J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2022.29.1.1 ISSN(Online) 2287-6081
Volume 29, Number 1 (February 2022), Pages 1–17

COMMON n-TUPLED FIXED POINT THEOREM UNDER GENERALIZED MIZOGUCHI-TAKAHASHI CONTRACTION FOR HYBRID PAIR OF MAPPINGS

Bhavana Deshpande [a,*] and Amrish Handa [b]

Abstract. We establish a common $n-$tupled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction. An example is given to validate our results. We improve, extend and generalize several known results.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space. We denote by 2^X the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. A functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d is given by

$$H(A,\ B) = \begin{cases} \max\left\{\sup\limits_{a\in A} D(a,\ B),\ \sup\limits_{b\in B} D(b,\ A)\right\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A,\ B \in CL(X)$, where $D(x,\ A) = \inf\limits_{a\in A} d(x,\ a)$ denote the distance from x to $A \subset X$. For simplicity, if $x \in X$, we denote $g(x)$ by gx.

The existence of fixed points for various multivalued contractions and non-expansive mappings has been studied by many authors under different conditions which was initiated by Markin [17]. For details, we refer [1, 5, 6, 7, 8, 9, 13, 14, 15, 18, 19, 21, 22] and the reference therein to the readers. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics.

Received by the editors August 31, 2020. Accepted October 06, 2021.
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. generalized Mizoguchi-Takahashi contraction, $n-$tupled fixed point, $n-$tupled coincidence point, $w-$compatibility, $F-$weakly commuting.
*Corresponding author.

In [2], Gnana-Bhaskar and Lakshmikantham established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [16] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces, extended and generalized the results of Gnana-Bhaskar and Lakshmikantham [2].

Nadler [19] extended the famous Banach Contraction Principle [3] from single-valued mapping to multi-valued mapping. Mizoguchi and Takahashi [18] proved the following generalization of Nadler's fixed point theorem for a weak contraction.

Theorem 1.1. *Let* $(X,\ d)$ *be a complete metric space and* $T : X \to CB(X)$ *be a multivalued mapping. Assume that*

$$H(Tx,\ Ty) \leq \psi(d(x,\ y))d(x,\ y),$$

for all $x,\ y \in X$, *where* ψ *is a function from* $[0,\ \infty)$ *into* $[0,\ 1)$ *satisfying* $\limsup_{s \to t+} \psi(s) < 1$ *for all* $t \geq 0$. *Then* T *has a fixed point.*

Amini-Harandi and O'Regan [1] obtained a generalization of Mizoguchi and Takahashi's fixed point theorem. Recently Ciric et al. [4] proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. Main results of Ciric et al. [4] extended and generalized the results of Gnana-Bhaskar and Lakshmikantham [2], Du [10] and Harjani et al. [11].

Imdad et al. [12] introduced the concept of n-tupled fixed point, n-tupled coincidence point and proved some n-tupled coincidence point and n-tupled fixed point results for single valued mapping.

These concepts were extended by Deshpande and Handa [8] to multivalued mappings and obtained n-tupled coincidence point and common n-tupled fixed point theorems involving hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction.

Definition 2.1 ([8])**.** Let X be a nonempty set, $F : X^r \to 2^X$ and g be a self-mapping on X. An element $(x^1,\ x^2,\ ...,\ x^r) \in X^r$ is called

(1) an $r-$*tupled fixed point* of F if $x^1 \in F(x^1,\ x^2,\ ...,\ x^r)$, $x^2 \in F(x^2,\ ...,\ x^r,\ x^1)$, ..., $x^r \in F(x^r,\ x^1,\ ...,\ x^{r-1})$.

(1) an $r-$*tupled coincidence point* of hybrid pair $(F,\ g)$ if $gx^1 \in F(x^1,\ x^2,\ ...,\ x^r)$, $gx^2 \in F(x^2,\ ...,\ x^r,\ x^1)$, ..., $gx^r \in F(x^r,\ x^1,\ ...,\ x^{r-1})$.

(2) a *common $r-$tupled fixed point* of hybrid pair $(F,\ g)$ if $x^1 = gx^1 \in F(x^1,\ x^2,\ ...,\ x^r)$, $x^2 = gx^2 \in F(x^2,\ ...,\ x^r,\ x^1)$, ..., $x^r = gx^r \in F(x^r,\ x^1,\ ...,\ x^{r-1})$.

We denote the set of $r-$tupled coincidence points of mappings F and g by $C(F,\ g)$. Note that if $(x^1,\ x^2,\ ...,\ x^r) \in C(F,\ g)$, then $(x^2,\ ...,\ x^r,\ x^1)$, ..., $(x^r,\ x^1,\ ...,\ x^{r-1})$ are also in $C(F,\ g)$.

Definition 2.2 ([8])**.** Let $F : X^r \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $(F,\ g)$ is called *$w-$compatible* if $gF(x^1,\ x^2,\ ...,\ x^r) \subseteq F(gx^1,\ gx^2,\ ...,\ gx^r)$ whenever $(x^1,\ x^2,\ ...,\ x^r) \in C(F,\ g)$.

Definition 2.3 ([8])**.** Let $F : X^r \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called *$F-$weakly commuting* at some point $(x^1,\ x^2,\ ...,\ x^r) \in X^r$ if $g^2x^1 \in F(gx^1,\ gx^2,\ ...,\ gx^r)$, $g^2x^2 \in F(gx^2,\ ...,\ gx^r,\ gx^1)$, ..., $g^2x^r \in F(gx^r,\ gx^1,\ ...,\ gx^{r-1})$.

