:

**Property (A)** ([2])**:** A non-constant entire function $f$ is said have the *Property (A)* if for any $\sigma > 1$ and for all sufficiently large values of $r$, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [2].

When $f$ is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of $f$, playing the same role as $M_f(r)$.

The integrated counting function $N_f(r,a)(\overline{N}_f(r,a))$ of $a$-points (distinct $a$-points) of $f$ is defined as

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$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(r,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

where we denote by $n_f(t,a)(\overline{n}_f(t,a))$ the number of $a$-points (distinct $a$-points) of $f$ in $|z| \leq t$ and an $\infty$-point is a pole of $f$. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function. On the other hand, the function $m_f(r) \equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$

and an $\infty$ -point is a pole of $f$ .

Analogously, $m_{\frac{1}{f-a}}(r) \equiv m_f(r,a)$ is defined when $a$ is not an $\infty$-point of $f$.

Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to $f$ is defined as

$$T_f(r) = N_f(r) + m_f(r).$$

When $f$ is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r) = 0$.

However, for a meromorphic function $f$, the Wronskian determinant $W(f) = W(a_1, a_2, ....a_k, f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where $a_1, a_2, ....a_k$ are linearly independent meromorphic functions and small with respect to $f$ ( i.e., $T_{a_i}(r) = S(r,f)$ for $i = 1, 2, 3...k$). From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leq 2$ (see [7, p. 43]), where $\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$. If in particular $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that $f$ has the maximum deficiency sum.

Moreover, if $f$ is non-constant entire then $T_f(r)$ is strictly increasing and continuous function of $r$. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of $f$ with respect to $g$ in terms of the Nevanlinna's Characteristic functions of the meromorphic functions $f$ and $g$.

However let us consider that $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers. We define $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper $l, p, q, m$ and $n$ always denote positive integers and $t \in \mathbb{N} \cup \{-1, 0\}$. Now considering this, we just recall that Shen et al. [12] defined the $(m, n)$-$\varphi$ order and $(m, n)$-$\varphi$ lower order of entire functions $f$ which are as follows:

**Definition 1.1** ([12])**.** Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function and $m \geq n$. The $(m, n)$-$\varphi$ order $\rho^{(m,n)}(f, \varphi)$ and $(m, n)$-$\varphi$ lower order $\lambda^{(m,n)}(f, \varphi)$ of entire functions $f$ are defined as:

$$\rho^{(m,n)}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} \text{ and } \lambda^{(m,n)}(f, \varphi) = \underline{\lim_{r\to\infty}} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)}.$$

If $f$ is a meromorphic function, then

$$\rho^{(m,n)}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)} \text{ and } \lambda^{(m,n)}(f, \varphi) = \underline{\lim_{r\to\infty}} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}.$$

Further for any non-decreasing unbounded function $\varphi : [0, +\infty) \to (0, +\infty)$, if we assume $\lim_{r\to+\infty} \frac{\log^{[n]} \varphi(\alpha r)}{\log^{[n]} \varphi(r)} = 1$ for all $\alpha > 0$, then for any entire function $f$, using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ cf.[7], one can easily verify that (see [12])

$$\rho^{(m,n)}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} = \overline{\lim_{r\to\infty}} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}$$

$$\left(\lambda^{(m,n)}(f, \varphi) = \underline{\lim_{r\to\infty}} \frac{\log^{[m]} M_f(r)}{\log^{[n]} \varphi(r)} = \underline{\lim_{r\to\infty}} \frac{\log^{[m-1]} T_f(r)}{\log^{[n]} \varphi(r)}\right)$$

when $m > 1$.

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r$, then the above definition reduces to the following definition:

**Definition 1.2.** The $(p,q)$-th order and $(p,q)$-th lower order of an entire function $f$ are defined as:

$$\rho^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If $f$ is a meromorphic function, then

$$\rho^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

Definition 1.2 avoids the restriction $p \geq q$ of the original definition of $(p,q)$-th order (respectively $(p,q)$-th lower order) of entire functions introduced by Juneja et al. [8].

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function $f$. For details about generalized order one may see [11]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function $f$.

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [8]):

**Definition 1.3.** An entire function $f$ is said to have *index-pair* $(p,q)$ if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots \end{cases}.$$

Analogously one can easily verify that Definition 1.3 of index-pair can also be applicable to a meromorphic function $f$.

