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The exponential generalized log-logistic model: Bagdonavičius-Nikulin test for validation and non-Bayesian estimation methods

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Abstract

A modified Bagdonavičius-Nikulin chi-square goodness-of-fit is defined and studied. The lymphoma data is analyzed using the modified goodness-of-fit test statistic. Different non-Bayesian estimation methods under complete samples schemes are considered, discussed and compared such as the maximum likelihood least square estimation method, the Cramer-von Mises estimation method, the weighted least square estimation method, the left tail-Anderson Darling estimation method and the right tail Anderson Darling estimation method. Numerical simulation studies are performed for comparing these estimation methods. The potentiality of the new model is illustrated using three real data sets and compared with many other well-known generalizations.

Keywords: Bagdonavičius-Nikulin test, Barzilai-Borwein, Anderson Darling estimation, log-logistic model, Cramér-von-Mises estimation

1. Introduction and motivation

In this paper, we extended the log-logistic (LL) model by proposing and studying a new probability distribution called the exponential generalized log-logistic (EG-LL) model. Different non-Bayesian well-known estimation methods under complete samples scheme are considered and discussed such as, the maximum likelihood estimation (MLE), the Anderson Darling estimation (ADE), the ordinary least square estimation (OLSE), the Cramér-von-Mises estimation (CVME), the weighted least square estimation (WLSE), the left tail-Anderson Darling estimation (ADE(L-T)), and the right tail Anderson Darling estimation (ADE(R-T)) methods. Numerical simulation studies are performed for comparing these estimation methods using different sample sizes and three different combinations of parameters. Some useful comments are made and recorded in Sections 4 and 5. The potentiality of the EG-LL model is illustrated using three real data sets. The EG-LL model is compared with many other well-known generalizations. The new model proved its worth in modeling breaking stress, survival times and medical data sets. The Barzilai-Borwein (BB) algorithm is employed via a simulation study for assessing the performance of the estimators with different sample sizes as sample size tends to ∞. Using the Bagdonavičius-Nikulin goodness-of-fit test for validation, we propose a modified chi-square GOF tests for the EG-LL model. We have analyzed a lymphoma data set consisting of

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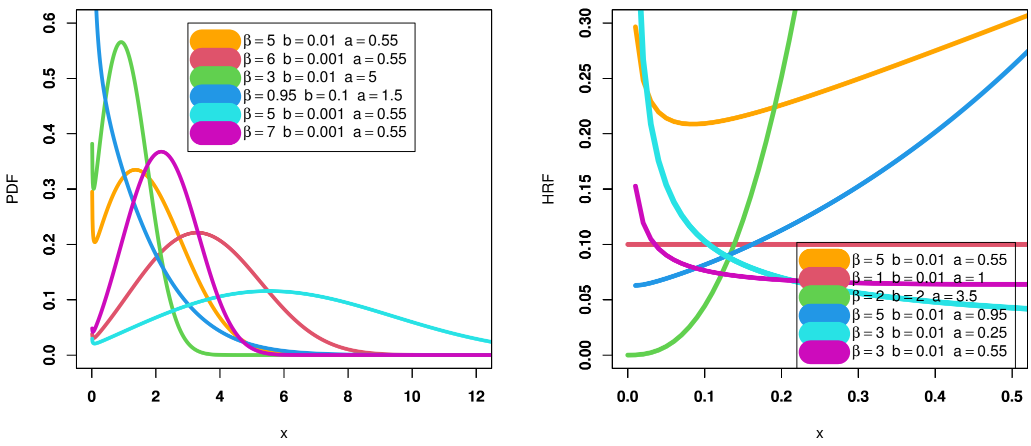

Figure 1: *Plots for the PDF and HRF for the EG-LL model.*

times (in months) from diagnosis to death for 31 individuals with advanced non Hodgkin's lymphoma clinical symptoms by using our model under the modified Bagdonavičius-Nikulin goodness-of-fit test statistic. Based on the MLEs, the modified Bagdonavičius-Nikulin goodness-of-fit test recovers the loss in information while the grouping data follow chi-square distributions. The corresponding elements of the modified Bagdonavičius-Nikulin goodness-of-fit criteria tests are explicitly derived.

A random variable (RV) Z is said to have the log-logistic (LL) distribution if its survival function (SF) is written as

$$\overline{S}_b(z) = 1 - G_b(z) = \frac{1}{1+z^b}, \tag{1.1}$$

where $z \geq 0$ and $b > 0$ is a shape parameter, and $G_b(z)$ refers to the cumulative distribution functions (CDF) of the LL model. The corresponding probability density function (PDF) of (1.1) is given by

$$g_b(z) = \frac{bz^{b-1}}{(1+z^b)^2}. \tag{1.2}$$

The PDF in (1.2) is a special case from the well-know Burr type XII (BXII) model (Burr 1942, 1968, 1973; Rodriguez, 1977; Tadikamalla, 1980). Based on the family of Yousof *et al.* (2018) and using (1.1), the CDF of the Exponential Generalized log-logistic (EG-LL) is then defined by,

$$F_{\beta,b,a}(z) = 1 - \exp\left\{-a\left[\left(1+z^b\right)^\beta - 1\right]\right\}, \tag{1.3}$$

where $z \geq 0$ and $\beta > 0, b > 0, a > 0$. The corresponding PDF is given by,

$$f_{\beta,b,a}(z) = \beta a b z^{b-1}\left(1+z^b\right)^{\beta-1} \exp\left\{-a\left[\left(1+z^b\right)^\beta - 1\right]\right\}. \tag{1.4}$$

For $a = 1$, the EG-LL model reduces to the two-parameter EG-LL model. For $\beta = 1$, the EG-LL model reduces to the two-parameter exponential log-logistic (E-LL) model. For $\beta = a = 1$, the

EG-LL model reduces to the one-parameter E-LL model. Figure 1 below gives some plots for the PDF and hazard rate function (HRF) for the EG-LL model. From Figure 1(a) shows that the new PDF can have many useful shapes. From Figure 1(b) we conclude that the new HRF can be "bathtub ($\beta = 5, b = 0.01, a = 0.55$)", "constant ($\beta = 1, b = 0.01, a = 1$)", "**J**-HRF ($\beta = 2, b = 2, a = 3.5$)", "increasing ($\beta = 5, b = 0.01, a = 0.95$)", "decreasing ($\beta = 3, b = 0.01, a = 0.25$)" and "decreasing-constant ($\beta = 3, b = 0.01, a = 0.55$)".

The pth ordinary moment of Z where Z ~EG-LL(β, b, a) is given by,

$$\mu'_p = \mathbf{E}(Z^p) = \int_{-\infty}^{\infty} z^p f(z)\, dz.$$

Then, using (1.5) we obtain

$$\mu'_p = \sum_{\kappa_4=0}^{\infty} u_{\kappa_4} \kappa_4^* B\left(1 + \frac{p}{b}, \kappa_4^* - \frac{p}{b}\right), \tag{1.5}$$

where $\kappa_4^* = 1 + \kappa_4,\ p < \kappa_4^* b,\ B(a_1, a_2) = \int_0^\infty t^{a_1-1}(1+t)^{-(a_1+a_2)} dt$ and

$$u_{\kappa_4} = \sum_{\kappa_1,\kappa_2,\kappa_3=0}^{\infty} \frac{(-1)^{1+\kappa_1+\kappa_2+\kappa_3+\kappa_4}}{\Gamma(1+\kappa_1)\Gamma\kappa_4^*} \binom{\kappa_1}{\kappa_2} \binom{\beta(\kappa_2-\kappa_1)}{\kappa_3} \binom{1+\kappa_3}{\kappa_4}$$

is the beta function of the second type. By setting $p = 1$ in (1.5), we get the mean of Z. The p^{th} incomplete moment ($\varsigma_p(t)$) of Z can be expressed as,

$$\varsigma_p(t) = \int_{-\infty}^{t} z^p f(z)\, dz = \sum_{\kappa_4=0}^{\infty} u_{\kappa_4} \kappa_4^* B\left(t^b; 1 + \frac{p}{b}, \kappa_4^* - \frac{p}{b}\right),$$

where $p < \kappa_4^* b$ and $B(q; a_1, a_2) = \int_0^q t^{a_1-1}(1+t)^{-(a_1+a_2)} dt$ is the incomplete beta function of the second type. The moment generating function $M_Z(t) = \mathbf{E}(\exp(tZ))$ of Z can be easily derived. Then, the p^{th} moment of the reversed residual life of Z becomes

$$A_p(t) = F^{-1}(t) \sum_{r=0}^{\infty} l_{\kappa_4} \kappa_4^* B\left(t^b; \kappa_4^* - \frac{p}{b}, 1 + \frac{p}{b}\right),$$

where

$$l_{\kappa_4} = u_{\kappa_4} \sum_{r=0}^{p} (-1)^r \binom{p}{r} t^{p-r}.$$

The proposed EG-LL model has only three parameters. However, in the applications to real data sets, all competitive distributions have at least three parameters (see comparing models under complete samples in Section 4). The EG-LL model with this advantage of a smaller number of parameters is the favorable statistical modeling, especially if it gives a better fit under some goodness-of-fit statistics tests (Tables 5, 6 and 7). So, it is recommended to use the EG-LL model instead of all other competitive models. In statistical modeling and applied fields, the EG-LL model could be useful in the following cases,

- Modeling the asymmetric right skewed real-life data sets especially the right skewed and heavy tail real-life data sets.
- Modeling the symmetric and asymmetric right skewed real-life data sets especially in case of modeling a certain real-life data for the first time.
- In reliability analysis, some physics studies and many engineering applications, the EG-LL model can be applied in modeling the breaking stress reliability data. As shown in Tables 5 and 6, the EG-LL model shows its superiority against the log-logistic, the exponential log-logistic, the Burr type XII, the Weibull log-logistic, the Marshall-Olkin Burr type XII, the Topp Leone Burr type XII, the Zografos-Balakrishnan Burr type XII, Five Parameters beta Burr type XII, Beta Burr Burr type XII, and the Beta exponentiated Burr type XII, the Five Parameters Kumaraswamy Burr type XII and Kumaraswamy Burr type XII distributions.
- In survival analysis of guinea pigs, the EG-LL model can be chosen in modeling the survival times data. Based on Table 8, the EG-LL model performs better than the log-logistic, the exponential log-logistic, the Burr type XII, the Weibull log-logistic, the Marshall-Olkin Burr type XII, the Topp Leone Burr type XII, the Zografos-Balakrishnan Burr type XII, the Five Parameters beta Burr type XII, the Beta Burr Burr type XII, the Beta exponentiated Burr type XII, the Five Parameters Kumaraswamy Burr type XII and the Kumaraswamy Burr type XII distributions.
- In the medicine, the EG-LL model can be applied for modeling the acute myelogenous leukemia data. The new model showed its superiority against many competitive models such as the log-logistic, the exponential log-logistic, the Burr type XII, the Weibull log-logistic, the Marshall-Olkin Burr type XII, the Topp Leone Burr type XII, the Zografos-Balakrishnan Burr type XII, the Five Parameters beta Burr type XII, the Beta Burr Burr type XII, the Beta exponentiated Burr type XII, the Five Parameters Kumaraswamy Burr type XII and the Kumaraswamy Burr type XII distributions as shown in Table 9 and Table 10.