Lemma 2.1 ([20])**.***Let $(X,\ d)$ be a metric space. Then, for each $a \in X$ and $B \in K(X)$, there is $b_0 \in B$ such that $D(a,\ B) = d(a,\ b_0)$, where $D(a,\ B) = \inf_{b \in B} d(a,\ b)$.*

In this paper, we establish a common $n-$tupled fixed point theorem for hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction. We improve, extend and generalize the results of Amini-Harandi and O'Regan [1], Bhaskar and Lakshmikantham [2], Ciric et al. [4], Du [10], Harjani et al. [11] and Mizoguchi and Takahashi [18]. An example which demonstrates the effectiveness of our result has also been cited.

2. Main Results

Let Φ denote the set of all functions $\varphi : [0,\ \infty) \to [0,\ \infty)$ satisfying

(i_φ) φ is non-decreasing,

(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,

(iii_φ) $\lim_{t \to 0+} \dfrac{t}{\varphi(t)} < \infty$.

Let Ψ denote the set of all functions $\psi : [0,\ \infty) \to [0,\ 1)$ which satisy $\lim_{r \to t+} \psi(r) < 1$ for all $t \geq 0$.

Theorem 2.1. *Let $(X,\ d)$ be a metric space. Suppose $F : X^r \to K(X)$ and $g : X \to X$ are two mappings for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))) \tag{2.1}$$

$$\leq \psi(\varphi(\max\{d(gx^1,\ gy^1),\ d(gx^2,\ gy^2),\ ...,\ d(gx^r,\ gy^r)\}))$$
$$\times\, \varphi(\max\{d(gx^1,\ gy^1),\ d(gx^2,\ gy^2),\ ...,\ d(gx^r,\ gy^r)\}),$$

for all $x^1, x^2, ..., x^r, y^1, y^2, ..., y^r \in X$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds.*

(*a*) F *and* g *are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$,...,$\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1, x^2, ..., x^r) \in C(F, g)$ *and for some* $y^1, y^2, ..., y^r \in X$ *and* g *is continuous at* $y^1, y^2, ..., y^r$.

(*b*) g *is* $F-$*weakly commuting for some* $(x^1, x^2, ..., x^r) \in C(F, g)$ *and* $gx^1, gx^2, ..., gx^r$ *are fixed points of* g, *that is,* $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.

(*c*) g *is continuous at* $x^1, x^2, ..., x^r$. $\lim_{i\to\infty} g^i y^1 = x^1$, $\lim_{i\to\infty} g^i y^2 = x^2$, ..., $\lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1, x^2, ..., x^r) \in C(F, g)$ *and for some* $y^1, y^2, ..., y^r \in X$.

(*d*) $g(C(F, g))$ *is a singleton subset of* $C(F, g)$.

Proof. Let $x_0^1, x_0^2, ..., x_0^r \in X$ be arbitrary. Then $F(x_0^1, x_0^2, ..., x_0^r)$, $F(x_0^2, ..., x_0^r, x_0^1)$, ..., $F(x_0^r, x_0^1, ..., x_0^{r-1})$ are well defined. Choose $gx_1^1 \in F(x_0^1, x_0^2, ..., x_0^r)$, $gx_1^2 \in F(x_0^2, ..., x_0^r, x_0^1)$, ..., $gx_0^r \in F(x_0^r, x_0^1, ..., x_0^{r-1})$ because $F(X^r) \subseteq g(X)$. Since $F : X^r \to K(X)$, therefore by Lemma 2.1, there exist $z^1 \in F(x_1^1, x_1^2, ..., x_1^r)$, $z^2 \in F(x_1^2, ..., x_1^r, x_1^1)$..., $z^r \in F(x_1^r, x_1^1, ..., x_1^{r-1})$ such that

$$\begin{aligned}
d(gx_1^1,\ z^1) &\leq H(F(x_0^1,\ x_0^2,\ ...,\ x_0^r),\ F(x_1^1,\ x_1^2,\ ...,\ x_1^r)),\\
d(gx_1^2,\ z^2) &\leq H(F(x_0^2,\ ...,\ x_0^r,\ x_0^1),\ F(x_1^2,\ ...,\ x_1^r,\ x_1^1)),\\
&\quad ...,\\
d(gx_1^r,\ z^r) &\leq H(F(x_0^r,\ x_0^1,\ ...,\ x_0^{r-1}),\ F(x_1^r,\ x_1^1,\ ...,\ x_1^{r-1})).
\end{aligned}$$

Since $F(X^r) \subseteq g(X)$, there exist $x_2^1, x_2^2, ..., x_2^r \in X$ such that $z^1 = gx_2^1$, $z^2 = gx_2^2$, ..., $z^r = gx_2^r$. Thus

$$\begin{aligned}
d(gx_1^1,\ gx_2^1) &\leq H(F(x_0^1,\ x_0^2,\ ...,\ x_0^r),\ F(x_1^1,\ x_1^2,\ ...,\ x_1^r)),\\
d(gx_1^2,\ gx_2^2) &\leq H(F(x_0^2,\ ...,\ x_0^r,\ x_0^1),\ F(x_1^2,\ ...,\ x_1^r,\ x_1^1)),\\
&\quad ...,\\
d(gx_1^r,\ gx_2^r) &\leq H(F(x_0^r,\ x_0^1,\ ...,\ x_0^{r-1}),\ F(x_1^r,\ x_1^1,\ ...,\ x_1^{r-1})).
\end{aligned}$$