However, the function $f$ is said to be of regular $(p,q)$ growth when $(p,q)$-th order and $(p,q)$-th lower order of $f$ are the same. Functions which are not of regular $(p,q)$ growth are said to be of irregular $(p,q)$ growth.

For entire functions, Somasundaram and Thamizharasi [13] introduced the notions of the growth indicators $L$-order and $L$-lower order where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant '$a$', i.e., $\lim_{r\to\infty} \frac{L(ar)}{L(r)} = 1$ where $L \equiv L(r)$ is a positive continuous function increasing slowly. The more generalized concept of $L$-order and $L$-lower order for entire function are $L^*$-order and $L^*$-lower order. Their definitions are as follows:

**Definition 1.4** ([13])**.** The $L^*$-order $\rho_f^{L^*}$ and the $L^*$-lower order $\lambda_f^{L^*}$ of an entire function $f$ are defined as

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log[re^{L(r)}]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log[re^{L(r)}]}.$$

When $f$ is meromorphic one can easily verify that

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log[re^{L(r)}]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log[re^{L(r)}]}.$$

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r \cdot \exp^{[t+1]} L(r)$, then Definition 1.1 turns into the definitions of $(p, q, t)L$-th order and $(p, q, t)L$-th lower order of an entire function $f$ which are as follows:

$$\rho_f^L(p, q, t) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)} \text{ and } \lambda_f^L(p, q, t) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

If $f$ is a meromorphic function, then

$$\rho_f^L(p, q, t) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)} \text{ and } \lambda_f^L(p, q, t) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

In order to compare the relative growth of two entire functions having same non zero finite $(p, q, t)L$-th order, one may introduce the definitions of $(p, q, t)L$-th type (respectively $(p, q, t)L$-th lower type) of entire functions having finite positive finite $(p, q, t)L$-th order in the following manner:

**Definition 1.5** ([5])**.** Let $f$ be an entire function with non-zero finite $(p, q, t)L$-th order $\rho_f^L(p, q, t)$. The $(p, q, t)L$-th type denoted by $\sigma_f^L(p, q, t)$ and $(p, q, t)L$-th lower type denoted by $\overline{\sigma}_f^L(p, q, t)$ are respectively defined as follows:

$$\sigma_f^L(p, q, t) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}$$

and

$$\overline{\sigma}_f^L(p, q, t) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_f^L(p,q,t)}}.$$

Analogously in order to determine the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th lower order one may introduce the definition of $(p,q,t)L$-th weak type of entire functions having finite positive $(p,q,t)L$-th lower order in the following way:

**Definition 1.6** ([5])**.** The $(p,q,t)L$-th weak type denoted by $\tau_f^L(p,q,t)$ of an entire function $f$ is defined as follows:

$$\tau_f^L(p,q,t) = \varliminf_{r\to+\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_f^L(p,q,t)}},\ 0 < \ \lambda_f^L(p,q,t) \ < \infty.$$

Also one may define the growth indicator $\overline{\tau}_f^L(p,q,t)$ of an entire function $f$ in the following manner :

$$\overline{\tau}_f^L(p,q,t) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_f^L(p,q,t)}},\ 0 < \ \lambda_f^L(p,q,t) \ < \infty.$$

Mainly the growth investigation of entire or meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire or meromorphic function with respect to a new entire function, the notions of relative growth indicators [2, 9] will come. Extending this notion,one may introduce the definitions of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of a meromorphic function $f$ with respect to another entire function $g$ in the following way:

**Definition 1.7** ([5])**.** Let $f$ be a meromorphic function and $g$ be an entire function. Then relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of a meromorphic function $f$ with respect to an entire function $g$ are defined by

$$\rho_g^{(p,q,t)L}(f) \ = \ \varlimsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}$$

and

$$\lambda_g^{(p,q,t)L}(f) \ = \ \varliminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Now to compare the relative growth of two meromorphic functions having same non zero finite relative $(p,q,t)L$-th order with respect to an entire function, one can introduce the notion of relative $(p,q,t)L$-th type (respectively relative $(p,q,t)L$-th lower type) of a meromorphic function with respect to an entire function which is as follows:

**Definition 1.8** ([5])**.** Let $f$ be a meromorphic function and $g$ be an entire function with $0 < \rho_g^{(p,q,t)L}(f) < \infty$. The relative $(p,q,t)L$-th type $\sigma_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower type $\overline{\sigma}_g^{(p,q,t)L}(f)$ of $f$ with respect to $g$ are defined as

$$\sigma_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}}$$

and

$$\overline{\sigma}_g^{(p,q,t)L}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}}.$$

