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ANALYTICAL AND NUMERICAL SOLUTIONS OF A CLASS OF GENERALISED LANE-EMDEN EQUATIONS

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ABSTRACT. The classical equation of Jonathan Homer Lane and Robert Emden, a nonlinear second-order ordinary differential equation, models the isothermal spherical clouded gases under the influence of the mutual attractive interaction between the gases' molecules. In this paper, the Adomian decomposition method (ADM) is presented to obtain highly accurate and reliable analytical solutions of a class of generalised Lane-Emden equations with strong nonlinearities. The nonlinear term $f(y(x))$ of the proposed problem is given by the integer powers of a continuous real-valued function $h(y(x))$, that is, $f(y(x)) = h^m(y(x))$, for integer $m \geq 0$, real $x > 0$. In the end, numerical comparisons are presented between the analytical results obtained using the ADM and numerical solutions using the eighth-order nested second derivative two-step Runge-Kutta method (NSDTSRKM) to illustrate the reliability, accuracy, effectiveness and convenience of the proposed methods. The special cases $h(y) = \sin y(x), \cos y(x)$; $h(y) = \sinh y(x), \cosh y(x)$ are considered explicitly using both methods. Interestingly, in each of these methods, a unified result is presented for an integer power of any continuous real-valued function - compared with the case by case computations for the nonlinear functions $f(y)$. The results presented in this paper are a generalisation of several published results. Several examples are given to illustrate the proposed methods. Tables of expansion coefficients of the series solutions of some special Lane-Emden type equations are presented. Comparisons of the two results indicate that both methods are reliably and accurately efficient in solving a class of singular strongly nonlinear ordinary differential equations.

1. INTRODUCTION

The standard and classical Lane-Emden equation, introduced by Jonathan Homer Lane and Robert Emden, models stellar structure and stellar dynamics which depend mainly on understanding the equilibrium of gases under their self-gravity. This equation is a singular nonlinear

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second-order ordinary differential equation with appropriate initial conditions ([1, 2]):

$$\begin{aligned} &y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad 0 < x \leq 1; \\ &y(0) = 1, \quad y'(0) = 0, \end{aligned} \tag{1.1}$$

where $m \geq 0$. This problem models the behaviour of spherical cloud and facilitates the studying of the self-gravitating gas ([1, 3]). Indeed, the gravitational potential of the self-gravitating gas sphere was considered a perfect suggestion to explain the process of the star-formation ([4]). The Lane-Emden equation is a significant tool in the modelling of a reaction–diffusion process and the solution of the Lane-Emden equation is useful in the optimisation of this process ([5]). It was stated in ([3]) that, in quantum mechanics and astrophysics, the values of m are physically meaningful and interesting and lie in the interval $[0, 5]$. As a result, the exact solutions of equation (1.1) when $m = 0, 1, 5$ are realisable (see, e.g., [1]).

A well-studied Lane-Emden equation is a pair of initial value problems (see, e. g., [2, 6])

$$\begin{aligned} &y''(x) + \frac{2}{x}y'(x) + e^{\pm y} = 0, \quad 0 < x \leq 1; \\ &y(0) = 0, \quad y'(0) = 0. \end{aligned} \tag{1.2}$$

The problem (1.2) with the nonlinear term e^y describes the isothermal gas spheres embedded in a pressurised medium at the maximum possible mass allowing for hydrostatic equilibrium; whereas, the initial value problem (1.2) with the nonlinear term e^{-y} describes Richardson's model of thermionic current, that is, the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium ([7, 8]). It was obtained in [6], using the Boubaker polynomials expansion scheme (BPES), the approximate analytical solutions to Lane–Emden equations (1.1) and (1.2). In [9], the analytical solution of the Lane-Emden equation (1.2) was obtained via the Lagrangian formulation and Noether symmetry classification and reduction.

Due to the significant applications of Lane-Emden equations in mathematical physics, mathematical chemistry and astrophysics, several authors and researchers in recent years studied a more general Lane-Emden type problem ([2, 10, 11, 12, 13])

$$\begin{aligned} &y'' + \frac{2}{x}y' + f(y(x)) = 0 \\ &y(0) = A, \quad y'(0) = B, \end{aligned} \tag{1.3}$$

where A and B are constants, and $f(y)$ is a continuous real-valued function. This Lane-Emden type problem (1.3) also models several phenomena such as the theory of stellar structure, thermal explosions, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and thermionic currents ([7, 8, 14]). There are many analytical methods employed for solving the Lane-Emden equation (1.3) (of which equations (1.1) and (1.2) are special, standard and classical cases), these methods include, but are not limited to: series methods ([1, 15, 16, 17]), homotopy perturbation method ([18, 19, 20, 21]), Adomian decomposition method ([2, 3, 22, 23, 24, 25, 26]), variational iteration method ([27, 28]), transformation method ([29]), Jacobian and Weierstrass elliptic function methods ([30]), reproducing kernel method and group preserving scheme ([31]), series expansion method, conformable Homotopy–Adomian decomposition method, conformable residual power series method ([32]), Homotopy-Adomian decomposition method ([33]).

In a recent paper [4], the authors applied the Laplace-residual power series method to construct a series solution of a class of Lane-Emden equations. The authors in [34] solved the singular second-order functional differential model of Lane-Emden type with the development of neuro-swarm intelligent computing solver based on mathematical modelling of artificial neural networks optimised globally search efficacy of particle swarm optimisation aided with local search efficiency of sequential quadratic programming. A well-known Bernoulli collocation scheme was used in [35] to solve a nonlinear Lane-Emden pantograph delay differential model; while the authors in [36] employed the Chebyshev pseudo-spectral approach to solve a class of Lane-Emden equations. The authors in [37] presented a spectral collocation method to obtain numerical solutions of a second-order nonlinear coupled functional Lane-Emden equation. For Ulam stability for a nonlinear differential equation of Lane-Emden type with anti periodic conditions, see [38] and for variable-order Lane-Emden type equations, consult [39], [36].