Continuing this process, we obtain sequences $\{x_i^1\} \subset X$, $\{x_i^2\} \subset X$, ..., $\{x_i^r\} \subset X$ such that for all $i \in \mathbb{N}$, we have $x_{i+1}^1 \in F(x_i^1, x_i^2, ..., x_i^r)$, $x_{i+1}^2 \in F(x_i^2, ..., x_i^r, x_i^1)$, ..., $x_{i+1}^r \in F(x_i^r, x_i^1, ..., x_i^{r-1})$ such that

$$\begin{aligned} d(gx_i^1,\ gx_{i+1}^1) &\leq H(F(x_{i-1}^1,\ x_{i-1}^2,\ ...,\ x_{i-1}^r),\ F(x_i^1,\ x_i^2,\ ...,\ x_i^r)), \\ d(gx_i^2,\ gx_{i+1}^2) &\leq H(F(x_{i-1}^2,\ ...,\ x_{i-1}^r,\ x_{i-1}^1),\ F(x_i^2,\ ...,\ x_i^r,\ x_i^1)), \\ &..., \\ d(gx_i^r,\ gx_{i+1}^r) &\leq H(F(x_{i-1}^r,\ x_{i-1}^1,\ ...,\ x_{i-1}^{r-1}),\ F(x_i^r,\ x_i^1,\ ...,\ x_i^{r-1})), \end{aligned}$$

which implies, by (i_φ) and (2.1), we have

$$\begin{aligned} &\varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right) \\ \leq\ &\varphi\left(H(F(x_{i-1}^1,\ x_{i-1}^2,\ ...,\ x_{i-1}^r),\ F(x_i^1,\ x_i^2,\ ...,\ x_i^r))\right) \\ \leq\ &\psi\left(\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\})\right) \\ &\times\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}). \end{aligned}$$

Thus

$$\begin{aligned} &\varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right) \\ &\leq \psi\left(\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\})\right) \\ &\times \varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}), \end{aligned} \tag{2.2}$$

which, by the fact that $\psi < 1$, implies

$$\varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right) \leq \varphi\left(\max\left\{\begin{array}{c} d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2), \\ ...,\ d(gx_{i-1}^r,\ gx_i^r) \end{array}\right\}\right).$$

Similarly

$$\begin{aligned} \varphi\left(d(gx_i^2,\ gx_{i+1}^2)\right) &\leq \varphi\left(\max\left\{\begin{array}{c} d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2), \\ ...,\ d(gx_{i-1}^r,\ gx_i^r) \end{array}\right\}\right), \\ &..., \\ \varphi\left(d(gx_i^r,\ gx_{i+1}^r)\right) &\leq \varphi\left(\max\left\{\begin{array}{c} d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2), \\ ...,\ d(gx_{i-1}^r,\ gx_i^r) \end{array}\right\}\right), \end{aligned}$$

Combining them, we get

$$\begin{aligned} &\max\{\varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right),\ \varphi\left(d(gx_i^2,\ gx_{i+1}^2)\right),\ ...,\ \varphi\left(d(gx_i^r,\ gx_{i+1}^r)\right)\} \\ \leq\ &\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}). \end{aligned}$$

Since φ is non-decreasing, it follows that

$$\begin{aligned}(2.3)\qquad & \varphi(\max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...\ d(gx_i^r,\ gx_{i+1}^r)\}) \\ & \leq \varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}),\end{aligned}$$

for all $i \geq 0$. Now (2.3) shows that $\{\varphi(\max\{d(gx_i^1, gx_{i+1}^1), d(gx_i^2, gx_{i+1}^2), ..., d(gx_i^r, gx_{i+1}^r)\})\}$ is a non-increasing sequence. Thus there exists $\delta \geq 0$ such that

$$(2.4)\qquad \lim_{i\to\infty} \varphi(\max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\}) = \delta.$$

Since $\psi \in \Psi$, we have $\lim_{r\to\delta+} \psi(r) < 1$ and $\psi(\delta) < 1$. Then there exist $\alpha \in [0, 1)$ and $\varepsilon > 0$ such that $\psi(r) \leq \alpha$ for all $r \in [\delta, \delta + \varepsilon)$. From (2.4), we can take $i_0 \geq 0$ such that $\delta \leq \varphi(\max\{d(gx_i^1, gx_{i+1}^1), d(gx_i^2, gx_{i+1}^2), ..., d(gx_i^r, gx_{i+1}^r)\}) \leq \delta + \varepsilon$ for all $i \geq i_0$. Then from (2.1), for all $i \geq i_0$, we have

$$\begin{aligned} & \varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right) \\ \leq\ & \psi\left(\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\})\right) \\ & \times\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}) \\ \leq\ & \alpha\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}).\end{aligned}$$

Thus

$$\varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right) \leq \alpha\varphi\left(\max\left\{\begin{array}{c} d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2), \\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\end{array}\right\}\right).$$

Similarly, for all $i \geq i_0$, we have

$$\begin{aligned}\varphi\left(d(gx_i^2,\ gx_{i+1}^2)\right) &\leq \alpha\varphi\left(\max\left\{\begin{array}{c} d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2), \\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\end{array}\right\}\right), \\ & ..., \\ \varphi\left(d(gx_i^r,\ gx_{i+1}^r)\right) &\leq \alpha\varphi\left(\max\left\{\begin{array}{c} d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2), \\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\end{array}\right\}\right).\end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\left\{\varphi\left(d(gx_i^1,\ gx_{i+1}^1)\right),\ \varphi\left(d(gx_i^2,\ gx_{i+1}^2)\right),\ ...,\ \varphi\left(d(gx_i^r,\ gx_{i+1}^r)\right)\right\} \\ \leq\ & \alpha\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}).\end{aligned}$$

Since φ is non-decreasing, it follows that

$$\begin{aligned}(2.5)\qquad & \varphi(\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\}) \\ & \leq \alpha\varphi(\max\left\{d(gx_{i-1}^1,\ gx_i^1),\ d(gx_{i-1}^2,\ gx_i^2),\ ...,\ d(gx_{i-1}^r,\ gx_i^r)\right\}),\end{aligned}$$

for all $i \geq i_0$. Letting $i \to \infty$ in (2.5) and using (2.4), we obtain that $\delta \leq \alpha\delta$. Since $\alpha \in [0, 1)$, therefore $\delta = 0$. Thus

$$\lim_{n\to\infty} \varphi(\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\}) = 0. \tag{2.6}$$