Similarly, one can define relative $(p,q,t)L$-th weak type to determine the relative growth of two meromorphic functions having same non zero finite relative $(p,q,t)L$-th lower order with respect to an entire function in the following manner:

**Definition 1.9** ([5])**.** Let $f$ be a meromorphic function and $g$ be an entire function with $0 < \lambda_g^{(p,q,t)L}(f) < \infty$. The relative $(p,q,t)L$-th weak type $\tau_g^{(p,q,t)L}(f)$ of $f$ with respect to $g$ is defined as:

$$\tau_g^{(p,q,t)L}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_g^{(p,q,t)L}(f)}}.$$

Further one may define the growth indicator $\overline{\tau}_g^{(p,q,t)L}(f)$ of an entire function $f$ with respect to an entire function $g$ in the following way :

$$\overline{\tau}_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_g^{(p,q,t)L}(f)}}$$

when $0 < \lambda_g^{(p,q,t)L}(f) < \infty$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. Actually in the paper we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions usingrelative $(p,q,t)L$-th order, relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type of meromorphic function with respect to another entire functionr where $p, q \in \mathbb{N}$ and $t \in \mathbb{N} \cup \{-1, 0\}$ and that of Wronskian generated by one of the factors.

## 2. PRELIMINARIES

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.1** ([1]). *Let $f$ be meromorphic and $g$ be entire then for all sufficiently large values of $r$,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

**Lemma 2.2** ([6]). *Let $f$ be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f(\alpha r^{\delta}).$$

**Lemma 2.3** ([4]). *Let $f$ be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f) = 2$ and $g$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$-th type where $m > 2$. Then*

$$\lim_{r\to\infty}\frac{\log^{[p-1]}T^{-1}_{W(g)}(T_{W(f)}(r))}{\log^{[p-1]}T^{-1}_{g}(T_{f}(r))} = 1.$$

**Lemma 2.4** ([3]). *Let $f$ be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f) = 2$ and $g$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth where $m > 1$. Then the relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of $W(f)$ with respect to $W(g)$ are same as those of $f$ with respect to $g$ i.e.,*

$$\rho^{(p,q,t)L}_{W(g)}(W(f)) = \rho^{(p,q,t)L}_{g}(f) \text{ and } \lambda^{(p,q,t)L}_{W(g)}(W(f)) = \lambda^{(p,q,t)L}_{g}(f).$$

**Lemma 2.5.** *Let $f$ be a transcendental meromorphic function with $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f) = 2$ and $g$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$-th type where $m > 2$. Then the relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th lower type of $W(f)$ with respect to $W(g)$ are same as those of $f$ with respect to $g$ if $\rho^{(p,q,t)L}_{g}(f)$ is positive finite, i.e.,*

$$\sigma^{(p,q,t)L}_{W(g)}(W(f)) = \sigma^{(p,q,t)L}_{g}(f) \text{ and } \overline{\sigma}^{(p,q,t)L}_{W(g)}(W(f)) = \overline{\sigma}^{(p,q,t)L}_{g}(f).$$

*Proof.* Now from Lemma 2.3 and Lemma 2.4, we get that

$$\begin{aligned}
&\sigma_{W(g)}^{(p,q,t)L}(W(f)) \\
&= \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{W(g)}^{-1}(T_{W(f)}(r))}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_{W(g)}^{(p,q,t)L}(W(f))}} \\
&= \lim_{r\to\infty} \frac{\log^{[p-1]} T_{W(g)}^{-1}(T_{W(f)}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \cdot \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}} \\
&= 1 \cdot \sigma_g^{(p,q)}(f) = \sigma_g^{(p,q)}(f).
\end{aligned}$$

Similarly, $\overline{\sigma}_{W(g)}^{(p,q,t)L}(W(f)) = \overline{\sigma}_g^{(p,q,t)L}(f)$. This completes the proof. □

**Lemma 2.6.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$ and $g$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and nonzero finite $(m,p)$-th type where $m > 2$. Then $\tau_{W(g)}^{(p,q,t)L}(W(f))$ and $\overline{\tau}_{W(g)}^{(p,q,t)L}(W(f))$ are same as those of $f$ with respect to $g$, i.e.,*

$$\tau_{W(g)}^{(p,q,t)L}(W(f)) = \tau_g^{(p,q,t)L}(f) \text{ and } \overline{\tau}_{W(g)}^{(p,q,t)L}(W(f)) = \overline{\tau}_g^{(p,q,t)L}(f).$$

*when $\lambda_g^{(p,q,t)L}(f)$ is positive finite.*

We omit the proof of the above lemma as it can be carried out in the line of Lemma 2.5.

