J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2022.29.1.51 ISSN(Online) 2287-6081
Volume 29, Number 1 (February 2022), Pages 51–57

A REFINEMENT OF THE JENSEN-SIMIC-MERCER INEQUALITY WITH APPLICATIONS TO ENTROPY

Yamin Sayyari

Abstract. The Jensen, Simic and Mercer inequalities are very important inequalities in theory of inequalities and some results are devoted to this inequalities. In this paper, firstly, we establish extension of Jensen-Simic-Mercer inequality. After that, we investigate bounds for Shannons entropy of a probability distribution. Finally, We give some new applications in analysis.

1. Introduction

Jensen, Simic and Mercer inequalities are important for obtaining bounds for entropies. In this paper, by applying an extensions of Simic's inequality and Mercer's inequality, we obtain some estimates for the Shannon's entropy. Let $I := [a, b]$ be an interval, $\mathbf{x} := \{x_i\}_{i=1}^n \subseteq I$ and $\mathbf{p} := \{p_i\}_1^n \subseteq [0, 1]$ with $\sum_{i=1}^n p_i = 1$. The following inequality is well known in the literature as Jensens inequality.

Theorem 1.1 ([10, Jensen's inequality]). *If f is a convex function on an interval I, $\mathbf{x} := \{x_i\}_{i=1}^n \subseteq I$ and $\sum_{i=1}^n p_i = 1$, then*

$$0 \leq \sum_{i=1}^n p_i f(x_i) - f(\sum_{i=1}^n p_i x_i) := J_f(\mathbf{p}, \mathbf{x}).$$

In [17], Simic proved the following extension of Jensens inequality known as the Jensen Simic inequality.

Theorem 1.2. *If f is a convex function on an interval I, $x_i \in I$, $1 \leq i \leq n$ and $\sum_{i=1}^n p_i = 1$, then*

Received by the editors July 27, 2021. Accepted January 06, 2022.
2010 *Mathematics Subject Classification.* 26B25, 26D15, 94A17.
Key words and phrases. Shannon's entropy, Jensen's inequality, Simic's inequality, Mercer's inequality, convex function.

$$0 \le \max_{1\le r<s\le n}\{p_r f(x_r) + p_s f(x_s) - (p_r+p_s) f(\frac{p_r x_r + p_s x_s}{p_r+p_s})\}$$
$$\le \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \le f(a) + f(b) - 2f(\frac{a+b}{2}).$$

Theorems 1.1 and 1.2 yield the following corollary.

Corollary 1.3. *If f is a convex function on an an interval I, $x_i \in I$, $1 \le i \le n$ and $\sum_{i=1}^{n} p_i = 1$, then*

$$0 \le \max_{r,s}\{p_r f(x_r) + p_s f(x_s) - (p_r+p_s) f(\frac{p_r x_r + p_s x_s}{p_r+p_s})\}$$
$$\le \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \le f(a) + f(b) - 2f(\frac{a+b}{2}).$$

A variant of Jensens inequality is obtained by Mercer [9].

Theorem 1.4 ([9]). *If f is a convex function on an interval $I := [a, b]$, $x_i \in I$, $1 \le i \le n$ and $\sum_{i=1}^{n} p_i = 1$, then*

$$I_f(\mathbf{p}, \mathbf{x}) := f(a + b - \sum_{i=1}^{n} p_i x_i) + \sum_{i=1}^{n} p_i f(x_i) \le f(a) + f(b). \tag{1.1}$$

2. Refinement of Jensen-Simic-Mercer Inequality

In this section, we extend the Jensen-Mercer inequality (1.1) for convex functions.

Theorem 2.1. *Let f be a convex function on an interval I, $x_i \in I$, $1 \le i \le n$ and $\sum_{i=1}^{n} p_i = 1$, then*

$$\begin{aligned} 2f(\frac{a+b}{2}) &\le f(a + b - \sum_{i=1}^{n} p_i x_i) + \sum_{i=1}^{n} p_i f(x_i) \le f(a) + f(b) \\ &- \max_{r,s}\{p_r f(a+b-x_r) + p_s f(a+b-x_s) - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\} \\ &\le f(a) + f(b). \end{aligned} \tag{2.1}$$

Corollary 2.2. *Let f be a convex function on I, then*

$$2f(\frac{a+b}{2}) \le f(a+b-\sum_{i=1}^{n} p_i x_i) + \sum_{i=1}^{n} p_i f(x_i) \le f(a)+f(b)$$
$$-\{p_r f(a+b-x_r) + p_s f(a+b-x_s) - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\}$$
$$\le f(a)+f(b),$$

for every $r, s \in \{1, ..., n\}$.