The author in [2], applied a reliable algorithm based on Adomian decomposition method to present approximate analytical solutions of a class of Lane-Emden equation whose nonlinear terms are given by notable elementary functions that include trigonometric and hyperbolic functions. Very recently, Awonusika in [40], presented, using a power series method, approximate analytical solutions of a class of fractional Lane-Emden equations whose nonlinear terms are given by Jacobi polynomials, of which Gegenbauer polynomials, Legendre polynomials, and Chebyshev polynomials of the first, second, third and fourth kinds are important special cases. In [41], which is also of recent, Awonusika and Mogbojuri presented, using the Mittag-Leffler function, approximate analytical solutions of a class of fractional Lane-Emden equation whose nonlinear terms are described by trigonometric functions $\sin y$, $\cos y$; hyperbolic functions $\sinh y$, $\cosh y$; and exponential functions $e^{\pm y}$.

In recent times, the development of numerical methods for finding the solution of nonlinear differential equations have gained popular interests, this can be attributed to the fact that several nonlinear differential equations cannot be solved exactly; an important special case is the Lane-Emden equation. Some of these numerical methods include, but are not limited to, collocation method ([42]), orthonormal Bernoulli's polynomials method ([43]), Chebyshev wavelets and a finite difference approach ([44, 45, 46, 47]), Jacobi-Gauss spectral collocation method ([48]), the multistep methods ([49, 50]), the two-step Runge-Kutta methods ([51, 52]), the block methods ([53, 54]) and the general linear methods ([55, 56, 57, 58]).

In this paper, the Adomian decomposition method (ADM) and the eighth-order nested second derivative two-step Runge-Kutta (NSDTSRK) method ([59]) are applied to solve a class of generalised Lane-Emden initial value problems with nonlinear terms $f(y(x)) = \sin^m y(x)$, $\cos^m y(x)$, $\sinh^m y(x)$, $\cosh^m y(x)$, for integer $m \geq 0$. Numerical comparisons are presented between the analytical results obtained using the ADM and the numerical solutions using the

NSDTSRKM to illustrate the reliability, accuracy, effectiveness and convenience of the proposed methods. It must be mentioned that the Adomian decomposition method employed in this paper is distinguished in that it avoids the case-by-case computations of the Adomian polynomials for the nonlinear term $f(y)$, as presented, in e.g., [2]; instead, a unified result is presented. Several examples are given to illustrate the proposed methods. Tables of expansion coefficients of the series solutions of some special Lane-Emden type equations are presented. The results of the two methods are tabulated and plotted. Comparisons of the two results indicate that both methods are reliably and accurately efficient in solving a class of singular strongly nonlinear ordinary differential equations. The motivation for the class of the nonlinear functions $f(y)$ considered in this paper is to generalise the results of the authors in [2, 3, 22, 41, 47, 60] as regard these elementary functions.

2. ADOMIAN POLYNOMIALS FOR NONLINEAR FUNCTIONS

In this section, we present the Adomian polynomials for the nonlinear function $f(y)$ appearing in the Lane-Emden equation

$$\begin{aligned} &y'' + \frac{n}{x}y' + f(y(x)) = 0 \\ &y(0) = y_0, \quad y'(0) = 0, \end{aligned} \tag{2.1}$$

where y_0 is a constant, $f(y)$ is a real-valued continuous function and $n \geq 0$, n real. It is known (see [2, 23, 24]) that the Adomian algorithm decomposes the nonlinear term $f(y)$ into an infinite series of polynomials.

Towards this end, one sees that the Adomian decomposition method expresses the solution $y(x)$ as an infinite series of components:

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \tag{2.2}$$

We then recover the nonlinear term $f(y(x))$ from the linear term $y(x)$ by the decomposition series of polynomials:

$$f(y(x)) = \sum_{k=0}^{\infty} F_k(Y), \tag{2.3}$$

where

$$F_k(Y) := F_k\left(y_0, y_1, y_2, \ldots, y_k\right).$$

The functions F_k are the Adomian polynomials which can be computed for several classes of nonlinear functions using the algorithms of Adomian [24, 25], and Wazwaz [2, 61].

Now, let $f(y)$ be a nonlinear function. Using algebraic operators, trigononetric identities and Taylor series as appropriate, the first few Adomian polynomials for the function $f(y)$ are given by ([2, 23, 24, 61])

$F_0(y_0) = f(y_0)$

$F_1(y_0, y_1) = y_1 f'(y_0)$

$F_2(y_0, y_1, y_2) = y_2 f'(y_0) + \frac{1}{2!}(y_1)^2 f''(y_0)$

$F_3(y_0, y_1, y_2, y_3) = y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{3!}(y_1)^3 f'''(y_0)$

$F_4(y_0, y_1, y_2, y_3, y_4) = y_4 f'(y_0) + \left(\frac{(y_2)^2}{2!} + y_1 y_3\right) f''(y_0) + \frac{(y_1)^2}{2!} y_2 f'''(y_0) + \frac{(y_1)^4}{4!} f^{(4)}(y_0)\,.$

See [2, 23, 24, 61, 62, 63, 64, 65, 66] for further discussions on Adomian polynomials and their explicit computations for various classes of nonlinear functions.