## 3. Main Results

In this section we present the main results of the paper.

**Theorem 3.1.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and nonzero finite $(m,p)$-th type such that $0 < \overline{\sigma}_h^{(p,q,t)L}(f\circ g) \leq \sigma_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < \overline{\sigma}_h^{(p,q,t)L}(f) \leq \sigma_h^{(p,q,t)L}(f) < \infty$ and $\rho_h^{(p,q,t)L}(f\circ g) = \rho_h^{(p,q,t)L}(f)$ where $m > 2$, then*

$$\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)} \leq \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}$$

$$\leq \min\Big\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}\Big\} \leq \max\Big\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}\Big\}$$

$$\leq \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \leq \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}.$$

*Proof.* From the definition of $\sigma_{W(h)}^{(p,q,t)L}(W(f))$ and $\overline{\sigma}_h^{(p,q,t)L}(f\circ g)$ and in view of Lemma 2.4 and Lemma 2.5, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$,

$$\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r)) \geq (\overline{\sigma}_h^{(p,q,t)L}(f\circ g)-\varepsilon)[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f\circ g)}, \tag{3.1}$$

and

$$\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) \leq (\sigma_{W(h)}^{(p,q,t)L}(W(f))+\varepsilon)[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_{W(h)}^{(p,q,t)L}(W(f))}$$

(3.2)

$$i.e.,\ \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) \leq (\sigma_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f)}.$$

Now from (3.1), (3.2) and the condition $\rho_h^{(p,q,t)L}(f\circ g) = \rho_h^{(p,q,t)L}(f)$, it follows for all sufficiently large values of $r$,

$$\frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \geqslant \frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)-\varepsilon}{\sigma_h^{(p,q,t)L}(f)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \geqslant \frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}. \tag{3.3}$$

Again for a sequence of values of $r$ tending to infinity,

$$\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r)) \leq (\overline{\sigma}_h^{(p,q,t)L}(f\circ g)+\varepsilon)[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f\circ g)}, \tag{3.4}$$

and for all sufficiently large values of $r$,

$$\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) \geq (\overline{\sigma}_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_{W(h)}^{(p,q,t)L}(W(f))},$$

(3.5)

$$i.e.,\ \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) \geq (\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f)}.$$

Combining (3.4) and (3.5) and the condition $\rho_h^{(p,q,t)L}(f\circ g) = \rho_h^{(p,q,t)L}(f)$, we get for a sequence of values of $r$ tending to infinity

$$\frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \leq \frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)+\varepsilon}{\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))}\leq\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}.\tag{3.6}$$

Also in view of Lemma 2.4 and Lemma 2.5 and for a sequence of values of $r$ tending to infinity, it follows that

$$\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\leq(\overline{\sigma}_{W(h)}^{(p,q,t)L}(W(f))+\varepsilon)[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_{W(h)}^{(p,q,t)L}(W(f))},$$

$$\text{i.e., }\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\leq(\overline{\sigma}_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f)}.\tag{3.7}$$

Now from (3.1), (3.7) and the condition $\rho_h^{(p,q,t)L}(f\circ g)=\rho_h^{(p,q,t)L}(f)$, we obtain for a sequence of values of $r$ tending to infinity

$$\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))}\geq\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)-\varepsilon}{\overline{\sigma}_h^{(p,q,t)L}(f)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we get from above

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))}\geq\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}.\tag{3.8}$$

Also for all sufficiently large values of $r$,

$$\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))\leq(\sigma_h^{(p,q,t)L}(f\circ g)+\varepsilon)[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f\circ g)}.\tag{3.9}$$

In view of the condition $\rho_h^{(p,q,t)L}(f\circ g)=\rho_h^{(p,q,t)L}(f)$, it follows from (3.5) and (3.9) for all sufficiently large values of $r$,

$$\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))}\leq\frac{\sigma_h^{(p,q,t)L}(f\circ g)+\varepsilon}{\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))}\leq\frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}.\tag{3.10}$$