Theorem 2.3. *If f is convex function on I, $\mu := \min\{x_i\}$ and $\nu := \max\{x_i\}$, then*

$$2f(\frac{\mu+\nu}{2}) \le f(\mu+\nu-\frac{1}{n}\sum_{i=1}^{n} x_i) + \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$
$$\le f(\mu)+f(\nu) - \frac{1}{n}\{f(\mu)+f(\nu)-2f(\frac{\mu+\nu}{2})\}.$$

3. Applications

In this section, we present some applications of Theorem 2.1 in information theory and analysis.

3.1. Applications in information theory

Definition 3.1. The Shannon entropy of a positive probability distribution $P = (p_1, ..., p_n)$ is defined by $H(\mathbf{p}) := \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$.

Proposition 3.2. *Define $\mu := \min_{1\le i\le n}\{p_i\}$ and $\nu := \max_{1\le i\le n}\{p_i\}$. Then*

$$\log(\frac{4\mu^2\nu^2}{(\mu+\nu)^2}) \le \log(\frac{\mu\nu}{\mu+\nu-n\mu\nu}) - H(\mathbf{p})$$
$$\le \log(\mu\nu) - \mu\log(\frac{\mu^2+\nu^2}{\mu(\mu+\nu)}) - \nu\log(\frac{\mu^2+\nu^2}{\nu(\mu+\nu)}) \tag{3.1}$$

Proposition 3.3. *Define $\mu := \min_{1\le i\le n}\{p_i\}$ and $\nu := \max_{1\le i\le n}\{p_i\}$. Then*

$$(\mu+\nu)\log(\frac{\mu+\nu}{2}) \le (\mu+\nu-\frac{1}{n})\log(\mu+\nu-\frac{1}{n}) - H(\mathbf{p})$$
$$\le \mu\log\mu + \nu\log\nu - \frac{1}{n}[\mu\log(\frac{2\mu}{\mu+\nu}) + \nu\log(\frac{2\nu}{\mu+\nu})]. \tag{3.2}$$

3.2. Applications in analysis Let $\mathbf{x} = \{x_i\}_{i=1}^n$ be a positive real sequence and

$$A := \frac{1}{n}\sum_{i=1}^n x_i \text{ and } G := (\prod_{i=1}^n x_i)^{\frac{1}{n}}$$

denote the usual arithmetic and geometric means of $\{x_i\}$, respectively. Denote $\mu := \min\{x_i\}$, $\nu := \max\{x_i\}$, $\tilde{A} := \mu + \nu - A$, $\tilde{G} := \frac{\mu\nu}{G}$, $A(\mu, \nu) := \frac{\mu+\nu}{2}$ and $G(\mu, \nu) := \sqrt{\mu\nu}$. From (2.1) we conclude the following result.

Proposition 3.4. *Let* $\mathbf{x} = \{x_i\}_{i=1}^n$ *and* $x_i > 0$ *for all* $i = 1, ..., n$, $\mu = \min\{x_i\}$ *and* $\nu = \max\{x_i\}$, *then*

$$\tilde{G} \leq \tilde{G}[\frac{A(\mu,\nu)}{G(\mu,\nu)}]^{\frac{2}{n}} \leq \tilde{A} \leq \frac{[A(\mu,\nu)]^2}{G}.$$

Remark 3.5. Proposition 3.4 is equivalent to

$$\tilde{G} \leq (\frac{\tilde{A}+A}{2})^{\frac{2}{n}} \frac{\tilde{G}}{\sqrt[n]{\tilde{G}G}} \leq \tilde{A} \leq \frac{(\tilde{A}+A)^2}{4G}.$$

4. Proofs

Proof of Theorem 2.1. Since $\{x_i\}_i \subseteq [a, b]$, there is a sequence $\{\lambda_i\}_i (0 \leq \lambda_i \leq 1)$, such that $x_i = \lambda_i a + (1 - \lambda_i)b$. Hence,

$$\begin{aligned} & f(a+b-\sum_{i=1}^n p_i x_i) + \sum_{i=1}^n p_i f(x_i) \\ &= f(a+b-\sum_{i=1}^n p_i(\lambda_i a + (1-\lambda_i)b)) + \sum_{i=1}^n p_i f(\lambda_i a + (1-\lambda_i)b) \\ &\geq f(a+b-a\sum_{i=1}^n p_i\lambda_i - b\sum_{i=1}^n p_i(1-\lambda_i)) + f(a\sum_{i=1}^n p_i\lambda_i + b\sum_{i=1}^n p_i(1-\lambda_i)). \end{aligned}$$