2. Non-Bayesian estimation methods under complete samples schemes

2.1. The MLE method

Suppose that $z_1, z_2, \ldots, z_n$ is a RS from the EG-LL model with parameter vector $\underline{\mathbf{\Phi}}=(\beta, a, b)^{\intercal}$. Then the log-likelihood function ($\mathcal{L}_n(\underline{\mathbf{\Phi}})$) for $\underline{\mathbf{\Phi}}$ is given by,

$$\mathcal{L}_n(\underline{\mathbf{\Phi}}) = n\log\beta + n\log a + n\log b + (\beta-1)\sum_{\varsigma=1}^{n}\log\left(1+z_{[\varsigma:n]}^{b}\right) - a\sum_{\varsigma=1}^{n}\left[\left(1+z_{[\varsigma:n]}^{b}\right)^{\beta}-1\right],$$

where the above $\mathcal{L}_n(\underline{\mathbf{\Phi}})$ can be maximized numerically via `SAS` (`PROC NLMIXED`) or `R` (`optim`) or `Ox` program (via sub-routine `MaxBFGS`), among others. The components of the score vector

$$\mathbf{U}(\underline{\mathbf{\Phi}}) = \frac{\partial\mathcal{L}}{\partial\underline{\mathbf{\Phi}}} = \left(\frac{\partial\mathcal{L}_n(\underline{\mathbf{\Phi}})}{\partial\beta}, \frac{\partial\mathcal{L}_n(\underline{\mathbf{\Phi}})}{\partial a}, \frac{\partial\mathcal{L}_n(\underline{\mathbf{\Phi}})}{\partial b}\right)^{\intercal},$$

can easily be derived.

2.2. CVME method

The CVME of the parameters β, b and a are obtained via minimizing the following expression with respect to the parameters β, b and a respectively, where

$$\mathbf{CVME}_{(\underline{\mathbf{\Phi}})} = \frac{1}{12}n^{-1} + \sum_{\varsigma=1}^{n}\left[F_{\underline{\mathbf{\Phi}}}\left(z_{[\varsigma:n]}\right) - c_{(\varsigma,n)}\right]^2,$$

and where $c_{(\varsigma,n)} = (2\varsigma - 1)/2n$ and

$$\mathbf{CVME}_{(\underline{\mathbf{\Phi}})} = \frac{1}{12}n^{-1} + \sum_{\varsigma=1}^{n}\left[\left(1 - \exp\left\{-a\left[\left(1 + z_{[\varsigma:n]}^{b}\right)^{\beta} - 1\right]\right\}\right) - c_{(\varsigma,n)}\right]^2.$$

The CVME of the parameters β, b and a are obtained by solving the two following non-linear equations as

$$\sum_{\varsigma=1}^{n}\left[\left(1 - \exp\left\{-a\left[\left(1 + z_{[\varsigma:n]}^{b}\right)^{\beta} - 1\right]\right\}\right) - c_{(\varsigma,n)}\right]\nabla_{(\beta)}\left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}}\right) = 0,$$
$$\sum_{\varsigma=1}^{n}\left[\left(1 - \exp\left\{-a\left[\left(1 + z_{[\varsigma:n]}^{b}\right)^{\beta} - 1\right]\right\}\right) - c_{(\varsigma,n)}\right]\nabla_{(b)}\left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}}\right) = 0,$$
$$\sum_{\varsigma=1}^{n}\left[\left(1 - \exp\left\{-a\left[\left(1 + z_{[\varsigma:n]}^{b}\right)^{\beta} - 1\right]\right\}\right) - c_{(\varsigma,n)}\right]\nabla_{(a)}\left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}}\right) = 0,$$

where

$$\nabla_{(\beta)}(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}}) = \frac{\partial F_{\underline{\mathbf{\Phi}}}\left(z_{[\varsigma:n]}\right)}{\partial \beta},$$
$$\nabla_{(b)}(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}}) = \frac{\partial F_{\underline{\mathbf{\Phi}}}\left(z_{[\varsigma:n]}\right)}{\partial b},$$
$$\nabla_{(a)}(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}}) = \frac{\partial F_{\underline{\mathbf{\Phi}}}\left(z_{[\varsigma:n]}\right)}{\partial a}.$$

2.3. **OLSE** method

Let $F_{\underline{\mathbf{\Phi}}}(z_{[\varsigma:n]})$ denote the CDF of the EG-LL model and let $z_{1:n} < z_{2:n} < \cdots < z_{n:n}$ be the n ordered RS. The OLSEs are obtained upon minimizing,

$$\mathbf{OLSE}(\underline{\mathbf{\Phi}}) = \sum_{\varsigma=1}^{n}\left[F_{\underline{\mathbf{\Phi}}}\left(z_{[\varsigma:n]}\right) - \upsilon_{(\varsigma,n)}\right]^2.$$

Then, we have

$$\mathbf{OLSE}(\underline{\mathbf{\Phi}}) = \sum_{\varsigma=1}^{n}\left[\left(1 - \exp\left\{-a\left[\left(1 + z_{[\varsigma:n]}^{b}\right)^{\beta} - 1\right]\right\}\right) - \upsilon_{(\varsigma,n)}\right]^2,$$

where $\upsilon_{(\varsigma,n)} = \varsigma/n + 1$. The LSEs are obtained via solving the following non-linear equations,

$$0 = \sum_{\varsigma=1}^{n} \left[\left(1 - \exp\left\{ -a \left[\left(1 + z_{[\varsigma:n]}^{b} \right)^{\beta} - 1 \right] \right\} \right) - \upsilon_{(\varsigma,n)} \right] \nabla_{(\beta)} \left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}} \right),$$

$$0 = \sum_{\varsigma=1}^{n} \left[\left(1 - \exp\left\{ -a \left[\left(1 + z_{[\varsigma:n]}^{b} \right)^{\beta} - 1 \right] \right\} \right) - \upsilon_{(\varsigma,n)} \right] \nabla_{(b)} \left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}} \right),$$

$$0 = \sum_{\varsigma=1}^{n} \left[\left(1 - \exp\left\{ -a \left[\left(1 + z_{[\varsigma:n]}^{b} \right)^{\beta} - 1 \right] \right\} \right) - \upsilon_{(\varsigma,n)} \right] \nabla_{(a)} \left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}} \right),$$

where $\nabla_{(\beta)}(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}})$, $\nabla_{(b)}(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}})$ and $\nabla_{(a)}(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}})$ are defined above.

2.4. WLSE method

The WLSE is obtained by minimizing the function $\mathbf{WLSE}(\underline{\mathbf{\Phi}})$ with respect to β, b and a

$$\mathbf{WLSE}\left(\underline{\mathbf{\Phi}}\right) = \sum_{\varsigma=1}^{n} \vartheta_{(\varsigma,n)} \left[F_{\underline{\mathbf{\Phi}}}\left(z_{[\varsigma:n]}\right) - \upsilon_{(\varsigma,n)} \right]^{2},$$

where

$$\vartheta_{(\varsigma,n)} = \frac{\left[(1+n)^{2}(2+n)\right]}{\left[\varsigma(1+n-\varsigma)\right]}.$$

The WLSEs are obtained by solving,

$$0 = \sum_{\varsigma=1}^{n} \left\{ \left[\left(1 - \exp\left\{ -a \left[\left(1 + z_{[\varsigma:n]}^{b} \right)^{\beta} - 1 \right] \right\} \right) - \upsilon_{(\varsigma,n)} \right] \vartheta_{(\varsigma,n)} \nabla_{(\beta)} \left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}} \right) \right\},$$

$$0 = \sum_{\varsigma=1}^{n} \left\{ \left[\left(1 - \exp\left\{ -a \left[\left(1 + z_{[\varsigma:n]}^{b} \right)^{\beta} - 1 \right] \right\} \right) - \upsilon_{(\varsigma,n)} \right] \vartheta_{(\varsigma,n)} \nabla_{(b)} \left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}} \right) \right\},$$

$$0 = \sum_{\varsigma=1}^{n} \left\{ \left[\left(1 - \exp\left\{ -a \left[\left(1 + z_{[\varsigma:n]}^{b} \right)^{\beta} - 1 \right] \right\} \right) - \upsilon_{(\varsigma,n)} \right] \vartheta_{(\varsigma,n)} \nabla_{(a)} \left(z_{[\varsigma:n]}; \underline{\mathbf{\Phi}} \right) \right\}.$$

2.5. The ADE method

The ADE $\hat{\beta}_{(\mathrm{ADE})}$, $\hat{b}_{(\mathrm{ADE})}$, $\hat{a}_{(\mathrm{ADE})}$ are obtained by minimizing the function,

$$\mathbf{ADE}_{\left(z_{[\varsigma:n]}, z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right) = -n - n^{-1} \sum_{\varsigma=1}^{n} \left[\begin{array}{c} (2\varsigma - 1) \\ \times \left\{ \begin{array}{c} \log F_{(\underline{\mathbf{\Phi}})}(z_{[\varsigma:n]}) \\ + \log\left[1 - F_{(\underline{\mathbf{\Phi}})}(z_{[-\varsigma+1+n:n]})\right] \end{array} \right\} \end{array} \right].$$

The parameter estimates $\hat{\beta}_{(\mathrm{ADE})}$, $\hat{b}_{(\mathrm{ADE})}$, $\hat{a}_{(\mathrm{ADE})}$ are derived by solving the nonlinear equations,

$$0 = \frac{\partial\left[\mathbf{ADE}_{\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial\beta},$$
$$0 = \frac{\partial\left[\mathbf{ADE}_{\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial b},$$
$$0 = \frac{\partial\left[\mathbf{ADE}_{\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial a}.$$

2.6. The ADE (R-T) method

The ADE(R-T) $\hat{\beta}_{(\mathrm{ADE}_{(\text{R-T})})}$, $\hat{b}_{(\mathrm{ADE}_{(\text{R-T})})}$, $\hat{a}_{(\mathrm{ADE}_{(\text{R-T})})}$ are obtained by minimizing,

$$\begin{aligned}\mathrm{ADE}_{(\text{R-T})\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right) = {} & \frac{1}{2}n - 2\sum_{\varsigma=1}^{n} F_{(\underline{\mathbf{\Phi}})}(z_{[\varsigma:n]}) \\ & - \frac{1}{n}\sum_{\varsigma=1}^{n}(2\varsigma-1)\left\{\log\left[1-F_{(\underline{\mathbf{\Phi}})}(z_{[-\varsigma+1+n:n]})\right]\right\}.\end{aligned}$$