Again from the definition of $\sigma_{W(h)}^{(p,q,t)L}(W(f))$ and in view of Lemma 2.4 and Lemma 2.5, we get for a sequence of values of $r$ tending to infinity

$$\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\geq(\sigma_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_{W(h)}^{(p,q,t)L}(W(f))},$$

$$\text{i.e., }\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\geq(\sigma_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f)}.\tag{3.11}$$

Now from (3.9), (3.11) and the condition $\rho_h^{(p,q,t)L}(f\circ g)=\rho_h^{(p,q,t)L}(f)$, it follows for a sequence of values of $r$ tending to infinity

$$\frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}\leq \frac{\sigma_h^{(p,q,t)L}(f\circ g)+\varepsilon}{\sigma_h^{(p,q,t)L}(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain

$$\lim_{r\to\infty}\frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}\leq \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}. \tag{3.12}$$

Again for a sequence of values of $r$ tending to infinity

(3.13)
$$\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))\geqslant(\sigma_h^{(p,q,t)L}(f\circ g)-\varepsilon)[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f\circ g)}.$$

Combining (3.2) and (3.13) and in view of the condition

$$\rho_h^{(p,q,t)L}(f\circ g)=\rho_h^{(p,q,t)L}(f),$$

we get for a sequence of values of $r$ tending to infinity

$$\frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}\geqslant \frac{\sigma_h^{(p,q,t)L}(f\circ g)-\varepsilon}{\sigma_h^{(p,q,t)L}(f)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}\geqslant \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}. \tag{3.14}$$

Thus the theorem follows from (3.3), (3.6), (3.8), (3.10), (3.12) and (3.14). □

The following theorem can be proved in the line of Theorem 3.1 and so its proof is omitted.

**Theorem 3.2.** *If $f$ be a meromorphic function, $g$ be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$-th type such that $0<\overline{\sigma}_h^{(p,q,t)L}(f\circ g)\leq\sigma_h^{(p,q,t)L}(f\circ g)<\infty$, $0<\overline{\sigma}_h^{(p,q,t)L}(g)\leq\sigma_h^{(p,q,t)L}(g)$ $<\infty$ and $\rho_h^{(p,q,t)L}(f\circ g)=\rho_h^{(p,q,t)L}(g)$ where $m>2$, then*

$$\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(g)} \le \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(g)}(r))}$$

$$\le \min\left\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(g)}, \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(g)}\right\} \le \max\left\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(g)}, \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(g)}\right\}$$

$$\le \overline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(g)}(r))} \le \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(g)}.$$

Now in the line of Theorem 3.1 and Theorem 3.2 respectively and in view of Lemma 2.4 and Lemma 2.6, one can easily prove the following two theorems using the notion of relative $(p,q,t)L$-th weak type and therefore their proofs are omitted.

**Theorem 3.3.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$-th type such that $0 < \tau_h^{(p,q,t)L}(f\circ g) \le \overline{\tau}_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < \tau_h^{(p,q,t)L}(f) \le \overline{\tau}_h^{(p,q,t)L}(f) < \infty$ and $\lambda_h^{(p,q,t)L}(f\circ g) = \lambda_h^{(p,q,t)L}(f)$ where $m > 2$, then*

$$\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(f)} \le \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}$$

$$\le \min\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(f)}, \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(f)}\right\} \le \max\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(f)}, \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(f)}\right\}$$

$$\le \overline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \le \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(f)}.$$

**Theorem 3.4.** *If $f$ be a meromorphic function, $g$ be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g) + \delta(\infty;g) = 2$ and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$-th type such that $0 < \tau_h^{(p,q,t)L}(f\circ g) \le \overline{\tau}_h^{(p,q,t)L}(f\circ g) < \infty$, $0 < \tau_h^{(p,q,t)L}(g) \le \overline{\tau}_h^{(p,q,t)L}(g) < \infty$ and $\lambda_h^{(p,q,t)L}(f\circ g) = \lambda_h^{(p,q,t)L}(g)$ where $m > 2$, then*

$$\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(g)} \le \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(g)}(r))}$$

$$\leq \min\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(g)}, \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(g)}\right\} \leq \max\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(g)}, \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(g)}\right\}$$

$$\leq \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(g)}(r))} \leq \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(g)}.$$