Now, one writes equation (2.1) in an operator form ([2, 23, 25, 61]):

$$\begin{aligned} &Ly = -f(y) \\ &y(0) = y_0, \quad y'(0) = 0, \end{aligned} \tag{2.4}$$

where the differential operator L is given by

$$L = \frac{d^2}{dx^2} + \frac{n}{x}\frac{d}{dx} = \frac{1}{x^n}\frac{d}{dx}\left(x^n \frac{d}{dx}\right).$$

The inverse operator L^{-1} is then defined (for a well-behaved function $g(y)$) by

$$L^{-1}g(y) = \int_0^x \frac{1}{x^n}\int_0^x x^n g(y)\,dxdx. \tag{2.5}$$

Applying the inverse operator L^{-1} defined in equation (2.5) on equation (2.4) gives the solution

$$y(x) = y_0 - L^{-1}f(y), \tag{2.6}$$

satisfying the conditions

$$y(0) = y_0, \quad y'(0) = 0.$$

It follows from equations (2.2), (2.3) and (2.6) the identity

$$\sum_{k=0}^{\infty} y_k(x) = y_0 - L^{-1}\sum_{k=0}^{\infty} F_k(y_0, y_1, y_2, \ldots, y_k),$$

and as a result one obtains the recurrence relation

$$\begin{aligned} &y_0(x) = y_0 \\ &y_{k+1}(x) = -L^{-1}F_k(y_0, y_1, y_2, \ldots, y_k), \qquad k = 0, 1, 2, 3, \ldots. \end{aligned} \tag{2.7}$$

The algorithm set in equation (2.7) computes the solution $y(x)$ as an infinite series of the components $y_k(x)$ as given in equation (2.2).

3. Analytical Solution of Lane-Emden Type Problem Using Adomian Decomposition Method

This section presents approximate analytical solutions of a class of generalised Lane-Emden equations whose nonlinear terms are given by powers of continuous real-valued functions. Indeed, the generalised Lane-Emden equation under consideration is given by the initial value problem

$$\begin{aligned} &y''(x) + \frac{n}{x}y'(x) + h^m(y(x)) = 0, \quad \text{real } n \geq 0 \\ &y(0) = A, \quad y'(0) = 0, \quad A \in [0,1], \end{aligned} \tag{3.1}$$

with $0 < x \leq 1, m \in \mathbb{N}_0$. Here h^m is the mth power of a continuous real-valued function h. In particular, if $h(y(x)) = y(x)$ and $A = 1$, then equation (3.1) reduces to the standard Lane-Emden equation (1.1). In this paper, we will be considering the nonlinear functions $h(y(x)) = \sin y(x), \cos y(x); h(y(x)) = \sinh y(x), \cosh y(x)$. Our approaches in solving the initial value problem (3.1) involve the Adomian decomposition method and the eighth-order nested second derivative two-step Runge-Kutta method. Recall that for the special case $n = 0$, the boundary $x = 0$ is an ordinary point; while for $n \neq 0$, the boundary $x = 0$ is a singular point ([2]). The generalised model (3.1) can handle several numbers of differential equations of the same type with and without a singular point. For extensive discussions of models involving these nonlinear elementary functions as well as the physical interpretations of their solutions, see [7, 8, 14].

Now, let us implement the Adomian decomposition method to obtain approximate analytical solutions of the problem (3.1). Towards this end, one sees from Section 2 that

$$y_{\ell+1}(x) = -L^{-1}F_\ell\left(A, y_1, y_2, \ldots, y_\ell\right), \quad \ell = 0, 1, 2, 3, \ldots,$$

where

$$\begin{aligned} &F_0\left(A\right) = h^m\left(A\right) \\ &F_1\left(A, y_1\right) = \alpha_1 y_1 \\ &F_2\left(A, y_1, y_2\right) = \alpha_1 y_2 + \frac{\alpha_2}{2}\left(y_1\right)^2 \\ &F_3\left(A, y_1, y_2, y_3\right) = \alpha_1 y_3 + \alpha_2 y_1 y_2 + \frac{\alpha_3}{6}\left(y_1\right)^3 \\ &F_4\left(A, y_1, y_2, y_3, y_4\right) = \alpha_1 y_4 + \alpha_2\left(\frac{\left(y_2\right)^2}{2} + y_1 y_3\right) + \frac{\alpha_3}{2}\left(y_1\right)^2 y_2 + \frac{\alpha_4}{24}\left(y_1\right)^4. \end{aligned} \tag{3.2}$$

Using the notations $h(A) = a_0, h'(A) = a_1, h''(A) = a_2, h'''(A) = a_3, h^{(4)} = a_4$; and $p_1 = m, p_2 = m(m-1), p_3 = m(m-1)(m-2), p_4 = m(m-1)(m-2)(m-3)$; we have

$$\begin{aligned} \alpha_1 =& p_1 a_0^{p_1-1}a_1, \ \alpha_2 = p_1 a_0^{p_1-1}a_2 + p_2 a_0^{p_1-2}a_1^2, \ \alpha_3 = p_1 a_0^{p_1-1}a_3 + 3p_2 a_0^{p_1-2}a_1 a_2 + p_3 a_0^{p_1-3}a_1^3 \\ \alpha_4 =& p_1 a_0^{p_1-1}a_4 + 3p_2 a_0^{p_1-2}a_2^2 + 4p_2 a_0^{p_1-2}a_1 a_3 + 6p_3 a_0^{p_1-3}a_1^2 a_2 + p_4 a_0^{p_1-4}a_1^4. \end{aligned} \tag{3.3}$$

We now compute the functions $y_{\ell+1}(x), \ell = 0, 1, 2, 3, \dots$ using the integral formula

$$y_{\ell+1}(x) = -L^{-1}F_\ell(A, y_1, y_2, \dots, y_\ell) = -\int_0^x \frac{1}{x^n}\int_0^x x^n F_\ell\, dxdx, \qquad \ell = 0, 1, 2, 3, \dots.$$

