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FRACTIONAL VECTOR CROSS PRODUCT

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Abstract. A definition of fractional vector cross product of two vectors in Euclidean 3-space is presented. The formulas for Euclidean norm of the fractional vector cross product of two vectors, and for fractional triple vector cross product are obtained.

1. Introduction

A good source of history of vector analysis is the well-known book presented by M. J. Crowe in 1967 [1]. Recently, in 2013, S. Das [2] gave a simple geometric method to derive fractional cross product operation and used the concept to derive fractional curl. Then, he noted down elaborate formulations that were useful for applications in electrodynamics, elastodynamics and fluid flow etc. In this paper, we present a suitable definition of fractional vector cross product of two vectors in Euclidean 3-space, which agrees with the definition given in [2]. Then, we obtain some basic properties of fractional vector cross product which include the formulas for Euclidean norm of the fractional vector cross product of two vectors, and for fractional triple vector cross product.

2. Fractional Vector Cross Product

In accordance with the concept of fractional vector cross product introduced in [2], we give the following definition.

Definition 2.1. Let $\mathbb{R}^3$ be the Euclidean 3-space equipped with the standard inner product denoted by $\langle \cdot, \cdot \rangle$. Let (e_1, e_2, e_3) be standard orthonormal basis of $\mathbb{R}^3$, and

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$\alpha \in [0,1]$ a real number. Then, for vectors $a = a_1e_1+a_2e_2+a_3e_3$, $b = b_1e_1+b_2e_2+b_3e_3$ in $\mathbb{R}^3$, the *α-fractional vector cross product* is defined by

$$\begin{aligned} a \times^\alpha b &= \left\{(a_2b_3 - a_3b_2)\sin\left(\frac{\alpha\pi}{2}\right) + (a_2+a_3)\, b_1 \cos\left(\frac{\alpha\pi}{2}\right)\right\} e_1 \\ &\quad + \left\{(a_3b_1 - a_1b_3)\sin\left(\frac{\alpha\pi}{2}\right) + (a_3+a_1)\, b_2 \cos\left(\frac{\alpha\pi}{2}\right)\right\} e_2 \\ &\quad + \left\{(a_1b_2 - a_2b_1)\sin\left(\frac{\alpha\pi}{2}\right) + (a_1+a_2)\, b_3 \cos\left(\frac{\alpha\pi}{2}\right)\right\} e_3. \end{aligned} \tag{2.1}$$

From (2.1), we immediately get

$$e_i \times^\alpha e_j = \cos\left(\frac{\alpha\pi}{2}\right) e_j + \sin\left(\frac{\alpha\pi}{2}\right) e_k, \tag{2.2}$$

$$e_j \times^\alpha e_i = \cos\left(\frac{\alpha\pi}{2}\right) e_i - \sin\left(\frac{\alpha\pi}{2}\right) e_k, \tag{2.3}$$

$$e_\ell \times^\alpha e_\ell = 0, \qquad \ell \in \{1,2,3\}, \tag{2.4}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$. The equations (2.2), (2.3) and (2.4) include the equations (2) and (3) of [2]. Assuming (2.2), (2.3), and (2.4), and linearity, one can recover (2.1). The equation (2.1) can be written as

$$\begin{aligned} a \times^\alpha b &= \sin\left(\frac{\alpha\pi}{2}\right)\{(a_2b_3 - a_3b_2)\, e_1 + (a_3b_1 - a_1b_3)\, e_2 + (a_1b_2 - a_2b_1)\, e_3\} \\ &\quad + \cos\left(\frac{\alpha\pi}{2}\right)\{(a_2+a_3)\, b_1e_1 + (a_3+a_1)\, b_2e_2 + (a_1+a_2)\, b_3e_3\}, \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} a \times^\alpha b &= \sin\left(\frac{\alpha\pi}{2}\right)\{(a_2b_3 - a_3b_2)\, e_1 + (a_3b_1 - a_1b_3)\, e_2 + (a_1b_2 - a_2b_1)\, e_3\} \\ &\quad + \cos\left(\frac{\alpha\pi}{2}\right)(a_1+a_2+a_3)\, b - \cos\left(\frac{\alpha\pi}{2}\right)(a_1b_1e_1 + a_2b_2e_2 + a_3b_3e_3). \end{aligned} \tag{2.6}$$