The estimates $\hat{\beta}_{(\mathrm{ADE}_{(\text{R-T})})}$, $\hat{b}_{(\mathrm{ADE}_{(\text{R-T})})}$, $\hat{a}_{(\mathrm{ADE}_{(\text{R-T})})}$ follow by solving the nonlinear equations,

$$0 = \frac{\partial\left[\mathrm{ADE}_{(\text{R-T})\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial\beta},$$
$$0 = \frac{\partial\left[\mathrm{ADE}_{(\text{R-T})\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial b},$$
$$0 = \frac{\partial\left[\mathrm{ADE}_{(\text{R-T})\left(z_{[\varsigma:n]},z_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial a}.$$

2.7. The ADE(L-T) method

The ADE(L-T) $\hat{\beta}_{(\mathrm{ADE}_{(\text{L-T})})}$, $\hat{b}_{(\mathrm{ADE}_{(\text{L-T})})}$, $\hat{a}_{(\mathrm{ADE}_{(\text{L-T})})}$ are obtained by minimizing,

$$\mathrm{ADE}_{(\text{L-T})\left(z_{[\varsigma:n]}\right)}\left(\underline{\mathbf{\Phi}}\right) = -\frac{3}{2}n + 2\sum_{\varsigma=1}^{n} F_{(\underline{\mathbf{\Phi}})}(z_{[\varsigma:n]}) - \frac{1}{n}\sum_{\varsigma=1}^{n}(2\varsigma-1)\log F_{(\underline{\mathbf{\Phi}})}(z_{[\varsigma:n]}).$$

The estimates $\hat{\beta}_{(\mathrm{ADE}_{(\text{L-T})})}$, $\hat{b}_{(\mathrm{ADE}_{(\text{L-T})})}$, $\hat{a}_{(\mathrm{ADE}_{(\text{L-T})})}$ follow by solving the nonlinear equations,

$$0 = \frac{\partial\left[\mathrm{ADE}_{(\text{L-T})\left(z_{[\varsigma:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial\beta},$$
$$0 = \frac{\partial\left[\mathrm{ADE}_{(\text{L-T})\left(z_{[\varsigma:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial b},$$
$$0 = \frac{\partial\left[\mathrm{ADE}_{(\text{L-T})\left(z_{[\varsigma:n]}\right)}\left(\underline{\mathbf{\Phi}}\right)\right]}{\partial a}.$$

3. Comparing the non-Bayesian estimation methods under complete samples schemes via a simulation study

A numerical simulation is performed to compare the classical estimation methods. The simulation study is based on N=1000 generated data sets from the EG-LL version where $n = 50, 100, 300$ and 500. The simulation is performed in terms of bias ($\text{BIAS}_{(\underline{\mathbf{\Phi}})}$), the root mean-standard error ($\text{RMSE}_{(\underline{\mathbf{\Phi}})}$), the mean of the absolute difference between the theoretical and the estimates (D_{abs}), and the maximum absolute difference between the true parameters and estimates ($D_{\max}$), where

$$\text{BIAS}_{(\beta)} = \frac{1}{B}\sum_{\varsigma=1}^{B}\left(\hat{\beta}-\beta\right),$$

$$\text{BIAS}_{(b)} = \frac{1}{B}\sum_{\varsigma=1}^{B}\left(\hat{b}-b\right),$$

$$\text{BIAS}_{(a)} = \frac{1}{B}\sum_{\varsigma=1}^{B}\left(\hat{a}-a\right),$$

$$\text{RMSE}_{(\beta)} = \sqrt{\frac{1}{B}\sum_{\varsigma=1}^{B}\left(\hat{\beta}-\beta\right)^2},$$

$$\text{RMSE}_{(a)} = \sqrt{\frac{1}{B}\sum_{\varsigma=1}^{B}\left(\hat{a}-a\right)^2},$$

$$\text{RMSE}_{(b)} = \sqrt{\frac{1}{B}\sum_{\varsigma=1}^{B}\left(\hat{b}-b\right)^2},$$

$$D_{\text{abs}} = \frac{1}{nB}\sum_{\varsigma=1}^{B}\sum_{j=1}^{n}|F_{\underline{\mathbf{\Phi}}}(z_{\varsigma j}) - F_{(\hat{\beta},\hat{b},\hat{a})}(\tau_{\varsigma j})|,$$

$$D_{\max} = \frac{1}{B}\sum_{\varsigma=1}^{B}\max_{j}|F_{\underline{\mathbf{\Phi}}}(z_{\varsigma j}) - F_{(\hat{\beta},\hat{b},\hat{a})}(z_{\varsigma j})|.$$

From Tables 1–3 we note that,

- The $\text{BIAS}_{(\underline{\mathbf{\Phi}})}$ tend to 0 when n increases and tent to ∞ which means that all estimators are non-biased.
- The $\text{RMSE}_{(\underline{\mathbf{\Phi}})}$ tend to 0, when n increases and tent to ∞ which means incidence of consistency property.
- The MLE method is the best among all the non-Bayesian estimations methods.
- The $\text{BIAS}_{(\underline{\mathbf{\Phi}})}$ takes positive and sometimes negative values.

The following results are based on Table 1,

- For $n = 50|\beta = 0.9, b = 0.9, a = 0.9$, the MLE method is the best with RMSE=0.07860,0.09449, 0.13275.

- For $n = 100|\beta = 0.9,\ b = 0.9,\ a = 0.9$, the MLE method is the best with RMSE=0.05537, 0.06685, 0.09235.
- For $n = 300|\beta = 0.9,\ b = 0.9,\ a = 0.9$, the MLE method is the best with RMSE=0.03037, 0.03696, 0.05179.
- For $n = 500|\beta = 0.9,\ b = 0.9,\ a = 0.9$, the MLE method is the best with RMSE=0.02306, 0.02795, 0.03919.

Based on Tables 2, the following results are,

- For $n = 50|\beta = 1.2,\ b = 1.5,\ a = 0.1$, the MLE method is the best with RMSE=0.05641, 0.07782, 0.01477.
- For $n = 100|\beta = 1.2,\ b = 1.5,\ a = 0.1$, the MLE method is the best with RMSE=0.04007, 0.05464, 0.01062.
- For $n = 300|\beta = 1.2,\ b = 1.5,\ a = 0.1$, the MLE method is the best with RMSE =0.02135, 0.02924, 0.00562.
- For $n = 500|\beta = 1.2,\ b = 1.5,\ a = 0.1$, the MLE method is the best with RMSE=0.01690, 0.02293,0.00446.

Based on Tables 3, the following results are,

- For $n = 50|\beta = 0.2, b = 0.6, a = 1.2$ the MLE method is the best with RMSE= 0.00959, 0.05527, 0.17868.
- For $n = 100|\beta = 0.2,\ b = 0.6,\ a = 1.2$, the MLE method is the best with RMSE=0.00634, 0.03677, 0.12063.
- For $n = 300|\beta = 0.2,\ b = 0.6,\ a = 1.2$, the MLE method is the best with RMSE=0.00373, 0.02192, 0.06960.
- For $n = 500|\beta = 0.2,\ b = 0.6,\ a = 1.2$, the MLE method is the best with RMSE=0.00284, 0.01649, 0.05288.

Based on the results of Tables 1–3, the MLE method is considered for modeling the real-life data sets in the following Section.

4. Comparing models under complete samples

For comparing models under complete samples case, we provide three real data applications. These applications, also illustrate the importance, potentiality and flexibility of the EG-LL model in modeling real data. For these three data applications, we compare the EG-LL distribution with the Burr type XII (BXII), the Marshall-Olkin Burr XII (MOBXII), the Topp Leone Burr XII (TLBXII), the Zografos-Balakrishnan Burr XII (ZBBXII), Five Parameters beta Burr XII (FBBXII), Beta Burr XII (BBXII), Beta exponentiated Burr XII (BEBXII), Five Parameters Kumaraswamy Burr XII (FKw-BXII), Kumaraswamy Burr XII (KwBXII), exponential log-logistic (E-LL). and Weibull log-logistic (WLL) distributions given in Yousof *et al.* (2018), Altun *et al.* (2018 a, b) and Yousof *et al.* (2019).

Table 1: Simulation results for parameters $\beta = 0.9, b = 0.9$ and $a = 0.9$

		BIAS			RMSE			D	
	n	$\hat{\beta}$	$\hat{b}$	$\hat{a}$	$\hat{\beta}$	$\hat{b}$	$\hat{a}$	D_{abs}	D_{max}
MLE	50	0.00880	0.01193	0.01097	0.07860	0.09449	0.13275	0.00642	0.01195
CVM		0.00577	−0.01521	0.00963	0.10507	0.12259	0.15231	0.00503	0.00802
OLS		−0.00971	0.00007	−0.01301	0.09766	0.12525	0.14005	0.00744	0.01109
WLS		−0.00910	0.07468	−0.01814	0.08258	0.14968	0.13148	0.01323	0.02385
ADE		0.00697	−0.01483	0.01361	0.08885	0.10349	0.13690	0.00661	0.01117
ADE(R-T)		0.00581	0.00622	0.01150	0.08320	0.12107	0.13258	0.00537	0.01060
ADE(L-T)		0.01167	0.00197	0.01786	0.12321	0.16005	0.17147	0.00936	0.01593
MLE	100	0.00699	0.01029	0.00898	0.05537	0.06685	0.09235	0.00521	0.00982
CVM		0.00442	−0.00288	0.00663	0.07274	0.08295	0.10538	0.00357	0.00501
OLS		−0.00717	−0.00225	−0.00996	0.06986	0.08312	0.10041	0.00557	0.00871
WLS		−0.00589	0.04671	−0.01328	0.05798	0.08998	0.09585	0.00862	0.01566
ADE		0.00140	−0.00832	0.00356	0.06291	0.07201	0.09619	0.00182	0.00452
ADE(R-T)		0.00105	0.00171	0.00278	0.05917	0.08698	0.09372	0.00117	0.00327
ADE(L-T)		0.00606	0.00430	0.00870	0.08373	0.10625	0.11633	0.00471	0.00919
MLE	300	0.00188	0.00189	0.00290	0.03037	0.03696	0.05179	0.00152	0.00264
CVM		0.00235	−0.00202	0.00360	0.04187	0.04856	0.06095	0.00192	0.00269
OLS		−0.00170	−0.00296	−0.00209	0.04081	0.04869	0.05939	0.00128	0.00249
WLS		−0.00148	0.02383	−0.00488	0.03248	0.04915	0.05480	0.00416	0.00724
ADE		0.00057	−0.00512	0.00179	0.03611	0.04102	0.05595	0.00101	0.00288
RTADE		−0.00005	−0.00224	0.00068	0.03297	0.04572	0.05275	0.00040	0.00175
LEADE		0.00328	0.00036	0.00475	0.04861	0.06292	0.06783	0.00258	0.00504
MLE	500	0.00070	0.00237	0.00038	0.02306	0.02795	0.039190	0.00056	0.00115
CVM		−0.00047	0.00011	−0.00068	0.03107	0.03619	0.04501	0.00037	0.00054
OLS		−0.00122	−0.00053	−0.00171	0.03283	0.03623	0.04760	0.00094	0.00151
WLS		0.00026	0.01510	−0.00087	0.02519	0.03543	0.04236	0.00260	0.00408
ADE		−0.00078	−0.00027	−0.00108	0.02783	0.03188	0.04259	0.00060	0.00199
ADE(R-T)		−0.00049	0.00037	−0.00067	0.02594	0.03742	0.04109	0.00038	0.00158
ADE(L-T)		−0.00049	0.00278	−0.00064	0.03543	0.04626	0.04928	0.00052	0.00202