Indeed, one sees that

$$y_1(x) = -\,L^{-1}F_0(A) = -a_0^{p_1}\int_0^x \frac{1}{x^n}\int_0^x x^n\, dxdx = -\frac{a_0^{p_1}x^2}{2(n+1)}$$

$$y_2(x) = -\,L^{-1}F_1(A, y_1) = \frac{a_0^{p_1}\alpha_1}{2(n+1)}\int_0^x \frac{1}{x^n}\int_0^x x^{n+2}\, dxdx = \frac{p_1 a_0^{2p_1-1}a_1 x^4}{2\cdot 4(n+1)(n+3)}$$

$$\begin{aligned}
y_3(x) &= -\,L^{-1}F_2(A, y_1, y_2) = -\int_0^x \frac{1}{x^n}\int_0^x x^n\left(\alpha_1 y_2 + \widetilde{\alpha}_2 (y_1)^2\right) dxdx \\
&= -\left(\frac{p_1\alpha_1 a_1 a_0^{2p_1-1}}{2\cdot 4(n+1)(n+3)} + \frac{\widetilde{\alpha}_2 a_0^{2p_1}}{4(n+1)^2}\right)\int_0^x \frac{1}{x^n}\int_0^x x^{n+4}\, dxdx \\
&= -\left(\frac{p_1\alpha_1 a_1 a_0^{2p_1-1}}{2\cdot 4\cdot 6(n+1)(n+3)(n+5)} + \frac{\widetilde{\alpha}_2 a_0^{2p_1}}{4\cdot 6(n+1)^2(n+5)}\right)x^6 \\
&= -\,P_3(n)x^6,
\end{aligned}$$

where $\widetilde{\alpha}_2 := \alpha_2/2$. Further simplifications give

$$P_3(n) = \frac{a_1^2 a_0^{3p_1-2}\left[n\left(p_1^2 + p_2\right) + p_1^2 + 3p_2\right] + (n+3)p_1 a_0^{3p_1-1}a_2}{48(n+1)^2(n+3)(n+5)}. \tag{3.4}$$

Next we see that

$$\begin{aligned}
y_4(x) &= -\,L^{-1}F_3(y_0, y_1, y_2, y_3) = -\int_0^x \frac{1}{x^n}\int_0^x x^n F_3\, dxdx \\
&= -\int_0^x \frac{1}{x^n}\int_0^x x^n\left(\alpha_1 y_3 + \alpha_2 y_1 y_2 + \widetilde{\alpha}_3 (y_1)^3\right) dxdx \\
&= \left(\alpha_1 P_3(n) + \alpha_2 P_1(n)P_2(n) + \widetilde{\alpha}_3 (P_1(n))^3\right)\int_0^x \frac{1}{x^n}\int_0^x x^{n+6}\, dxdx \\
&= \left(\alpha_1 P_3(n) + \alpha_2 P_1(n)P_2(n) + \widetilde{\alpha}_3 (P_1(n))^3\right)\frac{x^8}{8(n+7)} = P_4(n)x^8,
\end{aligned}$$

where $\widetilde{\alpha}_3 := \alpha_3/3!$. Simplifying further we have

$$\begin{aligned}
P_4(n) =& \frac{\left[n^2\left(p_1^3 + 4p_2p_1 + p_3\right) + 2n\left(p_1^3 + 11p_2p_1 + 4p_3\right) + p_1^3 + 18p_2p_1 + 15p_3\right]a_0^{4p_1-3}a_1^3}{384(n+1)^3(n+3)(n+5)(n+7)} \\
&+ \frac{\left[n^2\left(4p_1^2 + 3p_2\right) + 2n\left(11p_1^2 + 12p_2\right) + 18p_1^2 + 45p_2\right]a_0^{4p_1-2}a_1a_2}{384(n+1)^3(n+3)(n+5)(n+7)} \\
&+ \frac{(n+3)(n+5)p_1 a_0^{4p_1-1}a_3}{384(n+1)^3(n+3)(n+5)(n+7)}.
\end{aligned} \tag{3.5}$$

For the lengthier case $\ell = 5$, one sees that

$$\begin{aligned} y_5(x) =& - L^{-1} F_4 \left(1, y_1, y_2, y_3, y_4\right) \\ =& - \int_0^x \frac{1}{x^n} \int_0^x x^n \left(\alpha_1 y_4 + \alpha_2 \left(\frac{\left(y_2\right)^2}{2} + y_1 y_3 \right) + \alpha_3 \frac{\left(y_1\right)^2}{2} y_2 + \alpha_4 \frac{\left(y_1\right)^4}{24} \right) dx dx \\ =& - \int_0^x \frac{1}{x^n} \int_0^x \left(\beta_1 + \beta_2 + \beta_3 + \beta_4\right) x^{n+8} \, dx dx \\ =& - \frac{\left(\beta_1 + \beta_2 + \beta_3 + \beta_4\right) x^{10}}{10(n+9)} = -P_5(n) x^{10}, \end{aligned} \tag{3.6}$$

where

$$\beta_1 := \alpha_1 P_4(n), \ \beta_2 := \alpha_2 \left(\frac{P_2^2(n)}{2} + P_1(n) P_3(n) \right), \ \beta_3 := \frac{\alpha_3}{2} P_1^2(n) P_2(n), \ \beta_4 := \frac{\alpha_4}{24} P_1^4(n), \tag{3.7}$$

and $\alpha_i, 1 \leq i \leq 4$ are as given in equation (3.3). Substituting the coefficients in equation (3.7) into the last equation in (3.6) gives