Denoting $p := \sum_{i=1}^n p_i\lambda_i$ and $q := 1 - \sum_{i=1}^n p_i\lambda_i$. Consequently,

$$\begin{aligned} I_f(\mathbf{p}, \mathbf{x}) &\geq f(a+b-pa-qb) + f(pa+qb) \\ &= f(qa+qb) + f(pa+qb) \\ &\geq 2f(\frac{pa+qb}{2} + \frac{qa+pb}{2}) = 2f(\frac{a+b}{2}). \end{aligned} \tag{4.1}$$

Here the first inequality holds. On the other hand, by the use of Corollary 1.3, we have

$$
\begin{aligned}
& f(a+b-\sum_{i=1}^{n} p_i x_i) + \sum_{i=1}^{n} p_i f(x_i) = f(\sum_{i=1}^{n} p_i(a+b-x_i)) + \sum_{i=1}^{n} p_i f(x_i) \\
& \leq \sum_{i=1}^{n} p_i f(a+b-x_i) - \max_{r,s}\{p_r f(a+b-x_r) + p_s f(a+b-x_s) \\
& - (p_r+p_s) f(\frac{p_r(a+b-x_r)+p_s(a+b-x_s)}{p_r+p_s})\} + \sum_{i=1}^{n} p_i f(x_i) \\
& = \sum_{i=1}^{n} p_i f(a+b-x_i) - \max_{r,s}\{p_r f(a+b-x_r) + p_s f(a+b-x_s) \\
& - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\} + \sum_{i=1}^{n} p_i f(x_i).
\end{aligned}
$$

Then from Mercers inequality (1.1), it follows that

$$
\begin{aligned}
& f(a+b-\sum_{i=1}^{n} p_i x_i) + \sum_{i=1}^{n} p_i f(x_i) \\
& \leq \sum_{i=1}^{n} p_i (f(a)+f(b)-f(x_i)) - \max_{r,s}\{p_r f(a+b-x_r) + p_s f(a+b-x_s) \\
& - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\} + \sum_{i=1}^{n} p_i f(x_i) = f(a)+f(b) \\
& - \max_{r,s}\{p_r f(a+b-x_r) + p_s f(a+b-x_s) \\
& - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\},
\end{aligned}
$$

which completes the proof. □

Proof of Corollary 2.2. Since

$$
p_r f(a+b-x_r) + p_s f(a+b-x_s) - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s}) \geq 0
$$

for all $r, s = 1, ..., n$, the results follow from (2.1). □

Proof of Theorem 2.3. Let $1 \leq \mu \leq \nu \leq n$. Since

$$
\begin{aligned}
& p_\mu f(a+b-x_\mu) + p_\nu f(a+b-x_\nu) - (p_\mu+p_\nu) f(a+b-\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu+p_\nu}) \\
& \max_{r,s}\{p_r f(a+b-x_r) + p_s f(a+b-x_s) - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\},
\end{aligned}
$$

we have

$$2f(\frac{a+b}{2}) \leq f(a+b-\sum_{i=1}^{n} p_i x_i) + \sum_{i=1}^{n} p_i f(x_i) \leq f(a)+f(b)$$
$$-\{p_r f(a+b-x_r) + p_s f(a+b-x_s) - (p_r+p_s) f(a+b-\frac{p_r x_r + p_s x_s}{p_r+p_s})\},$$

for all $r, s \in \{1, ..., n\}\}$. Now, putting $p_i = \frac{1}{n}, i = 1, ..., n$, $x_r = x_1 = a$ and $x_s = x_n = b$, which provides the desired inequality. □

Proof of Proposition 3.2. Applying Corollary 2.2 with $f(x) = -\log(x)$ and putting $x_i = \frac{1}{p_i}$ for all $i = 1, ..., n$, $a = x_r = \frac{1}{\nu}$ and $b = x_s = \frac{1}{\mu}$, we get

$$-2\log(\frac{\mu+\nu}{2\mu\nu}) \leq -\log(\frac{\mu+\nu-n\mu\nu}{\mu\nu}) - \sum_{i=1}^{n} p_i \log(\frac{1}{p_i}) \leq \log\nu + \log\mu$$
$$-\{-\nu\log(\frac{1}{\mu}) - \mu\log(\frac{1}{\nu}) + (\nu+\mu)\log(\frac{1}{\nu}+\frac{1}{\mu}-\frac{2}{\nu+\mu})\},$$