Since $\{\varphi(\max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\})\}$ is a non-increasing sequence and φ is non-decreasing, $\{\max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\}\}$ is also a non-increasing sequence of positive numbers. Thus there exists $\theta \geq 0$ such that

$$\lim_{n\to\infty} \max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\} = \theta.$$

Since φ is non-decreasing, we have

$$\varphi(\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\}) \geq \varphi(\theta). \tag{2.7}$$

Letting $n \to \infty$ in (2.7), by using (2.6), we get $0 \geq \varphi(\theta)$ which implies, by (ii_φ), that $\theta = 0$. Thus

$$\lim_{n\to\infty} \max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\} = 0. \tag{2.8}$$

Suppose that $\max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\} = 0$ for some $i \geq 0$. Then, we have $d(gx_i^1,\ gx_{i+1}^1) = 0$, $d(gx_i^2,\ gx_{i+1}^2) = 0$, ..., $d(gx_i^r,\ gx_{i+1}^r) = 0$ which implies that $gx_i^1 = gx_{i+1}^1 \in F(x_i^1,\ x_i^2,\ ...,\ x_i^r)$, $gx_i^2 = gx_{i+1}^2 \in F(x_i^2,\ ...,\ x_i^r,\ x_i^1)$, ..., $gx_i^r = gx_{i+1}^r \in F(x_i^r,\ x_i^1,\ ...,\ x_i^{r-1})$, that is, $(x_i^1,\ x_i^2,\ ...,\ x_i^r)$ is an $r-$tupled coincidence point of F and g. Now, suppose that

$$\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\} \neq 0, \text{ for all } i \geq 0.$$

Suppose $a_i = \varphi(\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\})$, for all $i \geq 0$. From (2.5), we have $a_i \leq \alpha a_{i-1}$ for all $i \geq i_0$. Then, we have

$$\sum_{i=0}^{\infty} a_i \leq \sum_{i=0}^{i_0} a_i + \sum_{i=i_0+1}^{\infty} \alpha^{i-i_0} a_{i_0} < \infty. \tag{2.9}$$

On the other hand, by (iii_φ), we have

$$\lim_{i\to\infty} \frac{\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\}}{\varphi(\max\left\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\right\})} < \infty. \tag{2.10}$$

Thus, by (2.9) and (2.10), we have $\sum \max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\} < \infty$. It means that $\{gx_i^1\}_{i=0}^{\infty}$, $\{gx_i^2\}_{i=0}^{\infty}$, ..., $\{gx_i^r\}_{i=0}^{\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, this implies that there exist x^1, x^2, ..., $x^r \in X$ such that

$$\lim_{i\to\infty} gx_i^1 = gx^1,\ \lim_{i\to\infty} gx_i^2 = gx^2,\ ...,\ \lim_{i\to\infty} gx_i^r = gx^r. \tag{2.11}$$

Now, since $gx_{i+1}^1 \in F(x_i^1,\ x_i^2,\ ...,\ x_i^r)$, $gx_{i+1}^2 \in F(x_i^2,\ ...,\ x_i^r,\ x_i^1)$, ..., $gx_{i+1}^r \in F(x_i^r,$ $x_i^1,\ ...,\ x_i^{r-1})$, by using condition (2.1), we get

$$\begin{aligned} & \varphi\left(D(gx_{i+1}^1,\ F(x^1,\ x^2,\ ...,\ x^r))\right) \\ \leq\ & \varphi\left(H(F(x_i^1,\ x_i^2,\ ...,\ x_i^r),\ F(x^1,\ x^2,\ ...,\ x^r))\right) \\ \leq\ & \psi\left(\varphi(\max\left\{d(gx_i^1,\ gx^1),\ d(gx_i^2,\ gx^2),\ ...,\ d(gx_i^r,\ gx^r)\right\})\right) \\ & \times\varphi(\max\left\{d(gx_i^1,\ gx^1),\ d(gx_i^2,\ gx^2),\ ...,\ d(gx_i^r,\ gx^r)\right\}), \end{aligned}$$

which, by the fact that $\psi < 1$, implies

$$\begin{aligned} & \varphi\left(D(gx_{i+1}^1,\ F(x^1,\ x^2,\ ...,\ x^r))\right) \\ \leq\ & \varphi(\max\left\{d(gx_i^1,\ gx^1),\ d(gx_i^2,\ gx^2),\ ...,\ d(gx_i^r,\ gx^r)\right\}). \end{aligned}$$

Since φ is non-decreasing, we have

$$\begin{aligned} & D(gx_{i+1}^1,\ F(x^1,\ x^2,\ ...,\ x^r)) \\ & \leq \max\left\{d(gx_i^1,\ gx^1),\ d(gx_i^2,\ gx^2),\ ...,\ d(gx_i^r,\ gx^r)\right\}. \end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in (2.12), by using (2.11), we obtain

$$D(gx^1,\ F(x^1,\ x^2,\ ...,\ x^r)) = 0.$$

Similarly, we can get

$$D(gx^2,\ F(x^2,\ ...,\ x^r,\ x^1)) = 0,\ ...,\ D(gx^r,\ F(x^r,\ x^1,\ ...,\ x^{r-1})) = 0,$$

which implies that

$$\begin{aligned} gx^1 &\in F(x^1,\ x^2,\ ...,\ x^r), \\ gx^2 &\in F(x^2,\ ...,\ x^r,\ x^1), \\ ...,\ gx^r &\in F(x^r,\ x^1,\ ...,\ x^{r-1}), \end{aligned}$$

that is, $(x^1, x^2, ..., x^r)$ is an $r-$tupled coincidence point of F and g.