$$\begin{aligned} P_5(n) =& \frac{n^4 \left(p_1^4 + 11 p_2 p_1^2 + 7 p_3 p_1 + 4 p_2^2 + p_4\right) a_0^{5p_1-4} a_1^4}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{n^3 \left(3p_1^4 + 64 p_2 p_1^2 + 54 p_3 p_1 + 28 p_2^2 + 9 p_4\right) a_0^{5p_1-4} a_1^4}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{n^2 \left(6p_1^4 + 233 p_2 p_1^2 + 283 p_3 p_1 + 128 p_2^2 + 58 p_4\right) a_0^{5p_1-4} a_1^4}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{n \left(5p_1^4 + 296 p_2 p_1^2 + 570 p_3 p_1 + 228 p_2^2 + 159 p_4\right) a_0^{5p_1-4} a_1^4}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{\left(p_1^4 + 81 p_2 p_1^2 + 225 p_3 p_1 + 84 p_2^2 + 105 p_4\right) a_0^{5p_1-4} a_1^4}{1280(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{\left[n^4 \left(11 p_1^3 + 29 p_2 p_1 + 6 p_3\right) + 4 n^3 \left(32 p_1^3 + 109 p_2 p_1 + 27 p_3\right)\right] a_0^{5p_1-3} a_1^2 a_2}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{\left[n^2 \left(233 p_1^3 + 1105 p_1 p_2 + 348 p_3\right) + 2n \left(148 p_1^3 + 1083 p_2 p_1 + 477 p_3\right)\right] a_0^{5p_1-3} a_1^2 a_2}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{3 \left(27 p_1^3 + 281 p_2 p_1 + 210 p_3\right) a_0^{5p_1-3} a_1^2 a_2}{1280(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{\left(4 n p_1^2 + 3 n p_2 + 4 p_1^2 + 15 p_2\right) a_0^{5p_1-2} a_2^2}{3840(n+1)^4(n+5)(n+9)} + \frac{p_1 a_0^{5p_1-1} a_4}{3840(n+1)^4(n+9)} \\ &+ \frac{\left[n^2 \left(7 p_1^2 + 4 p_2\right) + 4n(13 p_1^2 + 10 p_2) + 45 p_1^2 + 84 p_2\right] a_0^{5p_1-2} a_1 a_3}{3840(n+1)^4(n+3)(n+7)(n+9)}. \end{aligned} \tag{3.8}$$

The computation of the component y_ℓ, $\ell = 6, 7, 8, \ldots$ can be done similarly. Thus we have the series solution

$$y(x) = 1 - P_1(n)x^2 + P_2(n)x^4 - P_3(n)x^6 + P_4(n)x^8 - P_5(n)x^{10} + \cdots .$$

We now summarise our computations to give the first main result of this paper. The second main result will be given in Section 4.

Theorem 3.1. *For $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $n \geq 0$, the Lane-Emden type problem*

$$y''(x) + \frac{n}{x}y'(x) + h^m(y(x)) = 0$$
$$y(0) = A, \quad y'(0) = 0, \quad A \in [0,1],$$

admits the analytical solution given by the series

$$y(x) = A - \frac{h^m(A)\,x^2}{2(n+1)} + \frac{mh'(A)h^{2m-1}(A)}{2\cdot 4(n+1)(n+3)}x^4 - P_3(n)x^6 + P_4(n)x^8 - P_5(n)x^{10} + \cdots ,$$

where the coefficients $P_3(n)$, $P_4(n)$ and $P_5(n)$ are given respectively by equations (3.4), (3.5) *and* (3.8)*, with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$; and $h(A) = a_0$, $h'(A) = a_1$, $h''(A) = a_2$, $h'''(A) = a_3$, $h^{(4)} = a_4$.*

In particular, for $m = 0$, one has the solution

$$y(x) = A - \frac{x^2}{2(n+1)}.$$

Remark 3.1. Using the Adomian decomposition method, Theorem 3.1 gives a unified general result on the approximate analytical solution of a generalised Lane-Emden type equation in which the nonlinear term is given by the power of a continous real-valued function. This unified result, to the best of our knowledge, is novel in the discussion of approximate analytical solutions of Lane-Emden type equations.

3.1. **The Special Case $f(y(x)) = \sin^m y(x), m = 1, 2, 3, \ldots$.** We consider the case of the sine function. This case is given as a corollary and is presented as follows with a set of examples.

Corollary 3.2. *For $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $n \geq 0$, the Lane-Emden type problem*

$$y''(x) + \frac{n}{x}y'(x) + \sin^m(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1,$$

admits the analytical solution given by the series

$$\begin{aligned}
y(x) =&1 - \frac{k_1^{p_1} x^2}{2(n+1)} + \frac{p_1 k_1^{2p_1-1} k_2 x^4}{8(n+1)(n+3)} - \frac{\left[n\left(p_1^2+p_2\right)+p_1^2+3p_2\right] k_1^{3p_1-2} k_2^2 - (n+3)p_1 k_1^{3p_1}}{48(n+1)^2(n+3)(n+5)} x^6 \\
&+ \frac{\left[n^2\left(p_1^3+4p_2p_1+p_3\right)+2n\left(p_1^3+11p_2p_1+4p_3\right)+p_1^3+18p_2p_1+15p_3\right] k_1^{4p_1-3} k_2^3}{2\cdot 4\cdot 6\cdot 8(n+1)^3(n+3)(n+5)(n+7)} x^8 \\
&- \frac{\left[n^2(4p_1^2+p_1+3p_2)+2n(11p_1^2+4p_1+12p_2)+18p_1^2+15p_1+45p_2\right] k_1^{4p_1-1} k_2}{2\cdot 4\cdot 6\cdot 8(n+1)^3(n+3)(n+5)(n+7)} x^8 \\
&- P_5(n) x^{10} + \cdots ,
\end{aligned}$$