which completes the proof. □

Proof of Proposition 3.3. Let $f(x) = x\log x$, $a = \mu$ and $b = \nu$. Then apply Theorem 2.3 with x_i replaced by p_i, we get

$$(\mu+\nu)\log(\frac{\mu+\nu}{2}) \leq (\mu+\nu-\frac{1}{n})\log(\mu+\nu-\frac{1}{n}) + \frac{1}{n}\sum_{i=1}^{n} p_i \log p_i$$
$$\leq \mu\log\mu + \nu\log\nu - \frac{1}{n}\{\mu\log\mu + \nu\log\nu - (\mu+\nu)\log(\frac{\mu+\nu}{2})\},$$

after some calculations the desired assertion follows. □

Proof of Proposition 3.4. Applying Theorem 2.1 with $f(x) = -\log x$, $p_i = \frac{1}{n}$ for all $i = 1, ..., n$, $a = \min\{x_i\}_{i=1}^{n}$ and $b = \max\{x_i\}_{i=1}^{n}$, the desired results follow. □

References

1. H. Barsam & A.R. Sattarzadeh: Hermite-Hadamard inequalities for uniformly convex function and its applications in means. Miskolc mathematical notes **21** (2020), no. 2, 621-630.
2. I. Budimir, S.S. Dragomir & J. Pecaric: Further reverse results for Jensen's discrete inequality and applications in information theory. Journal of Inequalities in Pure and Applied Mathematics **2** (2001), no. 1, Art. 5.

3. Ch. Corda, M. FatehiNia, M.R. Molaei & Y. Sayyari: Entropy of Iterated Function Systems and Their Relations with Black Holes and Bohr-Like Black Holes Entropies, Entropy **20** (2018), no. 56.
4. S.S. Dragomir: A converse result for Jensen's discrete inequality via Grss inequality and applications in information theory. An. Univ. Oradea. Fasc. Mat. **7** (1999-2000), 178-189.
5. S.S. Dragomir & C.J. Goh: Some bounds on entropy measures in Information Theory. Applied Mathematics Letters **10** (1997), no. 3, 23-28.
6. JLWV Jensen: Om konvekse Funktioner og Uligheder mellem Middelvrdier. Nyt Tidsskr Math B **16** (1905), 49-68 (in Danish).
7. A. Mehrpooya, Y. Sayyari & M.R. Molaei: Algebraic and Shannon entropies of commutative hypergroups and their connection with information and permutation entropies and with calculation of entropy for chemical algebras. Soft Computing **23** (2019), no. 24, 13035-13053.
8. D.S. Mitrinovic: Analytic Inequalities, Springer, New York, 1970.
9. A. McD. Mercer: A variant of Jensen's inequality. Journal of Inequalities in Pure and Applied Mathematics **4** (2003), no. 4, Article 73.
10. D.S. Mitrinovic, J.E. Pecaric & A.M. Fink: Classical and new inequalities in analysis. Kluwer Academic Publishers, Dordrecht, 1993.
11. Y. Sayyari: An improvement of the upper bound on the entropy of information sources. Journal of Mathematical Extension **15** (2021), no. 5, 1-12.
12. Y. Sayyari: New bounds for entropy of information sources. Wavelet and Linear Algebra **7** (2020), no. 2, 1-9.
13. Y. Sayyari: New entropy bounds via uniformly convex functions. Chaos, Solitons and Fractals **141** (2020), no. 1, (DOI: 10.1016/j.chaos.2020.110360).
14. Y Sayyari: New refinements of Shannons entropy upper bounds. Journal of Information and Optimization Sciences **42** (2021), no. 8, 1869-1883.
15. Y. Sayyari, M.R. Molaei & S.M. Moghayer: Entropy of continuous maps on quasi-metric spaces. Journal of Advanced Research in Dynamical and Control Systems **7** (2015), no. 4, 1-10.
16. S. Simic: On a global bound for Jensen's inequality, Journal of Mathematical Analysis and Applications **343** (2008), 414-419 (DOI 10.1016/j.jmaa.2008.01.060).
17. S. Simic: Jensens inequality and new entropy bounds. Applied Mathematics Letters **22** (2009), no. 8, 1262-1265.
18. P. Walters: An Introduction to Ergodic Theory. Springer Verlag, New York, 2000.

Department of Mathematics, Sirjan University Of Technology, Sirjan, Iran
Email address: `ysayyari@gmail.com, y.sayyari@sirjantech.ac.ir`