Suppose now that (a) holds. Assume that for some $(x^1, x^2, ..., x^r) \in C(F, g)$,

$$\lim_{i\to\infty} g^i x^1 = y^1,\ \lim_{i\to\infty} g^i x^2 = y^2,\ ...,\ \lim_{i\to\infty} g^i x^r = y^r, \tag{2.13}$$

where y^1, y^2, ..., $y^r \in X$. Since g is continuous at y^1, y^2, ..., y^r. We have

$$gy^1 = y^1,\ gy^2 = y^2, ..., gy^r = y^r. \tag{2.14}$$

As F and g are $w-$compatible, so, for all $i \geq 1$,

$$
\begin{aligned}
g^i x^1 &\in F(g^{i-1}x^1,\ g^{i-1}x^2,\ ...,\ g^{i-1}x^r), \\
g^i x^2 &\in F(g^{i-1}x^2,\ ...,\ g^{i-1}x^r,\ g^{i-1}x^1), \\
...,\ g^i x^r &\in F(g^{i-1}x^r,\ g^{i-1}x^1,\ ...,\ g^{i-1}x^{r-1}).
\end{aligned} \tag{2.15}
$$

Now, by using (2.1) and (2.15), we obtain

$$
\begin{aligned}
& \varphi\left(D(g^i x^1,\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\
\leq\ & \varphi\left(H(F(g^{i-1}x^1,\ g^{i-1}x^2,\ ...,\ g^{i-1}x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\
\leq\ & \psi\left(\varphi(\max\left\{d(g^i x^1,\ gy^1),\ d(g^i x^2,\ gy^2),\ ...,\ d(g^i x^r,\ gy^r)\right\})\right) \\
& \times\varphi(\max\left\{d(g^i x^1,\ gy^1),\ d(g^i x^2,\ gy^2),\ ...,\ d(g^i x^r,\ gy^r)\right\}),
\end{aligned}
$$

which implies, by (i_φ) and (2.1), we have

$$
\begin{aligned}
& \varphi\left(D(g^i x^1,\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\
\leq\ & \varphi(\max\left\{d(g^i x^1,\ gy^1),\ d(g^i x^2,\ gy^2),\ ...,\ d(g^i x^r,\ gy^r)\right\}).
\end{aligned}
$$

Since φ is non-decreasing, we have

$$
\begin{aligned}
& D(g^i x^1,\ F(y^1,\ y^2,\ ...,\ y^r)) \\
& \leq \max\left\{d(g^i x^1,\ gy^1),\ d(g^i x^2,\ gy^2),\ ...,\ d(g^i x^r,\ gy^r)\right\}.
\end{aligned} \tag{2.16}
$$

On taking limit as $n \to \infty$ in (2.16), by using (2.13) and (2.14), we get

$$D(gy^1,\ F(y^1,\ y^2,\ ...,\ y^r)) = 0.$$

Similarly, we can get

$$D(gy^2,\ F(y^2,\ ...,\ y^r,\ y^1)) = 0,\ \ ...,\ D(gy^r,\ F(y^r,\ y^1,\ ...,\ y^{r-1})) = 0,$$

which implies that

$$
\begin{aligned}
gy^1 &\in F(y^1,\ y^2,\ ...,\ y^r), \\
gy^2 &\in F(y^2,\ ...,\ y^r,\ y^1), \\
...,\ gy^r &\in F(y^r,\ y^1,\ ...,\ y^{r-1}).
\end{aligned}
$$

Now, by (2.14), we have

$$
\begin{aligned}
y^1 &= gy^1 \in F(y^1,\ y^2,\ ...,\ y^r), \\
y^2 &= gy^2 \in F(y^2,\ ...,\ y^r,\ y^1), \\
...,\ y^r &= gy^r \in F(y^r,\ y^1,\ ...,\ y^{r-1}),
\end{aligned}
$$

that is, $(y^1, y^2, ..., y^r)$ is a common $r-$tupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x^1, x^2, ..., x^r) \in C(F, g)$, g is $F-$weakly commuting, that is, $g^2x^1 \in F(gx^1, gx^2, ..., gx^r)$, $g^2x^2 \in F(gx^2, ..., gx^r, gx^1)$, ..., $g^2x^r \in F(gx^r, gx^1, ..., gx^{r-1})$ and $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$. Thus $gx^1 = g^2x^1 \in F(gx^1, gx^2, ..., gx^r)$, $gx^2 = g^2x^2 \in F(gx^2, ..., gx^r, gx^1)$, ..., $gx^r = g^2x^r \in F(gx^r, gx^1, ..., gx^{r-1})$, that is, $(gx^1, gx^2, ..., gx^r)$ is a common $r-$tupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x^1, x^2, ..., x^r) \in C(F, g)$ and for some $y^1, y^2, ..., y^r \in X$, $\lim_{i\to\infty} g^iy^1 = x^1$, $\lim_{i\to\infty} g^iy^2 = x^2$, ..., $\lim_{i\to\infty} g^iy^r = x^r$. Since g is continuous at $x^1, x^2, ..., x^r$. We have $gx^1 = x^1$, $gx^2 = x^2$, ..., $gx^r = x^r$. Since $(x^1, x^2, ..., x^r) \in C(F, g)$, we obtain $x^1 = gx^1 \in F(x^1, x^2, ..., x^r)$, $x^2 = gx^2 \in F(x^2, ..., x^r, x^1)$, ..., $x^r = gx^r \in F(x^r, x^1, ..., x^{r-1})$, that is, $(x^1, x^2, ..., x^r)$ is a common $r-$tupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F, g)) = \{(x^1, x^1, ..., x^1)\}$. Then $\{x^1\} = \{gx^1\} = F(x^1, x^1, ..., x^1)$. Hence $(x^1, x^1, ..., x^1)$ is an $r-$tupled fixed point of F and g. □

Example 2.1. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X^r \to K(X)$ be defined as

$$F(x^1,\ x^2,\ ...,\ x^r) = \begin{cases} \{0\},\ \text{for } x^1,\ x^2,\ ...,\ x^r = 1 \\ \left[0,\ \dfrac{1}{4}(x^1)^4\right],\ \text{for } x^1,\ x^2,\ ...,\ x^r \in [0,\ 1) \end{cases}$$

and $g : X \to X$ be defined as

$$gx = x^2,\ \text{for all } x \in X.$$

Define $\varphi : [0, \infty) \to [0, \infty)$ by

$$\varphi(t) = \begin{cases} \ln(t+1),\ \text{for } t \neq 1 \\ \dfrac{3}{4},\ \text{for } t = 1, \end{cases}$$

and $\psi : [0, \infty) \to [0, 1)$ by

$$\psi(t) = \frac{\varphi(t)}{t},\ \text{for all } t \geq 0.$$

Now, for all $x^1, x^2, ..., x^r, y^1, y^2, ..., y^r \in X$ with $x^1, x^2, ..., x^r, y^1, y^2, ..., y^r \in [0, 1)$.