Considering $a \times^\alpha b$, as a column vector in $\mathbb{R}^3$, we have

$$a \times^\alpha b = \begin{pmatrix} (a_2+a_3)\cos\left(\frac{\alpha\pi}{2}\right) & -a_3\sin\left(\frac{\alpha\pi}{2}\right) & a_2\sin\left(\frac{\alpha\pi}{2}\right) \\ a_3\sin\left(\frac{\alpha\pi}{2}\right) & (a_3+a_1)\cos\left(\frac{\alpha\pi}{2}\right) & -a_1\sin\left(\frac{\alpha\pi}{2}\right) \\ -a_2\sin\left(\frac{\alpha\pi}{2}\right) & a_1\sin\left(\frac{\alpha\pi}{2}\right) & (a_1+a_2)\cos\left(\frac{\alpha\pi}{2}\right) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{2.7}$$

Thus, considering $a \in \mathbb{R}^3$ as a fixed column vector in $\mathbb{R}^3$, we see that $a \times^\alpha b : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear function given by (2.7).

The 1-fractional vector cross product is simply the standard vector cross product, denoted by $a \times b$. Thus, in particular, we recover

$$
\begin{aligned}
a \times b &= (a_2b_3 - a_3b_2)\, e_1 + (a_3b_1 - a_1b_3)\, e_2 + (a_1b_2 - a_2b_1)\, e_3 \\
&= \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},
\end{aligned}
$$

$$
e_i \times e_j = -e_i \times e_j = e_k,
$$

$$
e_\ell \times e_\ell = 0, \qquad \ell \in \{1, 2, 3\},
$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

The equation (2.7) can also be expressed by

$$
\begin{aligned}
a \times^\alpha b &= \sin\left(\frac{\alpha\pi}{2}\right) \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \\
&\quad + \cos\left(\frac{\alpha\pi}{2}\right) \begin{pmatrix} (a_2 + a_3) & 0 & 0 \\ 0 & (a_3 + a_1) & 0 \\ 0 & 0 & (a_1 + a_2) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.
\end{aligned}
$$

Then we can see that the fractional curl $\nabla \times^\alpha F$ for a vector field $F = (f, g, h)$ is

$$
\begin{aligned}
\nabla \times^\alpha F &= \sin\left(\frac{\alpha\pi}{2}\right) \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^\alpha}{\partial x} & \frac{\partial^\alpha}{\partial y} & \frac{\partial^\alpha}{\partial z} \\ f & g & h \end{pmatrix} \\
&\quad + \cos\left(\frac{\alpha\pi}{2}\right) \begin{pmatrix} \left(\frac{\partial^\alpha}{\partial y} + \frac{\partial^\alpha}{\partial z}\right) & 0 & 0 \\ 0 & \left(\frac{\partial^\alpha}{\partial z} + \frac{\partial^\alpha}{\partial x}\right) & 0 \\ 0 & 0 & \left(\frac{\partial^\alpha}{\partial x} + \frac{\partial^\alpha}{\partial y}\right) \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}
\end{aligned}
$$

as given by S. Das in [2].

3. Some Basic Formulas

Throughout the paper $\alpha \in [0, 1]$ is a real number, and $\lambda \in \mathbb{R}$ is any real number. Also, $a = a_1e_1 + a_2e_2 + a_3e_3$, $b = b_1e_1 + b_2e_2 + b_3e_3$, and $c = c_1e_1 + c_2e_2 + c_3e_3$ are assumed to be arbitrary vectors in $\mathbb{R}^3$. The fractional vector cross product will mean the α-fractional vector cross product.

Theorem 3.1. *The fractional vector cross product satisfies*

$$
a \times^\alpha (b + c) = a \times^\alpha b + a \times^\alpha c, \tag{3.1}
$$

$$(a+b)\times^{\alpha} c = a\times^{\alpha} c + b\times^{\alpha} c, \tag{3.2}$$

$$(\lambda a)\times^{\alpha} b = a\times^{\alpha}(\lambda b) = \lambda\left(a\times^{\alpha} b\right), \tag{3.3}$$

$$\langle a, a\times^{\alpha} b\rangle = \{a_1b_1(a_2+a_3)+a_2b_2(a_3+a_1)+a_3b_3(a_1+a_2)\}\cos\left(\frac{\alpha\pi}{2}\right), \tag{3.4}$$