We may now state the following theorems without their proofs based on relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type:

**Theorem 3.5.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$ -th type such that $0<\overline{\sigma}_h^{(p,q,t)L}(f\circ g)\leq\sigma_h^{(p,q,t)L}(f\circ g)<\infty$, $0<\tau_h^{(p,q,t)L}(f)\leq\overline{\tau}_h^{(p,q,t)L}(f)<\infty$ and $\rho_h^{(p,q,t)L}(f\circ g)=\lambda_h^{(p,q,t)L}(f)$ where $m>2$, then*

$$\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(f)} \leq \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}$$

$$\leq \min\left\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(f)}, \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(f)}\right\} \leq \max\left\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(f)}, \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(f)}\right\}$$

$$\leq \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \leq \frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(f)}.$$

**Theorem 3.6.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$ -th type such that $0<\tau_h^{(p,q,t)L}(f\circ g)\leq\overline{\tau}_h^{(p,q,t)L}(f\circ g)<\infty$, $0<\overline{\sigma}_h^{(p,q,t)L}(f)\leq\sigma_h^{(p,q,t)L}(f)<\infty$ and $\lambda_h^{(p,q,t)L}(f\circ g)=\rho_h^{(p,q,t)L}(f)$ where $m>2$, then*

$$\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)} \leq \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}$$

$$\leq \min\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}\right\} \leq \max\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}, \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(f)}\right\}$$

$$\leq \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))} \leq \frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(f)}.$$

**Theorem 3.7.** *If $f$ be a meromorphic function, $g$ be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$ -th type such that $0<\overline{\sigma}_h^{(p,q,t)L}(f\circ g)\leq\sigma_h^{(p,q,t)L}(f\circ g)<\infty$, $0<\tau_h^{(p,q,t)L}(g)\leq\overline{\tau}_h^{(p,q,t)L}(g)$ $<\infty$ and $\rho_h^{(p,q,t)L}(f\circ g)=\lambda_h^{(p,q,t)L}(g)$ where $m>2$, then*

$$\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(g)}\leq\underline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(g)}(r))}$$
$$\leq\min\left\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(g)},\frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(g)}\right\}$$
$$\leq\max\left\{\frac{\overline{\sigma}_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(g)},\frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\overline{\tau}_h^{(p,q,t)L}(g)}\right\}$$
$$\leq\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(g)}(r))}\leq\frac{\sigma_h^{(p,q,t)L}(f\circ g)}{\tau_h^{(p,q,t)L}(g)}.$$

**Theorem 3.8.** *If $f$ be a meromorphic function, $g$ be a transcendental entire function with $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(m,p)$ growth and non zero finite $(m,p)$-th type such that $0<\tau_h^L(f\circ g)\leq\overline{\tau}_h^L(f\circ g)<\infty$, $0<\overline{\sigma}_h^{(p,q,t)L}(g)\leq\sigma_h^{(p,q,t)L}(g)<\infty$ and $\lambda_h^L(f\circ g)=\rho_h^{(p,q,t)L}(g)$ where $m>2$, then*

$$\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(g)}\leq\underline{\lim}_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(g)}(r))}$$
$$\leq\min\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(g)},\frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(g)}\right\}\leq\max\left\{\frac{\tau_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(g)},\frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\sigma_h^{(p,q,t)L}(g)}\right\}$$
$$\leq\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(g)}(r))}\leq\frac{\overline{\tau}_h^{(p,q,t)L}(f\circ g)}{\overline{\sigma}_h^{(p,q,t)L}(g)}.$$

**Theorem 3.9.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\rho_h^{(p,q,t)L}(f)=\rho_g^L(m,n,t)$, $0<\sigma_g^L(m,n,t)<\infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f)>0$ where $m-1=n=q$ and $l>2$. If $h$ satisfies the Property (A), then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \textit{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \rho_h^{(p,q,t)L}(f) & \textit{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