If $x^1 = y^1$, then

$$\begin{aligned} H(F(x^1,\ ...,\ x^r),\ F(y^1,\ ...,\ y^r)) &= \frac{1}{4}(y^1)^4 \\ &\leq \ln((y^1)^2+1) \\ &\leq \ln(\max\{(x^1)^2,\ (y^1)^2\}+1) \\ &\leq \ln(d(gx^1,\ gy^1)+1) \\ &\leq \ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1), \end{aligned}$$

which implies that

$$\begin{aligned} &\varphi\left(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\ =\ & \ln(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))+1) \\ \leq\ & \ln(\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1)+1) \\ \leq\ & \frac{\ln(\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1)+1)}{\ln(\max\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\}+1)} \\ & \times\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1) \\ \leq\ & \psi\left(\varphi(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\})\right) \\ & \times\varphi(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}). \end{aligned}$$

But if $x^1 \neq y^1$ and $x^1 < y^1$, then

$$\begin{aligned} H(F(x^1,\ ...,\ x^r),\ F(y^1,\ ...,\ y^r)) &= \frac{1}{4}(y^1)^4 \\ &\leq \ln((y^1)^2+1) \\ &\leq \ln(\max\{(x^1)^2,\ (y^1)^2\}+1) \\ &\leq \ln(d(gx^1,\ gy^1)+1) \\ &\leq \ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\ \right\}+1), \end{aligned}$$

which implies that

$$\begin{aligned} &\varphi\left(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\ =\ & \ln(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))+1) \\ \leq\ & \ln(\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1)+1) \\ \leq\ & \frac{\ln(\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1)+1)}{\ln(\max\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\}+1)} \\ & \times\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1) \end{aligned}$$

$$\begin{aligned} &\leq \psi\left(\varphi(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\})\right) \\ &\quad\times\varphi(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}). \end{aligned}$$

Similarly, we obtain the same result for $y^1 < x^1$. Thus the contractive condition (2.1) is satisfied for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in [0,\ 1)$. Again, for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., $x^r \in [0,\ 1)$ and y^1, y^2, ..., $y^r = 1$, we have

$$\begin{aligned} H(F(x^1,\ ...,\ x^r),\ F(y^1,\ ...,\ y^r)) &= \frac{1}{4}(x^1)^4 \\ &\leq \ln((x^1)^2+1) \\ &\leq \ln(\max\{(x^1)^2,\ (y^1)^2\}+1) \\ &\leq \ln(d(gx^1,\ gy^1)+1) \\ &\leq \ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1), \end{aligned}$$

which implies that

$$\begin{aligned} &\varphi\left(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\ &= \ln(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))+1) \\ &\leq \ln(\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1)+1) \\ &\leq \frac{\ln(\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1)+1)}{\ln(\max\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\}+1)} \\ &\quad\times\ln(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}+1) \\ &\leq \psi\left(\varphi(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\})\right) \\ &\quad\times\varphi(\max\left\{d(gx^1,\ gy^1),\ ...,\ d(gx^r,\ gy^r)\right\}). \end{aligned}$$

Thus the contractive condition (2.1) is satisfied for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., $x^r \in [0,\ 1)$ and y^1, y^2, ..., $y^r = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r = 1$. Hence, the hybrid pair $(F,\ g)$ satisfies the contractive condition (2.1), for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0,\ 0,\ ...,\ 0)$ is a common $r-$tupled fixed point of hybrid pair $(F,\ g)$. The function $F : X^r \to K(X)$ involved in this example is not continuous at the point $(1,\ 1,\ ...,\ 1) \in X^r$.

Remark 2. 1. We improve, extend and generalize the results of Ciric et al. [4] in the sense that

(i) We prove our result for hybrid pair of mappings.

(ii) We prove $n-$tupled coincidence and common $n-$tupled fixed point theorem while Ciric et al. [4] proved coupled coincidence and common coupled fixed point theorems.

(iii) We prove our result in the framework of noncomplete metric space $(X,\ d)$ and the product set X^r is not empowered with any order.

(iv) We prove our result without the assumption of continuity and mixed g-monotone property for mapping $F : X^r \to K(X)$.

(v) The functions $\varphi : [0,\ \infty) \to [0,\ \infty)$ and $\psi : [0,\ \infty) \to [0,\ 1)$ involved in our theorem and example are discontinuous.

If we put $g = I$ (the identity mapping) in Theorem 2.1, we get the following result:

Corollary 2.2. *Let $(X,\ d)$ be a complete metric space. Suppose $F : X^r \to K(X)$ is a mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} &\varphi\left(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\ \leq\ &\psi\left(\varphi(\max\left\{d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ d(x^r,\ y^r)\right\})\right) \\ &\times\varphi(\max\left\{d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ d(x^r,\ y^r)\right\}), \end{aligned}$$

for all $x^1,\ x^2,\ ...,\ x^r,\ y^1,\ y^2,\ ...,\ y^r \in X$. Then F has an $r-$tupled fixed point.