$$\langle a, a\times^{\alpha} b\rangle = \cos\left(\frac{\alpha\pi}{2}\right)(a_1+a_2+a_3)\langle a,b\rangle - \cos\left(\frac{\alpha\pi}{2}\right)\left(a_1^2b_1+a_2^2b_2+a_3^2b_3\right), \tag{3.5}$$

$$\langle b, a\times^{\alpha} b\rangle = \left\{b_1^2(a_2+a_3)+b_2^2(a_3+a_1)+b_3^2(a_1+a_2)\right\}\cos\left(\frac{\alpha\pi}{2}\right), \tag{3.6}$$

$$\langle b, a\times^{\alpha} b\rangle = \cos\left(\frac{\alpha\pi}{2}\right)(a_1+a_2+a_3)\|b\|^2 - \cos\left(\frac{\alpha\pi}{2}\right)\left(a_1b_1^2+a_2b_2^2+a_3b_3^2\right). \tag{3.7}$$

Proof. In view of (2.7), the relations (3.1), (3.2), and (3.3) are obvious. From (2.5), we get (3.4) and (3.6). From (2.6), we get (3.5) and (3.7). □

In view of (3.4) and (3.6), we can state the following

Corollary 3.2. *The fractional vector cross product satisfies*

$$\begin{aligned}\langle a+b, a\times^{\alpha} b\rangle &= \{(a_1+b_1)(a_2+a_3)b_1+(a_2+b_2)(a_3+a_1)b_2\\ &\qquad +(a_3+b_3)(a_1+a_2)b_3\}\cos\left(\frac{\alpha\pi}{2}\right),\end{aligned}$$

$$\begin{aligned}\langle a+b, a\times^{\alpha} b\rangle - \langle a+b, b\times^{\alpha} a\rangle &= \det\begin{pmatrix} a_3+b_3 & a_1+b_1 & a_2+b_2\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\cos\left(\frac{\alpha\pi}{2}\right)\\ &\quad -\det\begin{pmatrix} a_2+b_2 & a_3+b_3 & a_1+b_1\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\cos\left(\frac{\alpha\pi}{2}\right).\end{aligned}$$

Corollary 3.3. *The vector cross product in* $\mathbb{R}^3$ *satisfies*

$$\langle a, a\times b\rangle = \langle b, a\times b\rangle = 0.$$

Corollary 3.4. *The vector cross product in* $\mathbb{R}^3$ *satisfies*

$$\langle a, a\times b\rangle = \langle b, a\times b\rangle = 0.$$

Example 1. We have

$$\langle e_i, e_i\times^{\alpha} e_j\rangle = \langle e_j, e_i\times^{\alpha} e_j\rangle = 0.$$

Corollary 3.5. *The fractional vector cross product satisfies*

$$
\begin{aligned}
a\times^{\alpha} b + b\times^{\alpha} a &= \cos\left(\frac{\alpha\pi}{2}\right)\left(\left(a_1+a_2+a_3\right)b+\left(b_1+b_2+b_3\right)a\right)\\
&\quad -2\cos\left(\frac{\alpha\pi}{2}\right)\left(a_1b_1e_1+a_2b_2e_2+a_3b_3e_3\right). \qquad (3.8)
\end{aligned}
$$

Proof. In view of (2.6), we get (3.8). □

Corollary 3.6. *The vector cross product in* $\mathbb{R}^3$ *satisfies*

$$a\times b+b\times a=0.$$

Example 2. We have

$$e_i\times^{\alpha} e_j+e_j\times^{\alpha} e_i=\cos\left(\frac{\alpha\pi}{2}\right)\left(e_i+e_j\right).$$