*Proof.* Let us suppose that $\beta > 2$ and $\delta \to 1^+$ in Lemma 2.2. Since $T_h^{-1}(r)$ is an increasing function of $r$, it follows from Lemma 2.1, Lemma 2.2 and the inequality $T_g(r) \leq \log^+ M_g(r)$ {cf. [7] } for all sufficiently large values of $r$ that

$$\begin{aligned} T_h^{-1}(T_{f\circ g}(r)) &\leqslant T_h^{-1}[\{1+o(1)\}T_f(M_g(r))] \\ i.e.,\ T_h^{-1}(T_{f\circ g}(r)) &\leqslant \beta[T_h^{-1}T_f(M_g(r))]^\delta \\ i.e.,\ \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) &\leqslant \log^{[p]} T_h^{-1}T_f(M_g(r)) + O(1) \end{aligned}$$

$$i.e.,\ \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leq$$
$$(\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]} M_g(r) + \exp^{[t]} L(M_g(r))] + O(1)$$
$$i.e.,\ \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leq (\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[m-1]} M_g(r) + \exp^{[t]} L(M_g(r))] + O(1)$$
$$i.e.,\ \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leq (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot$$
$$[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)} + \exp^{[t]} L(M_g(r))] + O(1).$$

Since $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, we obtain from above for all sufficiently large values of $r$,

$$\log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leq (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot$$
$$[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f)} + \exp^{[t]} L(M_g(r))] + O(1). \tag{3.15}$$

Again in view of Lemma 2.4 and Lemma 2.5, we get for all sufficiently large values of $r$,

$$\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) \geq (\overline{\sigma}_{W(h)}^{(p,q,t)L}(W(f))-\varepsilon)[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_{W(h)}^{(p,q,t)L}(W(f))}$$
$$i.e.,\ \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) \geq (\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}$$
$$i.e.,\ [\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)} \leq \frac{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}{\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}$$
$$i.e.,\ [\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)} \leq \frac{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}{\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}. \tag{3.16}$$

Now from (3.15) and (3.16) it follows for all sufficiently large values of $r$ that

$$\log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \le (\rho_h^{(p,q,t)L}(f)+\varepsilon)\cdot \exp^{[t]} L(M_g(r)) + O(1) +$$

$$(\rho_h^{(p,q,t)L}(f)+\varepsilon)(\sigma_g^L(m,n,t)+\varepsilon)\cdot \frac{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}{\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}$$

*i.e.*,

$$\frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \le O(1) + \frac{\rho_h^{(p,q,t)L}(f)+\varepsilon}{1+\frac{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}{\exp^{[t]} L(M_g(r))}}$$

$$+\frac{\frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)(\sigma_g^L(m,n,t)+\varepsilon)}{(\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)}}{1+\frac{\exp^{[t]} L(M_g(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))}}. \tag{3.17}$$

If $\exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\}$ then from (3.17) we get

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \le \frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)(\sigma_g^L(m,n,t)+\varepsilon)}{\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \le \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)}$$

Again if $\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}$ then from (3.17) it follows that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \le \rho_h^{(p,q,t)L}(f)+\varepsilon.$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))} \le \rho_h^{(p,q,t)L}(f).$$

Thus the theorem is established. □

**Theorem 3.10.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\lambda_h^{(p,q,t)L}(f)<\infty$,*

$\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ *and* $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ *where* $m-1 = n = q$, *and* $l > 2$. *If* $h$ *satisfies the Property (A), then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

**Theorem 3.11.** *Let* $f$ *be a transcendental meromorphic function with* $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, $g$ *be an entire function and* $h$ *be a transcendental entire function having the maximum deficiency sum with regular* $(l,p)$ *growth such that* $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ *and* $\sigma_h^{(p,q,t)L}(f) > 0$ *where* $m-1 = n = q$ *and* $l > 2$. *If* $h$ *satisfies the Property (A), then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

**Theorem 3.12.** *Let* $f$ *be a transcendental meromorphic function with* $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, $g$ *be an entire function and* $h$ *be a transcendental entire function having the maximum deficiency sum with regular* $(l,p)$ *growth such that* $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \overline{\sigma}_g^L(m,n,t) < \infty$ *and* $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ *where* $m-1 = n = q$ *and* $l > 2$. *If* $h$ *satisfies the Property (A), then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

We omit the proof of the above three theorems as those can be carried out in the line of Theorem 3.9.

Similarly using the concept of the growth indicator $\tau_h^{(p,q,t)L}(f)$ and $\overline{\tau}_g^L(m,n,t)$ we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.9, Theorem 3.10, Theorem 3.11 and Theorem 3.12 respectively and with the help of Lemma 2.4 and Lemma 2.6.