Data set **I:** called breaking stress data {0.98, 5.56, 5.08, 0.39, 1.57, 3.19, 4.90, 2.93, 2.85, 2.77, 2.76, 1.73, 2.48, 3.68, 1.08, 3.22, 3.75, 3.22, 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.40, 3.15, 2.67,3.31, 2.81, 2.56, 2.17, 4.91, 1.59, 1.18, 2.48, 2.03, 1.69, 2.43, 3.39, 3.56, 2.83, 3.68, 2.00, 3.51, 0.85, 1.61, 3.28, 2.95, 2.81, 3.15, 1.92, 1.84, 1.22, 2.17, 1.61, 2.12, 3.09, 2.97, 4.20, 2.35, 1.41, 1.59, 1.12, 1.69, 2.79, 1.89, 1.87, 3.39, 3.33, 2.55, 3.68, 3.19, 1.71, 1.25, 4.70, 2.88, 2.96, 2.55, 2.59, 2.97, 1.57, 2.17, 4.38, 2.03, 2.82, 2.53, 3.31, 2.38, 1.36, 0.81, 1.17, 1.84, 1.80, 2.05, 3.65}. This data set consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006).

Data set **II:** called survival times in days of 72 guinea pigs infected with the virulent tubercle bacilli {0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55}, this data was originally observed and reported by Bjerkedal (1960).

Data set **III** {65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43} called the leukaemia data. This real data set gives the survival times, in weeks, of 33 patients suffering from acute myelogenous leukemia. Those real data sets were recently analysed by Altun *et al.* (2018a), Altun *et al.* (2018b), Abouelmagd *et al.* (2019), Gad *et al.* (2019) and Elsayed and Yousof (2019, 2021). The total time test (TTT) plot (Aarset, 1987) is an important

Table 2: Simulation results for parameters $\beta = 1.2, b = 1.5$ and $a = 0.1$

		BIAS			RMSE			D	
	n	$\hat{\beta}$	$\hat{b}$	$\hat{a}$	$\hat{\beta}$	$\hat{b}$	$\hat{a}$	D_{abs}	$D_{\max}$
MLE	50	0.00834	0.01278	0.00193	0.05641	0.07782	0.01477	0.01299	0.02089
CVM		0.00530	0.01033	0.00159	0.07635	0.11647	0.01727	0.00984	0.01581
OLS		−0.00803	−0.01005	−0.00126	0.07577	0.11459	0.01697	0.01042	0.01696
WLS		−0.00743	0.00133	−0.00244	0.06222	0.08810	0.01536	0.00966	0.01450
ADE		0.00027	−0.00059	0.00085	0.06888	0.09799	0.01664	0.00209	0.00323
ADE(R-T)		−0.00023	0.00041	0.00057	0.06344	0.09141	0.01580	0.00143	0.00228
ADE(L-T)		0.00779	0.00793	0.0214	0.09101	0.12964	0.01953	0.01174	0.01867
MLE	100	0.00388	0.00623	0.00088	0.04007	0.05464	0.01062	0.00611	0.00987
CVM		0.00192	0.00403	0.00064	0.05382	0.08093	0.01214	0.00382	0.00614
OLS		−0.00396	−0.00469	−0.00066	0.05367	0.08076	0.01196	0.00516	0.00833
WLS		−0.00050	0.00761	−0.00067	0.04276	0.05937	0.01098	0.00088	0.00175
ADE		0.00303	0.00369	0.00110	0.04623	0.06623	0.01106	0.00541	0.00863
ADE(R-T)		−0.00046	−0.00053	0.00021	0.04165	0.05978	0.01037	0.00068	0.00118
ADE(L-T)		0.00285	0.00290	0.00084	0.06320	0.090240	0.01354	0.00447	0.00719
MLE	300	0.00184	0.00235	0.00052	0.02135	0.02924	0.00562	0.00298	0.00473
CVM		0.00204	0.00326	0.00053	0.02935	0.04389	0.00660	0.00337	0.00543
OLS		−0.00262	−0.00360	−0.00050	0.02897	0.04344	0.00645	0.00372	0.00602
WLS		−0.00224	0.00128	−0.00085	0.02272	0.03136	0.00588	0.00290	0.00428
ADE		0.00026	0.00023	0.00018	0.02610	0.03708	0.00625	0.00066	0.00113
RTADE		0.00032	0.00049	0.00015	0.02488	0.03581	0.00612	0.00035	0.00064
LEADE		0.00283	0.00315	0.00071	0.03420	0.04943	0.00731	0.00420	0.00679
MLE	500	−0.00006	0.00005	−0.000001	0.01690	0.02293	0.00446	0.00002	0.00003
CVM		−0.00004	0.00012	0.00004	0.02322	0.03464	0.00520	0.00011	0.00017
OLS		−0.00059	−0.00092	−0.00005	0.02302	0.0343	0.00516	0.00071	0.00119
WLS		0.00023	0.00337	−0.00014	0.01837	0.02535	0.00477	0.00083	0.00162
ADE		−0.00047	−0.00086	−0.00002	0.02001	0.02843	0.00477	0.00056	0.00107
ADE(R-T)		0.00005	0.00019	0.00011	0.01898	0.02727	0.00468	0.00014	0.00039
ADE(L-T)		0.00019	−0.00012	0.00011	0.02696	0.03887	0.00574	0.00033	0.00060

graphical approach to verify whether the data can be applied to a specific distribution or not. The TTT plots the three real data sets as presented in Figure 2. This plot indicates that the empirical HRFs of data sets **I**, **II** are increasing and are bathtub for data set **III**.

We consider the following goodness-of-fit statistics: the Akaike information criterion (C_1), the Bayesian information criterion (C_2), the consistent Akaike information criterion (C_3) and the Hannan-Quinn information criterion (C_4), where

$$\mathrm{C}_1 = -2\mathcal{L}\left(\underline{\hat{\mathbf{\Phi}}}\right) + 2m,\ \mathrm{C}_2 = -2\mathcal{L}\left(\underline{\hat{\mathbf{\Phi}}}\right) + m\log(n),$$
$$\mathrm{C}_3 = -2\mathcal{L}\left(\underline{\hat{\mathbf{\Phi}}}\right) + 2nm/(n-m-1)\ \text{and}\ \mathrm{C}_4 = -2\mathcal{L}\left(\underline{\hat{\mathbf{\Phi}}}\right) + 2m\log[\log(n)],$$

and where m is the number of parameters, n is the sample size, and $-2\mathcal{L}(\underline{\hat{\mathbf{\Phi}}})$ is the maximized log-likelihood. Generally, the smaller these statistics are, the better the fit. Tables 5, 7 and 9 give the MLEs, standard errors (SEs) and confidence interval (CIs) with for the data set **I**, **II** and **III**, respectively. Tables 6, 8 and 10 give the statistics C_1, C_2, C_3 and C_4 values for the data set **I**, **II** and **III**. Based on the values in Tables 6, 8 and 10 and Figures 3-7 the EG-LL model provide the best fit as compared to other competitive models in the four applications with small values for C_1, C_2, C_3 and C_4. Figure 8 presents profile likelihood plots (the first row is for data set **I**, the second row is for data set **II** and the third row is for data set **III**).

Table 3: Simulation results for parameters β=0.1, b=0.6 and a=1.2

		BIAS			RMSE			D	
	n	$\hat{\beta}$	$\hat{b}$	$\hat{a}$	$\hat{\beta}$	$\hat{b}$	$\hat{a}$	D_{abs}	D_{max}
MLE	50	0.00095	0.00459	0.01347	0.00959	0.05527	0.17868	0.00821	0.01257
CVM		0.00082	0.00496	0.01449	0.01257	0.07630	0.20491	0.00819	0.01254
OLS		−0.00178	−0.01037	−0.0276	0.01224	0.07409	0.19862	0.01716	0.02620
WLS		−0.00175	−0.00196	−0.03531	0.00987	0.05883	0.17895	0.01428	0.02137
ADE		0.00040	0.00091	0.01231	0.01081	0.06415	0.18822	0.00434	0.00837
ADE(R-T)		0.00033	0.00191	0.00992	0.01018	0.06138	0.18222	0.00412	0.00848
ADE(L-T)		0.00142	0.00347	0.02336	0.01459	0.08306	0.22839	0.01118	0.01899
MLE	100	0.00066	0.00331	0.01114	0.00634	0.03677	0.12063	0.00615	0.00940
CVM		0.00068	0.00391	0.01193	0.00857	0.05175	0.14044	0.00667	0.01021
OLS		−0.00064	−0.00366	−0.00927	0.00855	0.05189	0.13901	0.00590	0.00907
WLS		−0.00074	0.00239	−0.01627	0.00702	0.04178	0.12902	0.00464	0.00676
ADE		0.00008	−0.00012	0.00353	0.00752	0.04507	0.12900	0.00094	0.00315
ADE(R-T)		0.00037	0.00217	0.00817	0.00709	0.04274	0.12695	0.00400	0.00783
ADE(L-T)		0.00115	0.00390	0.01920	0.01012	0.05760	0.16014	0.00968	0.01648
MLE	300	0.00023	0.00116	0.00385	0.00373	0.02192	0.06960	0.00216	0.00330
CVM		0.00026	0.00146	0.00472	0.00480	0.02909	0.07838	0.00256	0.00392
OLS		−0.00014	−0.00077	−0.00192	0.00478	0.02890	0.07812	0.00124	0.00190
WLS		−0.00005	0.00442	−0.00309	0.00388	0.02271	0.07280	0.00149	0.00248
ADE		−0.00002	−0.00040	0.00049	0.00438	0.02616	0.07534	0.00018	0.00180
RTADE		0.00023	0.00137	0.00441	0.00419	0.02529	0.07420	0.00236	0.00506
LEADE		0.00048	0.00188	0.00811	0.00549	0.03225	0.08634	0.00420	0.00796
MLE	500	0.00004	0.00003	0.00059	0.00284	0.01649	0.05288	0.00027	0.00041
CVM		0.00009	0.00052	0.00161	0.00364	0.02206	0.05951	0.00089	0.00135
OLS		−0.00019	−0.00113	−0.00296	0.00387	0.02342	0.06323	0.00183	0.00280
WLS		−0.00002	0.00330	−0.00221	0.00295	0.01744	0.05596	0.00119	0.00197
ADE		0.00001	−0.00007	0.00064	0.00332	0.01988	0.05722	0.00013	0.00158
ADE(R-T)		−0.00016	−0.00098	− 0.00234	0.00308	0.01855	0.05442	0.00151	0.00367
ADE(L-T)		0.00016	0.00038	0.00265	0.00419	0.02445	0.06600	0.00126	0.00332