Corollary 3.7. *The fractional vector cross product satisfies*

$$
\begin{aligned}
\left\|a\times^{\alpha} b\right\|^2 &= \|a\|^2\|b\|^2-\langle a,b\rangle^2+\left\{2a_2a_3\left(b_1^2-b_2b_3\right)\right.\\
&\quad \left.+2a_3a_1\left(b_2^2-b_3b_1\right)+2a_1a_2\left(b_3^2-b_1b_2\right)\right\}\cos^2\left(\frac{\alpha\pi}{2}\right)\\
&\quad -\det\begin{pmatrix} a_1b_1 & a_2b_2 & a_3b_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\sin\left(\alpha\pi\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\|b\times^{\alpha} a\right\|^2 &= \|a\|^2\|b\|^2-\langle a,b\rangle^2+\left\{2b_2b_3\left(a_1^2-a_2a_3\right)\right.\\
&\quad \left.+2b_3b_1\left(a_2^2-a_3a_1\right)+2b_1b_2\left(a_3^2-a_1a_2\right)\right\}\cos^2\left(\frac{\alpha\pi}{2}\right)\\
&\quad -\det\begin{pmatrix} a_1b_1 & a_2b_2 & a_3b_3\\ b_1 & b_2 & b_3\\ a & a_2 & a_3\end{pmatrix}\sin\left(\alpha\pi\right),
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\left\|a\times^{\alpha} b\right\|^2-\left\|b\times^{\alpha} a\right\|^2 &= 2\left\{a_2a_3b_1^2+a_3a_1b_2^2+a_1a_2b_3^2\right.\\
&\quad \left.-b_2b_3a_1^2-b_3b_1a_2^2-b_1b_2a_3^2\right\}\cos^2\left(\frac{\alpha\pi}{2}\right)\\
&\quad -2\det\begin{pmatrix} a_1b_1 & a_2b_2 & a_3b_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}\sin\left(\alpha\pi\right).
\end{aligned}
$$

Proof. From (2.1), we get

$$\begin{aligned}
\|a \times^{\alpha} b\|^2 &= (a_2b_3 - a_3b_2)^2 \sin^2\left(\frac{\alpha\pi}{2}\right) + (a_2 + a_3)^2 b_1^2 \cos^2\left(\frac{\alpha\pi}{2}\right) \\
&\quad + (a_2b_3 - a_3b_2)(a_2 + a_3) b_1 \sin(\alpha\pi) \\
&\quad + (a_3b_1 - a_1b_3)^2 \sin^2\left(\frac{\alpha\pi}{2}\right) + (a_3 + a_1)^2 b_2^2 \cos^2\left(\frac{\alpha\pi}{2}\right) \\
&\quad + (a_3b_1 - a_1b_3)(a_3 + a_1) b_2 \sin(\alpha\pi) \\
&\quad + (a_1b_2 - a_2b_1)^2 \sin^2\left(\frac{\alpha\pi}{2}\right) + (a_1 + a_2)^2 b_3^2 \cos\left(\frac{\alpha\pi}{2}\right) \\
&\quad + (a_1b_2 - a_2b_1)(a_1 + a_2) b_3 \sin(\alpha\pi) \\
&= \left\{\|a\|^2 \|b\|^2 - \langle a, b\rangle^2\right\} \sin^2\left(\frac{\alpha\pi}{2}\right) \\
&\quad + \left\{(a_2 + a_3)^2 b_1^2 + (a_3 + a_1)^2 b_2^2 + (a_1 + a_2)^2 b_3^2\right\} \cos^2\left(\frac{\alpha\pi}{2}\right) \\
&\quad + \{(a_2b_3 - a_3b_2)(a_2 + a_3) b_1 + (a_3b_1 - a_1b_3)(a_3 + a_1) b_2 \\
&\qquad + (a_1b_2 - a_2b_1)(a_1 + a_2) b_3\} \sin(\alpha\pi).
\end{aligned}$$

From

$$\begin{aligned}
&(a_2b_3 - a_3b_2)(a_2 + a_3) b_1 + (a_3b_1 - a_1b_3)(a_3 + a_1) b_2 \\
&\qquad + (a_1b_2 - a_2b_1)(a_1 + a_2) b_3 \\
&= \left(a_2^2b_1b_3 - a_3^2b_1b_2 - a_2a_3b_1b_2 + a_2a_3b_1b_3\right) \\
&\quad + \left(a_3^2b_1b_2 - a_1^2b_2b_3 + a_1a_3b_1b_2 - a_1a_3b_2b_3\right) \\
&\quad + \left(a_1^2b_2b_3 - a_2^2b_1b_3 - a_1a_2b_1b_3 + a_1a_2b_2b_3\right) \\
&= (a_2a_3b_1b_3 - a_2a_3b_1b_2) + (a_1a_3b_1b_2 - a_1a_3b_2b_3) + (a_1a_2b_2b_3 - a_1a_2b_1b_3) \\
&= a_2a_3b_1b_3 - a_2a_3b_1b_2 + a_1a_3b_1b_2 - a_1a_3b_2b_3 + a_1a_2b_2b_3 - a_1a_2b_1b_3 \\
&= -\det\begin{pmatrix} a_1b_1 & a_2b_2 & a_3b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},
\end{aligned}$$