If we put $\psi(t) = 1 - \dfrac{\widetilde{\psi}(t)}{t}$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

Corollary 2.3. *Let $(X,\ d)$ be a metric space. Assume $F : X^r \to K(X)$ and $g : X \to X$ are two mappings for which there exist $\varphi \in \Phi$ and $\widetilde{\psi} \in \Psi$ such that*

$$\begin{aligned} &\varphi\left(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\ \leq\ &\varphi(\max\left\{d(gx^1,\ gy^1),\ d(gx^2,\ gy^2),\ ...,\ d(gx^r,\ gy^r)\right\}) \\ &-\widetilde{\psi}\left(\varphi(\max\left\{d(gx^1,\ gy^1),\ d(gx^2,\ gy^2),\ ...,\ d(gx^r,\ gy^r)\right\})\right), \end{aligned}$$

for all $x^1,\ x^2,\ ...,\ x^r,\ y^1,\ y^2,\ ...,\ y^r \in X$. Furthermore assume that $F(X^r) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have an $r-$tupled coincidence point. Moreover, F and g have a common $r-$tupled fixed point, if one of the following conditions holds.

(*a*) *F and g are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$,...,$\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C(F,\ g)$ *and for some* $y^1,\ y^2,\ ...,\ y^r \in X$ *and* g *is continuous at* $y^1,\ y^2,\ ...,\ y^r$.

(*b*) g *is* $F-$*weakly commuting for some* $(x^1,\ x^2,\ ...,\ x^r) \in C(F,\ g)$ *and* $gx^1,\ gx^2,\ ...,\ gx^r$ *are fixed points of* g, *that is,* $g^2x^1 = gx^1,\ g^2x^2 = gx^2,\ ...,\ g^2x^r = gx^r$.

(*c*) g *is continuous at* $x^1,\ x^2,\ ...,\ x^r$. $\lim_{i\to\infty} g^i y^1 = x^1$, $\lim_{i\to\infty} g^i y^2 = x^2$, ..., $\lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C(F,\ g)$ *and for some* $y^1,\ y^2,\ ...,\ y^r \in X$.

(*d*) $g(C(F,\ g))$ *is a singleton subset of* $C(F,\ g)$.

If we put $g = I$ (the identity mapping) in Corollary 2.3, we get the following result:

Corollary 2.4. *Let* $(X,\ d)$ *be a complete metric space. Suppose* $F : X^r \to K(X)$ *is a mapping for which there exist* $\varphi \in \Phi$ *and* $\widetilde{\psi} \in \Psi$ *such that*

$$\begin{aligned} & \varphi\left(H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r))\right) \\ \leq\ & \varphi(\max\left\{d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ \ d(x^r,\ y^r)\right\}) \\ & -\widetilde{\psi}\left(\varphi(\max\left\{d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ d(x^r,\ y^r)\right\})\right), \end{aligned}$$

for all $x^1,\ x^2,\ ...,\ x^r,\ y^1,\ y^2,\ ...,\ y^r \in X$. *Then* F *has an* $r-$*tupled fixed point.*

If we put $\varphi(t) = 2t$ for all $t \geq 0$ in Theorem 2.1, then we get the following result:

Corollary 2.5. *Let* $(X,\ d)$ *be a metric space. Suppose* $F : X^r \to K(X)$ *and* $g : X \to X$ *are two mappings for which there exists* $\psi \in \Psi$ *such that*

$$\begin{aligned} & H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r)) \\ \leq\ & \psi\left(2\max\left\{\begin{array}{c} d(gx^1,\ gy^1), \\ d(gx^2,\ gy^2), \\ ..., \\ d(gx^r,\ gy^r) \end{array}\right\}\right)\max\left\{\begin{array}{c} d(gx^1,\ gy^1), \\ d(gx^2,\ gy^2), \\ ..., \\ d(gx^r,\ gy^r) \end{array}\right\}, \end{aligned}$$

for all $x^1,\ x^2,\ ...,\ x^r,\ y^1,\ y^2,\ ...,\ y^r \in X$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds.*

(*a*) *F and g are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$,...,$\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C(F,\ g)$ *and for some* $y^1,\ y^2,\ ...,\ y^r \in X$ *and* g *is continuous at* $y^1,\ y^2,\ ...,\ y^r$.

(b) g is $F-$weakly commuting for some $(x^1, x^2, ..., x^r) \in C(F, g)$ and gx^1, gx^2, ..., gx^r are fixed points of g, that is, $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.

(c) g is continuous at x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^iy^1 = x^1$, $\lim_{i\to\infty} g^iy^2 = x^2$, ..., $\lim_{i\to\infty} g^iy^r = x^r$ for some $(x^1, x^2, ..., x^r) \in C(F, g)$ and for some y^1, y^2, ..., $y^r \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

If we put $g = I$ (the identity mapping) in Corollary 2.5, we get the following result:

Corollary 2.6. *Let (X, d) be a complete metric space. Suppose $F : X^r \to K(X)$ is a mapping for which there exists $\psi \in \Psi$ such that*

$$\begin{aligned} & H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r)) \\ \leq\ & \psi\left(2\max\left\{d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ d(x^r,\ y^r)\right\}\right) \\ & \times\max\left\{d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ d(x^r,\ y^r)\right\}, \end{aligned}$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$. Then F has an $r-$tupled fixed point.

If we put $\psi(t) = k$ for all $t \geq 0$ in Corollary 2.5, then we get the following result:

Corollary 2.7. *Let (X, d) be a metric space. Assume $F : X^r \to K(X)$ and $g : X \to X$ are two mappings satisfying*

$$\begin{aligned} & H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r)) \\ \leq\ & k\max\left\{d(gx^1,\ gy^1),\ d(gx^2,\ gy^2),\ ...,\ d(gx^r,\ gy^r)\right\}, \end{aligned}$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, where $0 < k < 1$. Furthermore assume that $F(X^r) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have an $r-$tupled coincidence point. Moreover, F and g have a common $r-$tupled fixed point, if one of the following conditions holds.