5. Censored MLE

Suppose that $Z_1, Z_2, \ldots, Z_n$ is a random sample with right censoring from EG-LL($\underline{\boldsymbol{\Phi}}$) distribution where

$$z_\varsigma = \min(Z_\varsigma, C_\varsigma)|_{(\varsigma=1,2,\ldots,n)},$$

is the minimum of the survival time Z_ς and the censoring time C_ς for each subject in the sample. So, z_ς can be written in the form $(z_\varsigma, \nabla_\varsigma)_{\varsigma=1,\ldots,n}$, where $\nabla_\varsigma = 1$ if Z_ς is the moment of failure (complete observation) and $\nabla_\varsigma = 0$ if Z_ς is the moment of censoring. The right censoring is assumed to be non-informative, so the expression of the likelihood function is

$$(z, \underline{\boldsymbol{\Phi}}) = \prod_{\varsigma=1}^{n} \left[h_{\underline{\boldsymbol{\Phi}}}(z_\varsigma)\right]^{\nabla_\varsigma} S_{\underline{\boldsymbol{\Phi}}}(z_\varsigma)|_{(\nabla_\varsigma = 1_{Z_\varsigma < C_\varsigma})},$$

where $h_{\underline{\boldsymbol{\Phi}}}(z_\varsigma)$ refers to the HRF and $S_{\underline{\boldsymbol{\Phi}}}(z_\varsigma)$ refers to the survival function (SF). The log-likelihood function of EG-LL($\underline{\boldsymbol{\Phi}}$) distribution is

$$\mathcal{L}_n(\underline{\boldsymbol{\Phi}}) = \sum_{\varsigma=1}^{n} \nabla_\varsigma \left[\ln(\beta ab) + (b-1)\ln z_\varsigma + (\beta-1)\ln\left(1+z_\varsigma^b\right)\right] - a\sum_{\varsigma=1}^{n}\left[\left(1+z_\varsigma^b\right)^\beta - 1\right],$$

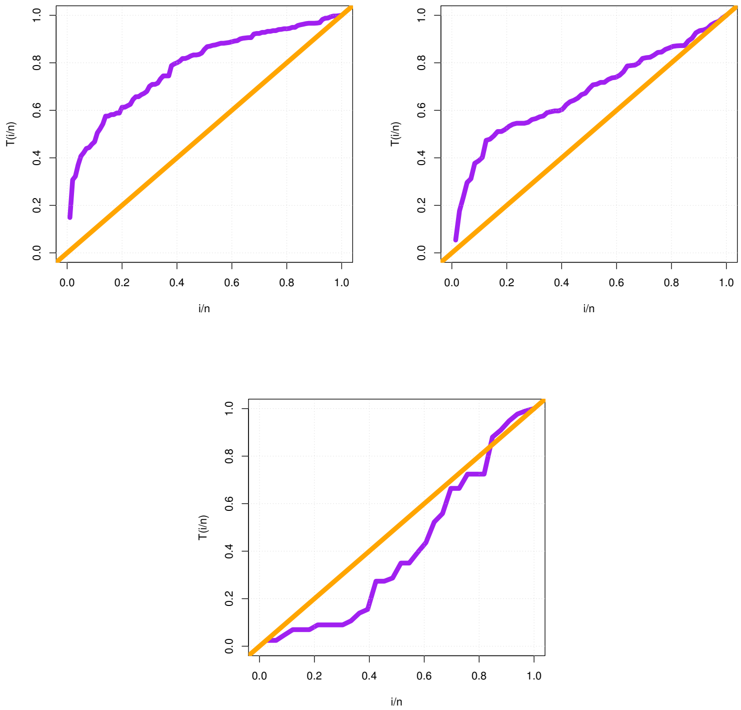

Figure 2: *TTT plots.*

and the score functions are obtained as follows

$$
\begin{aligned}
\frac{\partial \mathcal{L}_n(\underline{\mathbf{\Phi}})}{\partial \beta} &= \sum_{\varsigma=1}^{n} \nabla_{\varsigma} \left[\frac{1}{\beta} + \ln\left(1 + z_{\varsigma}^{b}\right) \right] - a \sum_{\varsigma=1}^{n} \left(1 + z_{\varsigma}^{b}\right)^{\beta} \ln\left(1 + z_{\varsigma}^{b}\right), \\
\frac{\partial \mathcal{L}_n(\underline{\mathbf{\Phi}})}{\partial b} &= \sum_{\varsigma=1}^{n} \nabla_{\varsigma} \left[\frac{1}{b} + \frac{\left(1 + \beta z_{\varsigma}^{b}\right) \ln z_{\varsigma}}{1 + z_{\varsigma}^{b}} \right] - a\beta \sum_{\varsigma=1}^{n} z_{\varsigma}^{b} \ln z_{\varsigma} \left(1 + z_{\varsigma}^{b}\right)^{\beta - 1}, \\
\frac{\partial \mathcal{L}_n(\underline{\mathbf{\Phi}})}{\partial a} &= \sum_{\varsigma=1}^{n} \frac{\nabla_{\varsigma}}{a} - \sum_{\varsigma=1}^{n} \left[\left(1 + z_{\varsigma}^{b}\right)^{\beta} - 1 \right],
\end{aligned}
$$

Maximum likelihood estimators of the unknown parameters can be obtained using various techniques, such as software R, EM algorithm or Newton Raphson methods. In this section we conduct an important simulation study to consolidate our results. For this, $10,000$ censored samples (with sizes: $n_1 = 15, n_2 = 25, n_3 = 50, n_4 = 130, n_5 = 350, n_6 = 500, n_7 = 1000$) from EG-LL($\underline{\mathbf{\Phi}}$) distribution are simulated. Simulated samples were generated with various parameters. Using R software and BB algorithm, the average values of the estimaotrs and their corresponding squred mean errors are given in Table 11. As shown in these results, the maximum likelihood estimators are convergent.

Table 4: MLEs, SEs and CIs for the data set **I**

Model		Estimates
BXII(α,β)	MLEs	5.941, 0.187
	SEs	(1.279) , (0.044)
	CIs	(3.43,8.45), (0.10,0.27)
MOBXII (α,β,γ)	MLEs	1.192, 4.834, 838.73
	SEs	(0.952), (4.896), (229.34)
	CIs	(0, 3.06), (0, 14.43), (389.22,1288.24)
TLBXII(α,β,γ)	MLEs	1.350, 1.061, 13.728
	SEs	(0.378), (0.384), (8.400)
	CIs	(0.61, 2.09), (0.31,1.81), (0, 30.19)
KwBXII ($\lambda,\theta,\alpha,$)	MLEs	48.103 ,79.516 ,0.351 ,2.730
	SEs	(19.348), (58.186), (0.098), (1.077)
	CIs	(10.18,86.03), (0,193.56), (0.16,0.54), (0.62,4.84)
BBXII($\lambda,\theta,\alpha,$)	MLEs	359.683, 260.097, 0.175, 1.123
	SEs	(57.941), (132.213), (0.013), (0.243)
	CIs	(246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6)
BEBXII($\lambda,\theta,\alpha,,\gamma$)	MLEs	0.381, 11.949, 0.937, 33.402, 1.705
	SEs	(0.078), (4.635), (0.267), (6.287),(0.478)
	CIs	(0.23,0.53) ,(2.86,21), (0.41,1.5), (21,45), (0.8,2.6)
FBBXII($\lambda,\theta,\alpha,\beta,\gamma$)	MLEs	0.421, 0.834, 6.111, 1.674, 3.450
	SEs	(0.011), (0.943), (2.314), (0.226), (1.957)
	CIs	(0.4,0.44), (0. 2.7), (1.57, 10.7), (1.23, 2.1), (0, 7)
FKwBXII($\lambda,\theta,\alpha,\beta,\gamma$)	MLEs	0.542, 4.223, 5.313, 0.411, 4.152
	SEs	(0.137), (1.882), (2.318), (0.497), (1.995)
	CIs	(0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)
ZBBXII(λ,α,β)	MLEs	123.101, 0.368, 139.247
	SEs	(243.011), (0.343), (318.546)
	CIs	(0, 599.40), (0, 1.04), (0, 763.59)
LL(a)	MLEs	1.629
	SEs	(0.1288)
	CIs	(1.36, 1.84)
E-LL	MLEs	5.221, 2.319
	SEs	(0.5983), (0.1496)
	CIs	(4.2, 6.2), (2, 2.6)
WLL(β,a)	MLEs	1.116, 0.71
	SEs	(6.5), (4.12)
	CIs	(0, 16.1), (0, 8.94)
EG-LL(β,b,a)	MLEs	**0.8076, 3.314, 0.0584**
	SEs	**(0.3504), (1.193), (0.025)**
	CIs	**(0.1, 1.5), (0.9, 5.7), (0.001, 0.11)**

6. Modified Bagdonavičius-Nikulin validation test

In this work, we are interested in the modified chi-squared type test proposed by Bagdonavičius and Nikulin (2011), Bagdonavičius *et al.* (2013) for parametric models with right censored data. Based on the maximum likelihood estimators of the non-grouped data, this test statistic is also based on the differences between the numbers of observed failures and the numbers of expected failures in the grouped intervals chosen. Thus, random grouping intervals are considered as data functions. The description of the construction for this chi-squared type test is developed in Voinov *et al.* (2013). The test statistic is defined as follows. Suppose that $Z_1, Z_2, \ldots, Z_n$ is a random sample with right censoring

Table 5: C_1, C_2, C_3 and C_4 values for the data set **I**

Model	C_1, C_2, C_3, C_4
BXII	382.94, 388.15, 383.06, 385.05
MOBXII	305.78, 313.61, 306.03, 308.96
TLBXII	323.52, 331.35, 323.77, 326.70
KwBXII	303.76, 314.20, 304.18, 308.00
BBXII	305.64, 316.06, 306.06, 309.85
BEBXII	305.82, 318.84, 306.46, 311.09
FBBXII	304.26, 317.31, 304.89, 309.56
FKwBXII	305.50, 318.55, 306.14, 310.80
ZBBXII	302.96, 310.78, 303.21, 306.13
LL	469.63, 472.23, 469.67, 470.68
E-LL	325.93, 331.14, 326.06, 328.04
WLL	510.69, 515.91, 510.82, 512.84
EG-LL	**288.84, 296.66, 289.09, 292.00**