it follows that

$$\begin{aligned}
\|a \times^{\alpha} b\|^2 &= \left\{\|a\|^2 \|b\|^2 - \langle a, b\rangle^2\right\} \sin^2\left(\frac{\alpha\pi}{2}\right) \\
&\quad + \left\{\|a\|^2 \|b\|^2 - \langle a, b\rangle^2 + 2a_2a_3\left(b_1^2 - b_2b_3\right)\right. \\
&\qquad \left. + 2a_3a_1\left(b_2^2 - b_3b_1\right) + 2a_1a_2\left(b_3^2 - b_1b_2\right)\right\} \cos^2\left(\frac{\alpha\pi}{2}\right) \\
&\quad - \det\begin{pmatrix} a_1b_1 & a_2b_2 & a_3b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \sin(\alpha\pi).
\end{aligned}$$

□

Corollary 3.8. *The vector cross product in* $\mathbb{R}^3$ *satisfies*

$$\|a \times b\|^2 = \|b \times a\|^2 = \|a\|^2 \|b\|^2 - \langle a, b\rangle^2 .$$

Example 3. We have

$$\|e_i \times^\alpha e_j\|^2 = \|e_j \times^\alpha e_i\|^2 = 1.$$

Corollary 3.9. *The fractional vector cross product satisfies*

$$\begin{aligned} \langle a \times^\alpha b, c\rangle &= \sin\left(\frac{\alpha\pi}{2}\right) \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \\ &\quad - \cos\left(\frac{\alpha\pi}{2}\right)(a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3) \\ &\quad + \cos\left(\frac{\alpha\pi}{2}\right)(a_1 + a_2 + a_3)\langle b, c\rangle . \end{aligned} \tag{3.9}$$

Proof. The proof follows from (2.6). □

Corollary 3.10. *The vector cross product satisfies*

$$\langle a \times b, c\rangle = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} . \tag{3.10}$$

Remark. From (3.10), it is well-known that $\langle a \times b, c\rangle = \langle b \times c, a\rangle = \langle c \times a, b\rangle$. Geometrically, $\langle a \times b, c\rangle$ is the volume of the parallelepiped consisting of three vectors a, b, c. Because of the last term in (3.9), in general, it follows that $\langle a \times^\alpha b, c\rangle \neq \langle b \times^\alpha c, a\rangle$. However, from (3.9), we have

$$\langle e_1 \times^\alpha e_2, e_3\rangle = \langle e_2 \times^\alpha e_3, e_1\rangle = \langle e_3 \times^\alpha e_1, e_2\rangle = \sin\left(\frac{\alpha\pi}{2}\right) = \sin\left(\frac{\alpha\pi}{2}\right)\langle e_1 \times e_2, e_3\rangle .$$

Now, we obtain a formula for fractional triple vector cross product.

Corollary 3.11. *The fractional vector triple product satisfies*

$$\begin{aligned} (a \times^\alpha b) \times^\alpha c &= \sin^2\left(\frac{\alpha\pi}{2}\right)(\langle a, c\rangle b - \langle b, c\rangle a) \\ &\quad + \cos\left(\frac{\alpha\pi}{2}\right)\left\{\sin\left(\frac{\alpha\pi}{2}\right)(a_1 + a_2 + a_3)\, b \times c \right. \\ &\quad - \sin\left(\frac{\alpha\pi}{2}\right)(a_1b_1e_1 + a_2b_2e_2 + a_3b_3e_3) \times c \end{aligned}$$

$$+\sin\left(\frac{\alpha\pi}{2}\right)\det\begin{pmatrix}1 & 1 & 1\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}c$$
$$+\left(\left(a_2+a_3\right)b_1+\left(a_3+a_1\right)b_2+\left(a_1+a_2\right)b_3\right)c$$
$$-\sin\left(\frac{\alpha\pi}{2}\right)\det\begin{pmatrix}c_1e_1 & c_2e_2 & c_3e_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}$$
$$-\cos\left(\frac{\alpha\pi}{2}\right)\left(a_1+a_2+a_3\right)\left(b_1c_1e_1+b_2c_2e_2+b_3c_3e_3\right)$$
$$+\cos\left(\frac{\alpha\pi}{2}\right)\left(a_1b_1c_1e_1+a_2b_2c_2e_2+a_3b_3c_3e_3\right)\Big\}. \tag{3.11}$$