**Theorem 3.13.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\rho_h^{(p,q,t)L}(f)<\infty$, $\lambda_h^{(p,q,t)L}(f)=\lambda_g^L(m,n,t)$, $0<\overline{\tau}_g^L(m,n,t)<\infty$ and $\tau_h^{(p,q,t)L}(f)>0$ where $m-1=n=q$ and $l>2$. If $h$ satisfies the Property (A), then*

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))+\exp^{[t]}L(M_g(r))}$$
$$\leq\begin{cases}\frac{\rho_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}L(M_g(r))=o\{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\}\\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))=o\{\exp^{[t]}L(M_g(r))\}.\end{cases}$$

**Theorem 3.14.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\lambda_h^{(p,q,t)L}(f)=\lambda_g^L(m,n,t)$, $0<\overline{\tau}_g^L(m,n,t)<\infty$ and $\tau_h^{(p,q,t)L}(f)>0$ where $m-1=n=q$ and $l>2$. If $h$ satisfies the Property (A), then*

$$\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))+\exp^{[t]}L(M_g(r))}$$
$$\leq\begin{cases}\frac{\lambda_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}L(M_g(r))=o\{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\}\\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))=o\{\exp^{[t]}L(M_g(r))\}.\end{cases}$$

**Theorem 3.15.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\rho_h^{(p,q,t)L}(f)<\infty$, $\lambda_h^{(p,q,t)L}(f)=\lambda_g^L(m,n,t)$, $0<\overline{\tau}_g^L(m,n,t)<\infty$ and $\overline{\tau}_h^{(p,q,t)L}(f)>0$ where $m-1=$*

*$n = q$ and $l > 2$. If $h$ satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

**Theorem 3.16.** *Let $f$ be a transcendental meromorphic function with $\sum\limits_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l, p)$ growth such that $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m, n, t)$, $0 < \tau_g^L(m, n, t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m - 1 = n = q$ and $l > 2$. If $h$ satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

Analogously we state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 3.9 and with the help of Lemma 2.4 and Lemma 2.6 respectively. Hence their proofs are omitted.

**Theorem 3.17.** *Let $f$ be a transcendental meromorphic function with $\sum\limits_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l, p)$ growth such that $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_h^{(p,q,t)L}(f) = \rho_g^L(m, n, t)$, $0 < \sigma_g^L(m, n, t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m - 1 = n = q$ and $l > 2$. If $h$ satisfies the Property (A), then*

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) + \exp^{[t]} L(M_g(r))}$$
$$\leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

**Theorem 3.18.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\lambda_h^{(p,q,t)L}(f)=\rho_g^L(m,n,t)$, $0<\sigma_g^L(m,n,t)<\infty$ and $\tau_h^{(p,q,t)L}(f)>0$ where $m-1=n=q$ and $l>2$. If $h$ satisfies the Property (A), then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))+\exp^{[t]}L(M_g(r))}$$
$$\leq\begin{cases}\frac{\lambda_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}L(M_g(r))=o\{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\}\\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))=o\{\exp^{[t]}L(M_g(r))\}.\end{cases}$$

**Theorem 3.19.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\rho_h^{(p,q,t)L}(f)=\lambda_g^L(m,n,t)$, $0<\overline{\tau}_g^L(m,n,t)<\infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f)>0$ where $m-1=n=q$ and $l>2$. If $h$ satisfies the Property (A), then*

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))+\exp^{[t]}L(M_g(r))}$$
$$\leq\begin{cases}\frac{\rho_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}L(M_g(r))=o\{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))\}\\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))=o\{\exp^{[t]}L(M_g(r))\}.\end{cases}$$

**Theorem 3.20.** *Let $f$ be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$, $g$ be an entire function and $h$ be a transcendental entire function having the maximum deficiency sum with regular $(l,p)$ growth such that $\rho_h^{(p,q,t)L}(f)=\lambda_g^L(m,n,t)$, $0<\overline{\tau}_g^L(m,n,t)<\infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f)>0$ where $m-1=n=q$ and $l>2$. If $h$ satisfies the Property (A), then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]}T_{W(h)}^{-1}(T_{W(f)}(r))+\exp^{[t]}L(M_g(r))}$$

$$\leq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \textit{if} \ \ \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r))\} \\ \lambda_h^{(p,q,t)L}(f) & \textit{if} \ \log^{[p-1]} T_{W(h)}^{-1}(T_{W(f)}(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases} .$$

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