Table 6: MLEs, SEs and CIs for the data set **II**

Model		Estimates
	MLEs	3.102, 0.465,
BXII(α,β)	SEs	(0.538), (0.077)
	CIs	(2.05,4.16), (0.31,0.62)
	MLEs	2.259,1.533, 6.760
MOBXII(α,β,γ)	SEs	(0.864), (0.907), (4.587)
	CIs	(0.57,3.95), (0,3.31), (0, 15.75)
	MLEs	2.393,0.458,1.796
TLBXII(α,β,γ)	SEs	(0.907), (0.244),(0.915)
	CIs	(0.62,4.17),(0, 0.94),(0.002,3.59)
	MLEs	14.105,7.424, 0.525, 2.274
KwBXII ($\lambda,\theta,\alpha,\beta$)	SEs	(10.805), (11.850), (0.279),(0.990)
	CIs	(0, 35.28), (0.30.65), (0, 1.07),(0.33, 4.21)
	MLEs	2.555, 6.058,1.800,0.294,
BBXII($\lambda,\theta,\alpha,\beta$)	SEs	(1.859), (10.391), (0.955),(0.466)
	CIs	(0, 6.28), (0, 26.42), (0, 3.67),(0, 1.21)
	MLEs	1.876,2.991, 1.780, 1.341, 0.572
BEBXII($\lambda,\theta,\alpha,\beta,\gamma$)	SEs	(0.094), (1.731), (0.702), (0.816), (0.325)
	CIs	(1.7,2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)
	MLEs	0.621, 0.549,3.838, 1.381, 1.665
FBBXII($\lambda,\theta,\alpha,\beta,\gamma$)	SEs	(0.541), (1.011), (2.785), (2.312), (0.436)
	CIs	(0, 1.7), (0, 2.5), (0, 9.3), (0, 5.9), (0.8, 4.5)
	MLEs	0.558, 0.308, 3.999, 2.131, 1.475
FKwBXII($\lambda,\theta,\alpha,\beta,\gamma$)	SEs	(0.442), (0.314), (2.082), (1.833), (0.361)
	CIs	(0, 1.4), (0, 0.9), (0, 3.1), (0, 5.7), (0.76, 2.2)
	MLEs	2.275
LL(a)	SEs	(0.223)
	CIs	(1.9, 2.7)
	MLEs	1.951, 2.25
E-LL	SEs	(0.229), (0.2069)
	CIs	(1.5, 2.3), (2.1, 2.9)
	MLEs	0.785, 1.254
WLL(β,a)	SEs	(0.00), (0.00)
	CIs	–, –
	MLEs	**0.3706, 2.9555, 0.9115,**
EG-LL(β,b,a)	SEs	**(0.1376), (0.581), (0.6984)**
	CIs	**(0.14, 0.66), (1.7, 4.1), (0, 2.3)**

Table 7: C_1, C_2, C_3 and C_4 values for the data set **II**

Model	C_1, C_2, C_3, C_4
BXII	209.60, 214.15, 209.77, 211.40
MOBXII	209.74, 216.56, 210.09, 212.44
TLBXII	211.80, 218.63, 212.15, 214.52
KwBXII	208.76, 217.86, 209.36, 212.38
BBXII	210.44, 219.54, 211.03, 214.06
BEBXII	212.10, 223.50, 213.00, 216.60
FBBXII	206.80, 218.20, 207.71, 211.30
FKwBXII	206.50, 217.90, 207.41, 211.00
LL	231.85, 234.13, 231.91, 232.76
E-LL	207.83, 212.38, 208.00, 209.64
WLL	270.44, 274.98, 270.60, 272.23
EG-LL	**203.89, 210.72, 204.24, 206.61**

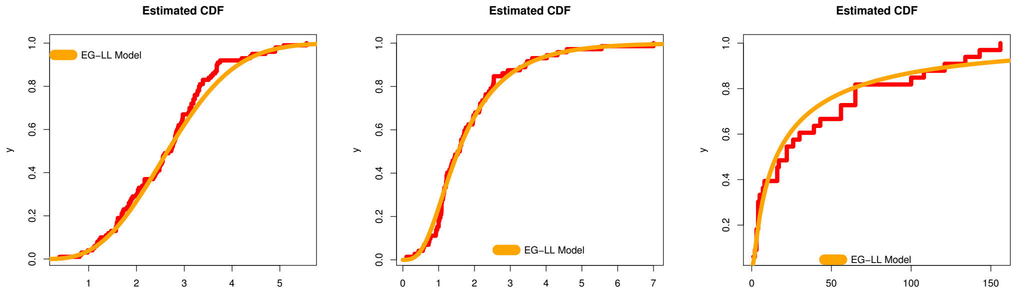

Figure 3: *The estimated CDFs.*

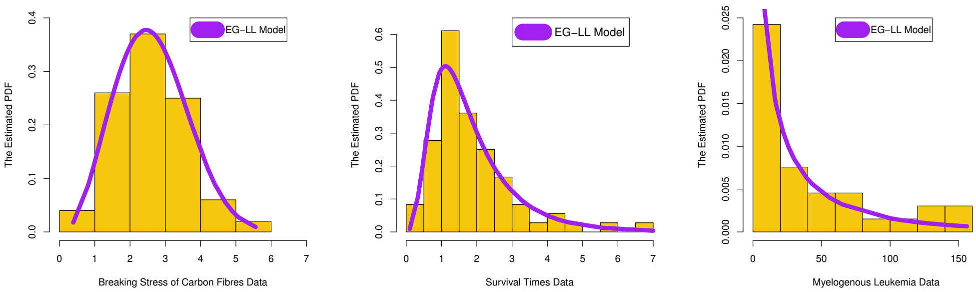

Figure 4: *The estimated PDFs.*

from a parametric model and a finite time τ. The test statistic is defined as

$$Y_n^2 = \sum_{j=1}^{n} \frac{\left(\mathbf{U}_j - e_j\right)^2}{\mathbf{U}_j} + \mathbf{Q},$$

Table 8: MLEs, SEs and CIs for the data set **III**

Model		Estimates
BXII(α,β)	MLEs	58.711,0.006
	SEs	(42.382), (0.004)
	CIs	(0, 141.78), (0, 0.01)
MOBXII(α,β,γ)	MLEs	11.838, 0.078, 12.251
	SEs	(4.368), (0.013), (7.770)
	CIs	(0, 141.78), (0, 0.01), (0, 27.48)
TLBXII(α,β,γ)	MLEs	0.281, 1.882 ,50.215
	SEs	(0.288), (2.402), (176.50)
	CIs	(0, 0.85), (0, 6.59), (0, 396.16)
KwBXII($\lambda,\theta,\alpha,\beta$)	MLEs	9.201, 36.428, 0.242,0.941
	SEs	(10.060), (35.650), (0.167), (1.045)
	CIs	(0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99)
BBXII($\lambda,\theta,\alpha,\beta$)	MLEs	96.104, 52.121, 0.104, 1.227
	SEs	(41.201), (33.490), (0.023), (0.326)
	CIs	(15.4,176.8),(0, 117.8), (0.6, 0.15), (0.59,1.9)
BEBXII($\lambda,\theta,\alpha,\beta,\gamma$)	MLEs	0.087, 5.007, 1.561, 31.270, 0.318
	SEs	(0.077), (3.851), (0.012), (12.940), (0.034)
	CIs	(0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)
FBBXII($\lambda,\theta,\alpha,\beta,\gamma$)	MLEs	15.194, 32.048, 0.233, 0.581, 21.855
	SEs	(11.58), (9.867), (0.091), (0.067), (35.548)
	CIs	(0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)
FKwBXII($\lambda,\theta,\alpha,\beta,\gamma$)	MLEs	14.732, 15.285, 0.293, 0.839, 0.034
	SEs	(12.390), (18.868), (0.215), (0.854), (0.075)
	CIs	(0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)
ZBBXII($\lambda,\alpha,\beta,\gamma$)	MLEs	41.973,0.157, 44.263
	SEs	(38.787),(0.082), (47.648)
	CIs	(0, 117.99),(0, 0.32,) (0, 137.65)
LL(a)	MLEs	0.507
	SEs	(0.071)
	CIs	(0.36, 0.64)
E-LL	MLEs	5.594, 0.7647
	SEs	(1.1834), (0.093)
	CIs	(3.1, 7.9), (0.52, 0.88)
WLL(β,a)	MLEs	1.069, 0.224
	SEs	(0.00), (0.00)
	CIs	–,–
EG-LL(β,b,a)	MLEs	**0.1269, 4.78, 0.141**
	SEs	**(0.046), (1.32), (0.090)**
	CIs	**(0.02, 0.22), (2.16, 7.44), (0, 0.32)**

where $\mathbf{U}_j$ and e_j are the observed and the expected numbers of failure in grouping intervals, and $\mathbf{Q}$ is

$$\mathbf{Q} = \mathbf{W}^T\hat{\mathbf{G}}^{-}\mathbf{W},$$

$$\hat{\mathbf{W}} = \hat{\mathbf{C}}\hat{\mathbf{A}}^{-1}\mathbf{T} = \left(\hat{\mathbf{W}}_1, \ldots, \hat{\mathbf{W}}_s\right)^T,$$

$$\mathbf{T}_j = \frac{1}{\sqrt{n}}\left(\mathbf{U}_j - e_j\right)$$

$$\mathbf{W} = \sum_{j=1}^{r} \hat{\mathbf{C}}_j\hat{\mathbf{A}}_j^{-1}\mathbf{T}_j,$$

$$\hat{\mathbf{G}} = [\hat{g}_{\imath}]_{s\times s},$$

$$\hat{g}_{\imath} = \hat{\imath}_{\imath} - \sum_{j=1}^{r} \hat{\mathbf{C}}_j\hat{\mathbf{C}}_{\imath j}\hat{\mathbf{A}}_j^{-1}|_{(\varsigma=1,\ldots,n,\ j=1,\ldots,r \text{ and } \imath=1,\ldots,s)}.$$

Table 9: C_1, C_2, C_3 and C_4 values for the data set **III**

Model	C_1, C_2, C_3, C_4
BXII	328.20, 331.19, 328.60, 329.19
MOBXII	315.54, 320.01, 316.37, 317.04
TLBXII	316.26, 320.73, 317.09, 317.76
KwBXII	317.36, 323.30, 318.79, 319.34
BBXII	316.46, 322.45, 317.89, 318.47
BEBXII	317.58, 325.06, 319.80, 320.09
FBBXII	317.86, 325.34, 320.08, 320.36
FKwBXII	317.76, 325.21, 319.98, 320.26
ZBBXII	313.86, 318.35, 314.39, 315.36
WLL	378.82, 381.79, 379.27, 379.84
LL	362.71, 364.20, 362.83, 363.21
E-LL	315.46, 318.45, 315.85, 316.47
EG-LL	**309.84, 314.33, 310.67, 311.35**