where $P_5(n)$ *is given by*

$$\begin{aligned}
P_5(n) =&\frac{n^4\left(p_1^4+11p_2p_1^2+7p_3p_1+4p_2^2+p_4\right) k_1^{5p_1-4} k_2^4}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&+ \frac{n^3\left(3p_1^4+64p_2p_1^2+54p_3p_1+28p_2^2+9p_4\right) k_1^{5p_1-4} k_2^4}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&+ \frac{n^2\left(6p_1^4+233p_2p_1^2+283p_3p_1+128p_2^2+58p_4\right) k_1^{5p_1-4} k_2^4}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&+ \frac{n\left(5p_1^4+296p_2p_1^2+570p_3p_1+228p_2^2+159p_4\right) k_1^{5p_1-4} k_2^4}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&+ \frac{\left(p_1^4+84p_2^2+81p_1^2p_2+225p_1p_3+105p_4\right) k_1^{5p_1-4} k_2^4}{1280(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&- \frac{n^4\left(11p_1^3+7p_1^2+29p_2p_1+4p_2+6p_3\right) k_1^{5p_1-2} k_2^2}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&- \frac{n^3\left(32p_1^3+27p_1^2+109p_2p_1+18p_2+27p_3\right) k_1^{5p_1-2} k_2^2}{960(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&- \frac{n^2\left(233p_1^3+283p_1^2+1105p_2p_1+232p_2+348p_3\right) k_1^{5p_1-2} k_2^2}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&- \frac{n\left(148p_1^3+285p_1^2+1083p_2p_1+318p_2+477p_3\right) k_1^{5p_1-2} k_2^2}{960(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&- \frac{3\left(27p_1^3+75p_1^2+281p_2p_1+140p_2+210p_3\right) k_1^{5p_1-2} k_2^2}{1280(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\
&+ \frac{\left(n\left(4p_1^2+p_1+3p_2\right)+4p_1^2+5p_1+15p_2\right) k_1^{5p_1}}{3840(n+1)^4(n+5)(n+9)},
\end{aligned}$$

with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$, $k_1 = \sin 1, k_2 = \cos 1$.

Proof. One sees in this case that the nonlinear function is given by $f(y) = \sin^m(y(x))$ with $A = 1$. Now, setting $a_0 = h(1) = \sin 1 = k_1$, $a_1 = h'(1) = k_2 = \cos 1$, $a_2 = h''(1) = -k_1$, $a_3 = h'''(1) = -k_2$, $a_4 = h^{(4)}(1) = k_1$; with $p_1 = m, p_2 = m(m-1), p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$, we obtain the set of expansion coefficients $P_\ell(n), \ell = 1, 2, 3, 4, 5, \ldots$ as required. □

Next we give some special cases of Corollary 3.2 for clearer and more explicit expression of these expansion coefficients. These special cases are given as examples.

Example 3.1 ($m = 1$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \sin(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.9}$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{k_1 x^2}{2(n+1)} + \frac{k_1 k_2 x^4}{8(n+1)(n+3)} - \frac{(n+1)k_1k_2^2 - (n+3)k_1^3}{48(n+1)^2 n+3)(n+5)}x^6 \\ &+ \frac{\left(n^2+2n+1\right)k_1k_2^3 - \left(5n^2+30n+33\right)k_1^3k_2}{384(n+1)^3(n+3)(n+5)(n+7)}x^8 \\ &- \frac{(n+3)^2(n+7)(5n+9)k_1^5 + \left(n^4+6n^3+12n^2+10n+3\right)k_1k_2^4}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10} \\ &+ \frac{\left(18n^4+236n^3+1032n^2+1732n+918\right)k_1^3k_2^2}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10} + \cdots. \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem ([2, 22, 60])

$$y''(x) + \frac{2}{x}y'(x) + \sin(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.10}$$

admits the series

$$\begin{aligned} y(x) =& 1 - \frac{k_1}{6}x^2 + \frac{k_1k_2}{120}x^4 - \left(\frac{k_1k_2^2}{5040} - \frac{k_1^3}{3024}\right)x^6 + \left(\frac{k_1k_2^3}{362880} - \frac{113k_1^3k_2}{3265920}\right)x^8 \\ &- \left(\frac{19k_1^5}{23950080} - \frac{1781k_1^3k_2^2}{898128000} + \frac{k_1k_2^4}{39916800}\right)x^{10} + \cdots, \end{aligned}$$

which agrees with Wazwaz [2] (see also [22]).

Example 3.2 ($m = 2$). Consider the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \sin^2(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1. \tag{3.11}$$

The solution of this problem is given by

$$\begin{aligned} y(x) =&1 - \frac{k_1^2 x^2}{2(n+1)} + \frac{k_1^3 k_2 x^4}{4(n+1)(n+3)} - \frac{(3n+5)k_1^4 k_2^2 - (n+3)k_1^6}{24(n+1)^2(n+3)(n+5)} x^6 \\ &+ \frac{(n+1)(3n+10)k_1^5 k_2^3 - \left(3n^2+19n+24\right) k_1^7 k_2}{48(n+1)^3(n+3)(n+5)(n+7)} x^8 \\ &- \frac{(n+3)^2(n+7)(3n+7)k_1^{10} + 3(n+1)(n+5)\left(5n^2+26n+25\right) k_1^6 k_2^4}{480(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} x^{10} \\ &+ \frac{\left(15n^4+209n^3+985n^2+1823n+1080\right) k_1^8 k_2^2}{240(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} x^{10} + \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x) + \frac{2}{x} y'(x) + \sin^2(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1,$$

is given by

$$\begin{aligned} y(x) =&1 - \frac{k_1^2}{6} x^2 + \frac{k_1^3 k_2}{60} x^4 - \left(\frac{11 k_1^4 k_2^2}{7560} - \frac{k_1^6}{1512} \right) x^6 + \left(\frac{k_1^5 k_2^3}{8505} - \frac{37 k_1^7 k_2}{204120} \right) x^8 \\ &- \left(\frac{13 k_1^{10}}{2993760} - \frac{1763 k_1^8 k_2^2}{56133000} + \frac{97 k_1^6 k_2^4}{10692000} \right) x^{10} + \cdots . \end{aligned}$$