Proof. For any vectors $c = c_1e_1 + c_2e_2 + c_3e_3$ and $d = d_1e_1 + d_2e_2 + d_3e_3$ in $\mathbb{R}^3$, in view of (2.6), we have

$$\begin{aligned}
d \times^{\alpha} c &= \sin\left(\frac{\alpha\pi}{2}\right)\left\{\left(d_2c_3-d_3c_2\right)e_1+\left(d_3c_1-d_1c_3\right)e_2+\left(d_1c_2-d_2c_1\right)e_3\right\}\\
&\quad+\cos\left(\frac{\alpha\pi}{2}\right)\left(d_1+d_2+d_3\right)c-\cos\left(\frac{\alpha\pi}{2}\right)\left(d_1c_1e_1+d_2c_2e_2+d_3c_3e_3\right)\\
&= \sin\left(\frac{\alpha\pi}{2}\right)d\times c+\cos\left(\frac{\alpha\pi}{2}\right)\left(d_1+d_2+d_3\right)c
\end{aligned}$$
$$-\cos\left(\frac{\alpha\pi}{2}\right)\left(d_1c_1e_1+d_2c_2e_2+d_3c_3e_3\right). \tag{3.12}$$

Letting $d = a \times^{\alpha} b$, in view of (2.1), using

$$\begin{aligned}
d_1 &= \left(a_2b_3-a_3b_2\right)\sin\left(\frac{\alpha\pi}{2}\right)+\left(a_1+a_2+a_3\right)b_1\cos\left(\frac{\alpha\pi}{2}\right)-a_1b_1\cos\left(\frac{\alpha\pi}{2}\right),\\
d_2 &= \left(a_3b_1-a_1b_3\right)\sin\left(\frac{\alpha\pi}{2}\right)+\left(a_1+a_2+a_3\right)b_2\cos\left(\frac{\alpha\pi}{2}\right)-a_2b_2\cos\left(\frac{\alpha\pi}{2}\right),\\
d_3 &= \left(a_1b_2-a_2b_1\right)\sin\left(\frac{\alpha\pi}{2}\right)+\left(a_1+a_2+a_3\right)b_3\cos\left(\frac{\alpha\pi}{2}\right)-a_3b_3\cos\left(\frac{\alpha\pi}{2}\right),
\end{aligned}$$

in (3.12), we get

$$\begin{aligned}
\left(a\times^{\alpha}b\right)\times^{\alpha}c &= \sin^2\left(\frac{\alpha\pi}{2}\right)\left(a\times b\right)\times c\\
&\quad+\cos\left(\frac{\alpha\pi}{2}\right)\Big\{\sin\left(\frac{\alpha\pi}{2}\right)\left(a_1+a_2+a_3\right)b\times c\\
&\qquad-\sin\left(\frac{\alpha\pi}{2}\right)\left(a_1b_1e_1+a_2b_2e_2+a_3b_3e_3\right)\times c\\
&\qquad+\sin\left(\frac{\alpha\pi}{2}\right)\left(\left(a_2b_3-a_3b_2\right)+\left(a_3b_1-a_1b_3\right)+\left(a_1b_2-a_2b_1\right)\right)c\\
&\qquad+\cos\left(\frac{\alpha\pi}{2}\right)\left(\left(a_2+a_3\right)b_1+\left(a_3+a_1\right)b_2+\left(a_1+a_2\right)b_3\right)c
\end{aligned}$$

$$-\left(\sin\left(\frac{\alpha\pi}{2}\right)(a_2b_3-a_3b_2)\,c_1e_1+\cos\left(\frac{\alpha\pi}{2}\right)(a_2+a_3)\,b_1c_1e_1\right)$$
$$-\left(\sin\left(\frac{\alpha\pi}{2}\right)(a_3b_1-a_1b_3)\,c_2e_2+\cos\left(\frac{\alpha\pi}{2}\right)(a_3+a_1)\,b_2c_2e_2\right)$$
$$-\left(\sin\left(\frac{\alpha\pi}{2}\right)(a_1b_2-a_2b_1)\,c_3e_3+\cos\left(\frac{\alpha\pi}{2}\right)(a_1+a_2)\,b_3c_3e_3\right)\Big\}\,,$$