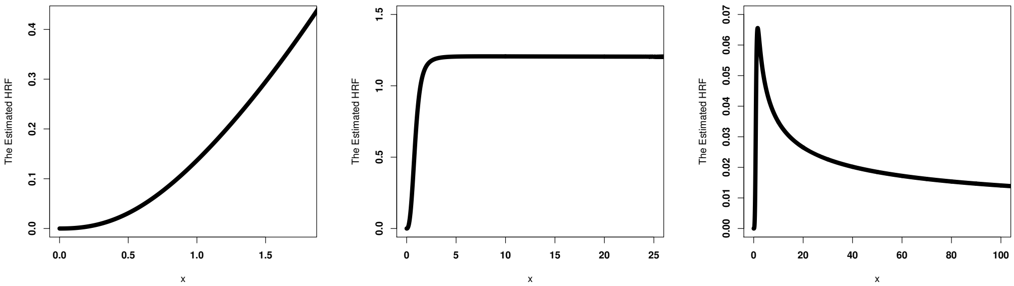

Figure 5: *The estimated HRFs.*

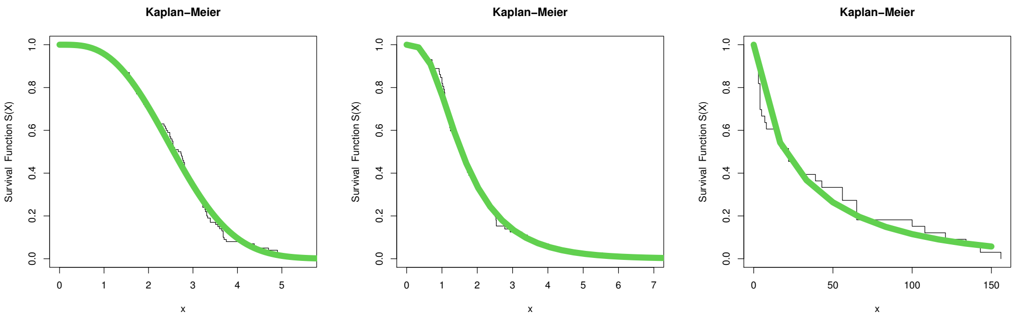

Figure 6: *The Kaplan-Meier survival plots.*

The limits ρ_j of r random grouping intervals $\varsigma_j = [\rho_{j-1}, \rho_j[$ are chosen so that the expected failure times falling into these intervals are the same for each $j|_{(j=1,\ldots,r-1)}$ and

$$\hat{\rho}_r = \max\left(z_{()}, \tau\right).$$

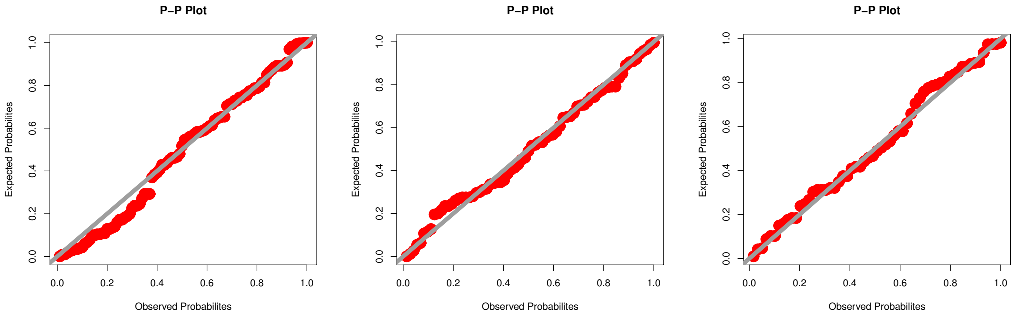

Figure 7: *The PP plots.*

Table 10: MLEs for $\hat{\beta}, \hat{a}, \hat{b}$ and their MSEs

$N = 10.000$	$\hat{\beta} = 2$	MSE	$\hat{a} = 0.8$	MSE	$\hat{b} = 2.5$	MSE
$n_1 = 15$	1.9623	0.0072	0.7734	0.0068	2.5163	0.0086
$n_2 = 25$	1.9684	0.0067	0.7796	0.0058	2.5142	0.0072
$n_3 = 50$	1.9736	0.0046	0.7812	0.0049	2.5106	0.0059
$n_4 = 130$	1.9794	0.0038	0.7876	0.0032	2.5094	0.0042
$n_5 = 350$	1.9878	0.0027	0.7912	0.0024	2.5057	0.0034
$n_6 = 500$	1.9945	0.0019	0.7998	0.0015	2.5023	0.0027
$n_7 = 1000$	1.9992	0.0012	0.8006	0.0009	2.5007	0.0015
	$\hat{\beta} = 1$	MSE	$\hat{a} = 1.5$	MSE	$\hat{b} = 1.3$	MSE
$n_1 = 15$	1.0463	0.0086	1.5236	0.0092	1.2624	0.0079
$n_2 = 25$	1.0425	0.0073	1.5194	0.0083	1.2694	0.0068
$n_3 = 50$	1.0362	0.0065	1.5137	0.0071	1.2732	0.0057
$n_4 = 130$	1.0247	0.0052	1.5092	0.0062	1.2876	0.0043
$n_5 = 350$	1.0123	0.0037	1.5067	0.0048	1.2943	0.0032
$n_6 = 500$	1.0067	0.0022	1.5034	0.0036	1.2976	0.0017
$n_7 = 1000$	1.0011	0.0010	1.5002	0.0012	1.3006	0.0008
	$\hat{\beta} = 0.7$	MSE	$\hat{a} = 3$	MSE	$\hat{b} = 0.6$	MSE
$n_1 = 15$	0.7234	0.0098	2.9516	0.0103	0.5764	0.0109
$n_2 = 25$	0.7189	0.0079	2.9634	0.0089	0.5823	0.0092
$n_3 = 50$	0.7126	0.0061	2.9718	0.0077	0.5886	0.0078
$n_4 = 130$	0.7079	0.0047	2.9834	0.0051	0.5916	0.0059
$n_5 = 350$	0.7065	0.0034	2.9927	0.0034	0.5943	0.0043
$n_6 = 500$	0.7027	0.0023	2.9976	0.0027	0.5986	0.0028
$n_7 = 1000$	0.7009	0.0010	3.0002	0.0018	0.6004	0.0013

The estimated $\hat{\rho}_j$ is defined by

$$\hat{\rho}_j = H^{-1}\left(\frac{E_j - \sum_{=1}^{\varsigma-1} H_{\underline{\Phi}}(z)}{n-\varsigma+1}, \underline{\Phi}\right)|_{(\hat{\rho}_k=\max(z_{(n)},\tau))}$$

where $H_{\underline{\Phi}}(z)$ is the cumulative hazard function (CHRF) of the model distribution. This test statistic Y_n^2 follows a chi-squared distribution. For more details see Goual *et al.* (2019), Ibrahim *et al.* (2019, 2020), Yadav *et al.* (2020), Mansour *et al.* (2020a,b), Salah *et al.* (2020), Goual and Yousof (2020) and Yousof *et al.* (2021).

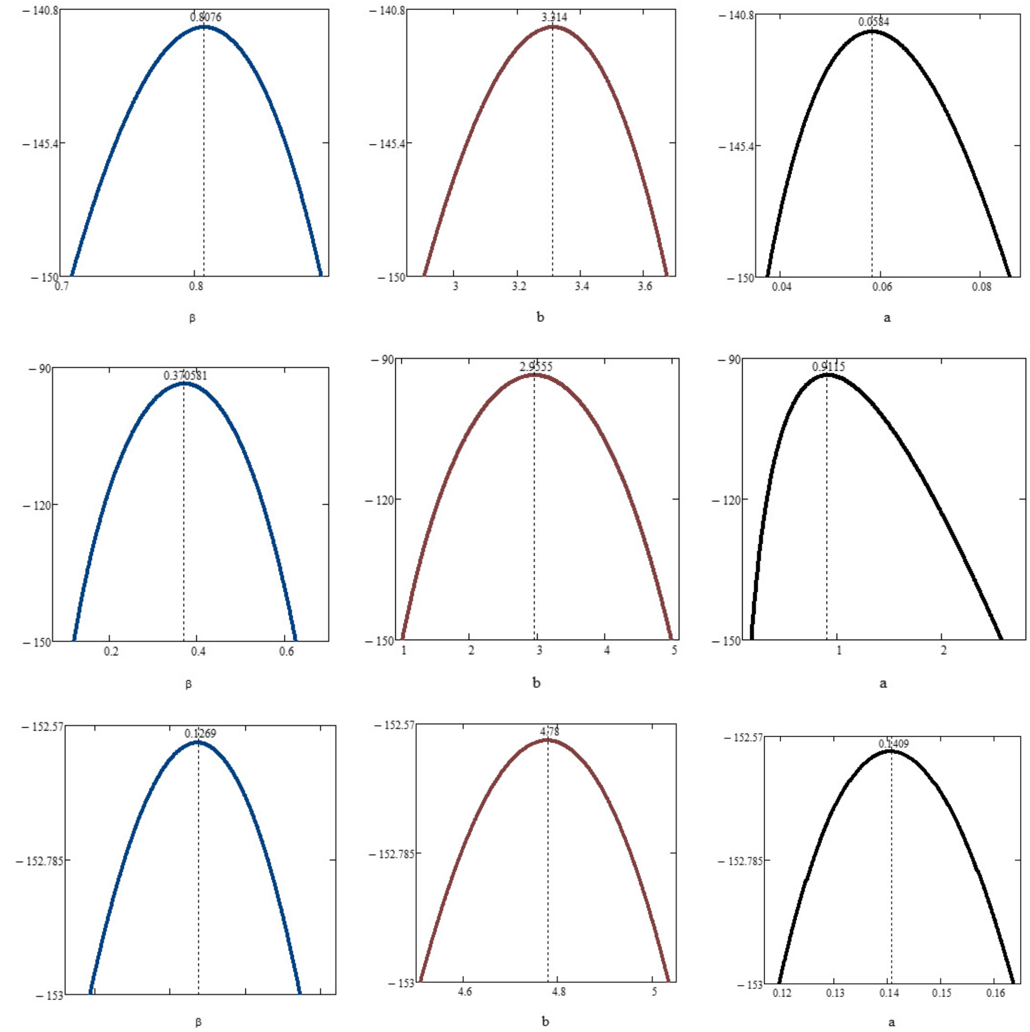

Figure 8: *Profile likelihood plots.*

6.1. Choice of random grouping intervals

Suppose that $Z_1, Z_2, \ldots, Z_n$ is a random sample with right censoring from EG-LL($\underline{\mathbf{\Phi}}$) distribution and a finite time τ . In our case, the estimated $\hat{\rho}_j$ is obtained as follows

$$\hat{\rho}_j = \left[\left(\frac{E_j - \sum_{=1}^{\varsigma-1} H_{\underline{\mathbf{\Phi}}}(z)}{\hat{a}\,(n - \varsigma + 1)} + 1 \right)^{1/\hat{\beta}} - 1 \right]^{1/\hat{b}},$$

where $\underline{\mathbf{\Phi}} = (\hat{\beta}, \hat{a}, \hat{b})^T$ are the maximum likelihood estimators of the unknown parameters $\underline{\mathbf{\Phi}} = (\beta, a, b)^T$ of the initial data, and $H_{\underline{\mathbf{\Phi}}}(z)$ is the cumulative HRF of the EG-LL($\underline{\mathbf{\Phi}}$) distribution.