Example 3.3 ($m = 3$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x} y'(x) + \sin^3(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.12}$$

admits the series formulation

$$\begin{aligned} y(x) =&1 - \frac{k_1^3 x^2}{2(n+1)} + \frac{3 k_1^5 k_2 x^4}{8(n+1)(n+3)} - \frac{(5n+9)k_1^7 k_2^2 - (n+3)k_1^9}{16(n+1)^2(n+3)(n+5)} x^6 \\ &+ \frac{\left(35n^2+166n+147\right) k_1^9 k_2^3 - \left(19n^2+122n+159\right) k_1^{11} k_2}{128(n+1)^3(n+3)(n+5)(n+7)} x^8 \\ &- \frac{3\left(105n^3+1157n^2+3903n+3843\right) k_1^{11} k_2^4}{1280(n+1)^3(n+3)^2(n+5)(n+7)(n+9)} x^{10} \\ &+ \frac{\left(157n^4+2226n^3+10736n^2+20550n+12843\right) k_1^{13} k_2^2}{640(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} x^{10} \\ &- \frac{(19n+47)k_1^{15}}{1280(n+1)^4(n+5)(n+9)} x^{10} + \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x) + \frac{2}{x} y'(x) + \sin^3(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1,$$

is given by

$$y(x) = 1 - \frac{k_1^3}{6}x^2 + \frac{1}{40}k_1^5 k_2 x^4 - \left(\frac{19k_1^7k_2^2}{5040} - \frac{k_1^9}{1008}\right)x^6 + \left(\frac{619k_1^9k_2^3}{1088640} - \frac{479k_1^{11}k_2}{1088640}\right)x^8$$
$$- \left(\frac{17k_1^{15}}{1596672} - \frac{1447k_1^{13}k_2^2}{11088000} + \frac{17117k_1^{11}k_2^4}{199584000}\right)x^{10} + \cdots .$$

Example 3.4 ($m = 4$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \sin^4(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1, \tag{3.13}$$

admits the series

$$y(x) = 1 - \frac{k_1^4 x^2}{2(n+1)} + \frac{k_1^7 k_2 x^4}{2(n+1)(n+3)} - \frac{(7n+13)k_1^{10}k_2^2 - (n+3)k_1^{12}}{12(n+1)^2(n+3)(n+5)}x^6$$
$$+ \frac{\left(35n^2+172n+161\right)k_1^{13}k_2^3 - \left(13n^2+84n+111\right)k_1^{15}k_2}{48(n+1)^3(n+3)(n+5)(n+7)}x^8$$
$$- \frac{\left(455n^4+5622n^3+23316n^2+37370n+19509\right)k_1^{16}k_2^4}{480(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10}$$
$$+ \frac{\left(75n^4+1072n^3+5226n^2+10160n+6507\right)k_1^{18}k_2^2}{120(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10}$$
$$- \frac{(13n+33)k_1^{20}}{480(n+1)^4(n+5)(n+9)}x^{10} + \cdots .$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x) + \frac{2}{x}y'(x) + \sin^4(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1,$$

is given by

$$y(x) = 1 - \frac{k_1^4}{6}x^2 + \frac{1}{30}k_1^7 k_2 x^4 - \left(\frac{1}{140}k_1^{10}k_2^2 - \frac{k_1^{12}}{756}\right)x^6 + \left(\frac{43k_1^{13}k_2^3}{27216} - \frac{331k_1^{15}k_2}{408240}\right)x^8$$
$$- \left(\frac{59k_1^{20}}{2993760} - \frac{19169k_1^{18}k_2^2}{56133000} + \frac{26641k_1^{16}k_2^4}{74844000}\right)x^{10} + \cdots .$$

The series solutions of the Lane-Emden problems (3.9), (3.11), (3.12), (3.13) for $n = 0, 1/2, 1, 3/2, 2$ are presented in Table 1.

3.2. **The Special Case** $f(y(x)) = \cos^m y(x), m = 1, 2, 3, \ldots$**.** We consider the case of the cosine function. This case is given as a corollary and is presented as follows.

Corollary 3.3. *For* $0 < x \leq 1$*,* $m \in \mathbb{N}_0$*, real* $n \geq 0$*, the Lane-Emden type problem*

$$y''(x) + \frac{n}{x}y'(x) + \cos^m(y(x)) = 0, \quad y(0) = 1, \ y'(0) = 0, \quad 0 < x \leq 1,$$

TABLE 1. Illustration of the expansion coefficients of the series solutions of the problems (3.9), (3.11), (3.12), (3.13) for $n = 0, 1/2, 1, 3/2, 2$