(a) F and g are $w-$compatible. $\lim_{i\to\infty} g^ix^1 = y^1$, $\lim_{i\to\infty} g^ix^2 = y^2$,...,$\lim_{i\to\infty} g^ix^r = y^r$, for some $(x^1, x^2, ..., x^r) \in C(F, g)$ and for some y^1, y^2, ..., $y^r \in X$ and g is continuous at y^1, y^2, ..., y^r.

(b) g is $F-$weakly commuting for some $(x^1, x^2, ..., x^r) \in C(F, g)$ and gx^1, gx^2, ..., gx^r are fixed points of g, that is, $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.

(c) g is continuous at x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^iy^1 = x^1$, $\lim_{i\to\infty} g^iy^2 = x^2$, ..., $\lim_{i\to\infty} g^iy^r = x^r$ for some $(x^1, x^2, ..., x^r) \in C(F, g)$ and for some y^1, y^2, ..., $y^r \in X$.

(d) $g(C(F,\, g))$ is a singleton subset of $C(F,\, g)$.

If we put $g = I$ (the identity mapping) in Corollary 2.7, we get the following result:

Corollary 2.8. *Let $(X,\, d)$ be a complete metric space. Assume $F : X^r \to K(X)$ is a mapping satisfying*

$$\begin{aligned} & H(F(x^1,\ x^2,\ ...,\ x^r),\ F(y^1,\ y^2,\ ...,\ y^r)) \\ \leq\ & k \max\left\{ d(x^1,\ y^1),\ d(x^2,\ y^2),\ ...,\ d(x^r,\ y^r) \right\}, \end{aligned}$$

for all $x^1,\, x^2,\, ...,\, x^r,\, y^1,\, y^2,\, ...,\, y^r \in X$, where $0 < k < 1$. Then F has an $r-$tupled fixed point.

REFERENCES

1. A. Amini-Harandi & D. O'Regan: Fixed point theorems for set-valued contraction type mappings in metric spaces. Fixed Point Theory Appl. **2010**, 7 (2010). Article ID 390183.
2. T.G. Bhaskar & V. Lakshmikantham: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 65 (2006), no. 7, 1379-1393.
3. S. Banach: Sur les Operations dans les Ensembles Abstraits et leur. Applications aux Equations Integrales, Fund. Math. **3** (1922), 133-181.
4. L. Ciric, B. Damjanovic, M. Jleli & B. Samet: Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications. Fixed Point Theory Appl. **2012**, 51.
5. B. Deshpande: Common fixed point for set and single valued functions without continuity and compatibility. Math. Morav. **11** (2007), 27-38.
6. B. Deshpande & S. Chouhan: Fixed points for two hybrid pairs of mappings satisfying some weaker conditions on noncomplete metric spaces. Southeast Asian Bull. Math. **35** (2011), 851-858.
7. B. Deshpande & A. Handa: Common n-tupled fixed point for hybrid pair of mappings under new contractive condition. J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **21** (2014), no. 3, 165-181.
8. B. Deshpande & A. Handa, n-tupled coincidence and fixed point result for hybrid pair of mappings under generalized nonlinear contraction. Southeast Asian Bull. Math. **40** (2016), 655-670.
9. B. Deshpande & R. Pathak. Fixed point theorems for noncompatible discontinuous hybrid pairs of mappings on 2- metric spaces. Demonstr. Math. **XLV** 2012, no. 1, 143-154.

10. W.S. Du: Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashi's condition in quasi ordered metric spaces. Fixed Point Theory Appl. **2010**, 9 (2010) Article ID 876372.
11. J. Harjani, B. Lopez & K. Sadarangani: Fixed point theorems for mixed monotone operators and applications to integral equations. Nonlinear Anal. **74** (2011), 1749-1760.
12. M. Imdad, A.H. Soliman, B.S. Choudhury & P. Das: On n-tupled coincidence point results in metric spaces. Journal of Operators Volume **2013**, Article ID 532867, 8 pages.
13. I. Kubiaczyk & B. Deshpande: A common fixed point theorem for multivalued mappings through T-weak commutativity. Math. Morav. **10** (2006), 55-60.
14. I. Kubiaczyk & B. Deshpande: Common fixed point of multivalued mappings without continuity. Fasc. Mathe. **37** (2007), no. 9, 19-26.
15. I. Kubiaczyk & B. Deshpande: Noncompatibility, discontinuity in consideration of common fixed point of set and single valued mappings. Southeast Asian Bull. Math. **32** (2008), 467-474.
16. V. Lakshmikantham & L. Ciric: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70** (2009), no. 12, 4341-4349.
17. J.T. Markin: Continuous dependence of fixed point sets. Proc. Amer. Math. Soc. **38** (1947), 545-547.
18. N. Mizoguchi & W. Takahashi: Fixed point theorems for multivalued mappings on complete metric spaces. J. Math. Anal. Appl. **141** (1989), 177-188.
19. S.B. Nadler: Multivalued contraction mappings. Pacific J. Math. **30** (1969), 475-488.
20. J. Rodriguez-Lopez & S. Romaguera: The Hausdorff fuzzy metric on compact sets. Fuzzy Sets Syst. **147** (2004), 273-283.
21. S. Sharma & B. Deshpande. Fixed point theorems for set and single valued mappings without continuity and compatibility. Demonstr. Math. **XL** (2007), no. 3, 649-658.
22. S. Sharma & B. Deshpande: Compatible multivalued mappings satisfying an implicit relation. Southeast Asian Bull. Math. **30** (2006), 535-540.

[a]Department of Mathematics, Govt. Arts and Science College, Ratlam (M. P.), India
Email address: bhavnadeshpande@yahoo.com

[b]Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam (M. P.), India
Email address: amrishhanda83@gmail.com