6.2. Quadratic form $\mathbf{Q}$

To calculate the quadratic form $\mathbf{Q}$ of the statistic Y_n^2, and since its distribution does not depend on the parameters, we can use the estimated matrices $\hat{\mathbf{W}}$, $\hat{\mathbf{C}}$ and the estimated information matrix $\hat{\mathbf{I}}$. The elements of $\hat{\mathbf{C}}$ are

$$\hat{\mathbf{C}}_j = \frac{1}{n} \sum_{\varsigma: z_\varsigma \in \varsigma_j}^{n} \nabla_\varsigma \frac{\partial}{\partial \underline{\mathbf{\Phi}}} \ln h_{\underline{\mathbf{\Phi}}}(z_\varsigma)$$

and are obtained as shown below

$$\hat{\mathbf{C}}_{1j} = \frac{1}{n} \sum_{\varsigma: z_\varsigma \in \varsigma_j}^{n} \nabla_\varsigma \left[\frac{1}{\beta} + \ln\left(1 + z_\varsigma^b\right) \right],$$

$$\hat{\mathbf{C}}_{2j} = \frac{1}{n} \sum_{\varsigma: z_\varsigma \in \varsigma_j}^{n} \nabla_\varsigma \left[\frac{1}{b} + \frac{\left(1 + \beta z_\varsigma^b\right) \ln z_\varsigma}{1 + z_\varsigma^b} \right],$$

$$\hat{\mathbf{C}}_{3j} = \frac{1}{n} \sum_{\varsigma: z_\varsigma \in \varsigma_j}^{n} \frac{\nabla_\varsigma}{a}.$$

Therefore, the estimated matrix $\hat{\mathbf{W}}$ can be deduced from $\hat{\mathbf{C}}$.

6.3. Estimated information matrix $\hat{\mathbf{I}}$

We also need the information matrix $\hat{\mathbf{I}}$ of the EG-LL($\underline{\mathbf{\Phi}}$) distribution with right censoring. The elements of the matrix can be obtained as follows,

$$\hat{\imath}_{11} = \frac{1}{n} \sum_{\varsigma=1}^{n} \nabla_\varsigma \left(\frac{1}{\beta} + \ln\left(1 + z_\varsigma^b\right) \right)^2,$$

$$\hat{\imath}_{22} = \frac{1}{n} \sum_{\varsigma=1}^{n} \nabla_\varsigma \left(\frac{1}{b} + \frac{\left(1 + \beta z_\varsigma^b\right) \ln z_\varsigma}{1 + z_\varsigma^b} \right)^2,$$

$$\hat{\imath}_{33} = \frac{1}{n} \sum_{\varsigma=1}^{n} \frac{\nabla_\varsigma}{a^2},$$

$$\hat{\imath}_{12} = \frac{1}{n} \sum_{\varsigma=1}^{n} \nabla_\varsigma \left(\frac{1}{\beta} + \ln\left(1 + z_\varsigma^b\right) \right) \left(\frac{1}{b} + \frac{\left(1 + \beta z_\varsigma^b\right) \ln z_\varsigma}{1 + z_\varsigma^b} \right),$$

$$\hat{\imath}_{13} = \frac{1}{n} \sum_{\varsigma=1}^{n} \frac{\nabla_\varsigma}{a} \left(\frac{1}{\beta} + \ln\left(1 + z_\varsigma^b\right) \right),$$

$$\hat{\imath}_{23} = \frac{1}{n} \sum_{\varsigma=1}^{n} \frac{\nabla_\varsigma}{a} \left(\frac{1}{b} + \frac{\left(1 + \beta z_\varsigma^b\right) \ln z_\varsigma}{1 + z_\varsigma^b} \right)$$

As all the components of the statistic are given explicitly, then the test statistic for the EG-LL($\underline{\mathbf{\Phi}}$) distribution with unknown parameters and right censored data are obtained. This statistic follows a

Table 11: Simulated levels of significance for $Y_n^2(\underline{\boldsymbol{\Phi}})$ test under the EG-LL model against their theoretical values(ε 0.01, 0.05, 0.10)

$N = 10,000$	$n = 15$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$	$n = 1000$
$\varepsilon = 1\%$	0.0065	0.0078	0.0082	0.0085	0.0092	0.0098	0.0103
$\varepsilon = 5\%$	0.0413	0.0445	0.0462	0.0475	0.0481	0.0497	0.0502
$\varepsilon = 10\%$	0.0937	0.0959	0.0965	0.0979	0.0983	0.0998	0.1004

chi-squared distribution with r degrees of freedom.

$$Y_n^2(\underline{\boldsymbol{\Phi}}) = \sum_{j=1}^{r} \frac{\left(\mathbf{U}_j - e_j\right)^2}{\mathbf{U}_j} + \hat{\mathbf{W}}^T \mathbf{M}^{-1} \hat{\mathbf{W}},$$

where

$$\mathbf{M}^{-1} = \left(\hat{\imath}' - \sum_{j=1}^{r} \hat{\mathbf{C}}_j \hat{\mathbf{C}}'_j \hat{\mathbf{A}}_j^{-1}\right)^{-1}.$$

6.4. Simulated samples (right censored case)

For testing the null hypothesis $\mathbb{H}_0$ right censored data that arises from the EG-LL model, we compute the criteria statistic $Y_n^2(\underline{\boldsymbol{\Phi}})$ as defined above for 10,000 simulated samples from the hypothesized distribution with different sizes($n = 15, 25, 50, 130, 350, 500, 1000$). Then, we calculate empirical levels of significance, when $Y > \chi^2_\varepsilon(r)$ correponds to theoretical levels of significance($\varepsilon = 0.10, 0.05, 0.01$), and where $r = 5$. The results are reported in Table 11. The null hypothesis $\mathbb{H}_0$ for which simulated samples are fitted by the EG-LL distribution, is widely validated for the different levels of significance. Therefore, the test proposed in this work, can be used to fit data from this new distribution.

7. Data analysis (right censored case)

We have analyzed a lymphoma data set consisting of times (in months) from diagnosis to death for 31 individuals with advanced non Hodgkin's lymphoma clinical symptoms, using our model. This data has been analyzed by Gijbels and Gurler (2003) using the exponential change point model. Among these 31 observations, 11 of the times are censored, because the patients were alive at the last time of follow-up.

2.5, 4.1, 4.6, 6.4, 6.7, 7.4, 7.6, 7.7, 7.8, 8.8, 13.3, 13.4, 18.3, 19.7, 21.9, 24.7, 27.5, 29.7, 30.1*, 32.9, 33.5, 35.4*, 37.7*, 40.9*, 42.6*, 45.4*, 48.5*, 48.9*, 60.4*, 64.4*, 66.4*, where * denotes a censored observation. We use the test statistic provided above to verify if data is modeled by EG-LL distribution and to that end, we first calculate the maximum likelihood estimators of the unknown parameters

$$\underline{\boldsymbol{\Phi}} = \left(\hat{\beta}, \hat{b}, \hat{a}\right)^T = (0.8469, 1.486, 2.9568)^T.$$

Data are grouped into $r = 5$ intervals ς_j. We provide the necessary calculus in the following Table 12. Then, we obtain the value of the test statistic Y_n^2,

$$Y_n^2 = X^2 + \mathbf{Q} = 4.9563 + 3.8496 = 8.8059.$$

Table 12: values of $\hat{\rho}_j, e_j, \mathbf{U}_j, \hat{\mathbf{C}}_{1j}, \hat{\mathbf{C}}_{2j}$ and $\hat{\mathbf{C}}_{3j}$

$\hat{\rho}_j$	7.2	8.5	15.9	31.5	66.4
$\mathbf{U}_J$	5	4	3	7	12
$\hat{\mathbf{C}}_{1j}$	1.3648	1.6342	1.2131	0.9362	0.8376
$\hat{\mathbf{C}}_{2j}$	0.9881	0.8734	0.7823	1.0234	0.9236
$\hat{\mathbf{C}}_{3j}$	1.6910	1.3528	1.0146	2.0292	0.6764
e_j	2.1836	2.1836	2.1836	2.1836	2.1836

Table 13: Y_n^2 for all models

Model	Y_n^2
EG-LL	**8.8059**
MOBXII	9.1748
TLBXII	9.3021
KwBXII	9.4536
BEBXII	9.6748
BBXII	9.3214
FKwBXII	9.8695
ZBBXII	9.0345
WLL	11.054
LL	10.956
E-LL	9.1356
FBBXII	10.075
BXII	10.265

For significance level $\varepsilon = 0.05$, the critical value $\chi_5^2 = 11.0705$ is greater than the value of $Y_n^2 = 8.8059$. Thus, we can say that the proposed model EG-LL fit the data. We also calculated the test statistic Y_n^2 to fit the data to the competing models. The results are given in Table 14. Table 14 shows that the EG-LL has the lowest value of Y_n^2 among all the other competitive models.

8. Conclusions

In this paper, we extended the log-logistic (LL) model by proposing and studying a new probability distribution called the exponential generalized log-logistic (EG-LL) model. Different non-Bayesian estimation methods under complete model scheme are considered such as the maximum likelihood estimation, the Anderson Darling estimation, the ordinary least square estimation, the Cramér-von-Mises estimation, the weighted least square estimation, the left tail-Anderson Darling estimation, and the right tail Anderson Darling estimation methods. Numerical simulation studies were performed for comparing these estimation methods using different sample sizes and three different combinations of parameters. The potentiality of the EG-LL model is illustrated using three real data sets and the model is compared with many other well-known generalizations. The new model was proven worthy in modeling breaking stress, survival times and medical data sets. The Barzilai-Borwein algorithm is employed via a simulation study for assessing the performance of the estimators with different sample sizes as sample size tends to ∞. Using the Bagdonavičius-Nikulin goodness-of-fit test for validation, we propose a modified chi-square GOF tests for the EG-LL model. We have analyzed a lymphoma data set consisting of times (in months) from diagnosis to death for 31 individuals with advanced non Hodgkin's lymphoma clinical symptoms by using our model under the modified Bagdonavičius-Nikulin goodness-of-fit test statistic. Based on the MLEs, the modified Bagdonavičius-Nikulin goodness-of-fit test recovered the loss of information for the grouping data and follows the chi-square distribution. The corresponding elements of the modified Bagdonavičius-Nikulin goodness-of-fit criteria tests are, therefore, explicitly derived.

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