or

$$\begin{aligned}
&(a\times^{\alpha} b)\times^{\alpha} c\\
&= \sin^2\left(\frac{\alpha\pi}{2}\right)(\langle a,c\rangle\, b-\langle b,c\rangle\, a)\\
&\quad+\cos\left(\frac{\alpha\pi}{2}\right)\Big\{\sin\left(\frac{\alpha\pi}{2}\right)(a_1+a_2+a_3)\,b\times c\\
&\qquad-\sin\left(\frac{\alpha\pi}{2}\right)(a_1b_1e_1+a_2b_2e_2+a_3b_3e_3)\times c\\
&\qquad+\sin\left(\frac{\alpha\pi}{2}\right)((a_2b_3-a_3b_2)+(a_3b_1-a_1b_3)+(a_1b_2-a_2b_1))\,c\\
&\qquad+\cos\left(\frac{\alpha\pi}{2}\right)((a_2+a_3)\,b_1+(a_3+a_1)\,b_2+(a_1+a_2)\,b_3)\,c\\
&\qquad-\sin\left(\frac{\alpha\pi}{2}\right)((a_2b_3-a_3b_2)\,c_1e_1+(a_3b_1-a_1b_3)\,c_2e_2+(a_1b_2-a_2b_1)\,c_3e_3)\\
&\qquad-\cos\left(\frac{\alpha\pi}{2}\right)((a_2+a_3)\,b_1c_1e_1+(a_3+a_1)\,b_2c_2e_2+(a_1+a_2)\,b_3c_3e_3)\Big\}\,.
\end{aligned}$$

or

$$\begin{aligned}
&(a\times^{\alpha} b)\times^{\alpha} c\\
&= \sin^2\left(\frac{\alpha\pi}{2}\right)(\langle a,c\rangle\, b-\langle b,c\rangle\, a)\\
&\quad+\cos\left(\frac{\alpha\pi}{2}\right)\Big\{\sin\left(\frac{\alpha\pi}{2}\right)(a_1+a_2+a_3)\,b\times c\\
&\qquad-\sin\left(\frac{\alpha\pi}{2}\right)(a_1b_1e_1+a_2b_2e_2+a_3b_3e_3)\times c\\
&\qquad+\sin\left(\frac{\alpha\pi}{2}\right)\det\begin{pmatrix}1&1&1\\a_1&a_2&a_3\\b_1&b_2&b_3\end{pmatrix}c\\
&\qquad+\cos\left(\frac{\alpha\pi}{2}\right)((a_2+a_3)\,b_1+(a_3+a_1)\,b_2+(a_1+a_2)\,b_3)\,c\\
&\qquad-\sin\left(\frac{\alpha\pi}{2}\right)\det\begin{pmatrix}c_1e_1&c_2e_2&c_3e_3\\a_1&a_2&a_3\\b_1&b_2&b_3\end{pmatrix}\\
&\qquad-\cos\left(\frac{\alpha\pi}{2}\right)((a_2+a_3)\,b_1c_1e_1+(a_3+a_1)\,b_2c_2e_2+(a_1+a_2)\,b_3c_3e_3)\Big\}\,.
\end{aligned}$$

From the above equation, we get (3.11). □

Example 4. We have

$$(e_1 \times^\alpha e_2) \times^\alpha e_3 = \cos\left(\frac{\alpha\pi}{2}\right)\left\{\sin\left(\frac{\alpha\pi}{2}\right) e_1 + \cos\left(\frac{\alpha\pi}{2}\right) e_3\right\}$$

and, in particular, we have $(e_1 \times e_2) \times e_3 = 0$.

4. CONCLUSION

In this paper, we present basic mathematical formulas of fractional cross product, which can be applied in fractional calculus. By assuming the equations of (2.2), (2.3) and (2.4) geometrically for the definition of fractional cross product, the fractional curl of Electromagnetic vector field can be easily derived. So these formulas could be very useful for fractional calculus.

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