$P_\ell(n)/f(y)$	$P_1(0)$	$P_2(0)$	$P_3(0)$	$P_4(0)$
$\sin y$	$\frac{k_1}{2}$	$\frac{k_1k_2}{24}$	$\frac{k_1\left(k_2^2-3k_1^2\right)}{720}$	$\frac{k_1k_2\left(k_2^2-33k_1^2\right)}{40320}$
$\sin^2 y$	$\frac{k_1^2}{2}$	$\frac{k_1^3k_2}{12}$	$\frac{k_1^4\left(5k_2^2-3k_1^2\right)}{360}$	$\frac{k_1^5k_2\left(5k_2^2-12k_1^2\right)}{2520}$
$\sin^3 y$	$\frac{k_1^3}{2}$	$\frac{k_1^5k_2}{8}$	$\frac{k_1^7\left(3k_2^2-k_1^2\right)}{80}$	$\frac{k_1^9k_2\left(49k_2^2-53k_1^2\right)}{4480}$
$\sin^4 y$	$\frac{k_1^4}{2}$	$\frac{k_1^7k_2}{6}$	$\frac{k_1^{10}\left(13k_2^2-3k_1^2\right)}{180}$	$\frac{k_1^{13}k_2\left(161k_2^2-111k_1^2\right)}{5040}$
$P_\ell(n)\,/f(y)$	$P_1\left(\frac{1}{2}\right)$	$P_2\left(\frac{1}{2}\right)$	$P_3\left(\frac{1}{2}\right)$	$P_4\left(\frac{1}{2}\right)$
$\sin y$	$\frac{k_1}{3}$	$\frac{k_1k_2}{42}$	$\frac{k_1\left(3k_2^2-7k_1^2\right)}{4158}$	$\frac{k_1k_2\left(9k_2^2-197k_1^2\right)}{748440}$
$\sin^2 y$	$\frac{k_1^2}{3}$	$\frac{k_1^3k_2}{21}$	$\frac{k_1^4\left(13k_2^2-7k_1^2\right)}{2079}$	$\frac{k_1^5k_2\left(69k_2^2-137k_1^2\right)}{93555}$
$\sin^3 y$	$\frac{k_1^3}{3}$	$\frac{k_1^5k_2}{14}$	$\frac{k_1^7\left(23k_2^2-7k_1^2\right)}{1386}$	$\frac{k_1^9k_2\left(955k_2^2-899k_1^2\right)}{249480}$
$\sin^4 y$	$\frac{k_1^4}{3}$	$\frac{2k_1^7k_2}{21}$	$\frac{2k_1^{10}\left(33k_2^2-7k_1^2\right)}{2079}$	$\frac{k_1^{13}k_2\left(1023k_2^2-625k_1^2\right)}{93555}$
$P\ell(n)/f(y)$	$P_1(1)$	$P_2(1)$	$P_3(1)$	$P_4(1)$
$\sin y$	$\frac{k_1}{4}$	$\frac{k_1k_2}{64}$	$\frac{k_1\left(k_2^2-2k_1^2\right)}{2304}$	$\frac{k_1k_2\left(k_2^2-17k_1^2\right)}{147456}$
$\sin^2 y$	$\frac{k_1^2}{4}$	$\frac{k_1^3k_2}{32}$	$\frac{k_1^4\left(2k_2^2-k_1^2\right)}{576}$	$\frac{k_1^5k_2\left(13k_2^2-23k_1^2\right)}{36864}$
$\sin^3 y$	$\frac{k_1^3}{4}$	$\frac{3k_1^5k_2}{64}$	$\frac{k_1^7\left(7k_2^2-2k_1^2\right)}{768}$	$\frac{k_1^9k_2\left(29k_2^2-25k_1^2\right)}{16384}$
$\sin^4 y$	$\frac{k_1^4}{4}$	$\frac{k_1^7k_2}{16}$	$\frac{k_1^{10}\left(5k_2^2-k_1^2\right)}{288}$	$\frac{k_1^{13}k_2\left(23k_2^2-13k_1^2\right)}{4608}$
$P_\ell(n)/f(y)$	$P_1\left(\frac{3}{2}\right)$	$P_2\left(\frac{3}{2}\right)$	$P_3\left(\frac{3}{2}\right)$	$P_4\left(\frac{3}{2}\right)$
$\sin y$	$\frac{k_1}{5}$	$\frac{k_1k_2}{90}$	$\frac{k_1\left(5k_2^2-9k_1^2\right)}{17550}$	$\frac{k_1k_2\left(25k_2^2-357k_1^2\right)}{5967000}$
$\sin^2 y$	$\frac{k_1^2}{5}$	$\frac{1}{45}k_1^3k_2$	$\frac{k_1^4\left(19k_2^2-9k_1^2\right)}{8775}$	$\frac{k_1^5k_2\left(145k_2^2-237k_1^2\right)}{745875}$
$\sin^3 y$	$\frac{k_1^3}{5}$	$\frac{1}{30}k_1^5k_2$	$\frac{k_1^7\left(11k_2^2-3k_1^2\right)}{1950}$	$\frac{k_1^9k_2\left(211k_2^2-171k_1^2\right)}{221000}$
$\sin^4 y$	$\frac{k_1^4}{5}$	$\frac{2}{45}k_1^7k_2$	$\frac{2k_1^{10}\left(47k_2^2-9k_1^2\right)}{8775}$	$\frac{k_1^{13}k_2\left(1991k_2^2-1065k_1^2\right)}{745875}$
$P_\ell(n)/f(y)$	$P_1(2)$	$P_2(2)$	$P_3(2)$	$P_4(2)$
$\sin y$	$\frac{k_1}{6}$	$\frac{k_1k_2}{120}$	$\frac{k_1\left(3k_2^2-5k_1^2\right)}{15120}$	$\frac{k_1k_2\left(9k_2^2-113k_1^2\right)}{3265920}$
$\sin^2 y$	$\frac{k_1^2}{6}$	$\frac{1}{60}k_1^3k_2$	$\frac{k_1^4\left(11k_2^2-5k_1^2\right)}{7560}$	$\frac{k_1^5k_2\left(24k_2^2-37k_1^2\right)}{204120}$
$\sin^3 y$	$\frac{k_1^3}{6}$	$\frac{1}{40}k_1^5k_2$	$\frac{k_1^7\left(19k_2^2-5k_1^2\right)}{5040}$	$\frac{k_1^9k_2\left(619k_2^2-479k_1^2\right)}{1088640}$
$\sin^4 y$	$\frac{k_1^4}{6}$	$\frac{1}{30}k_1^7k_2$	$\frac{k_1^{10}\left(27k_2^2-5k_1^2\right)}{3780}$	$\frac{k_1^{13}k_2\left(645k_2^2-331k_1^2\right)}{408240}$

admits an analytical solution given by the series

$$\begin{aligned} y(x) =& 1 - \frac{k_2^{p_1} x^2}{2(n+1)} - \frac{p_1 k_1 k_2^{2p_1-1} x^4}{8(n+1)(n+3)} - \frac{\left[n\left(p_1^2+p_2\right)+p_1^2+3p_2\right] k_1^2 k_2^{3p_1-2} - (n+3)p_1 k_2^{3p_1}}{48(n+1)^2(n+3)(n+5)} x^6 \\ &+ \frac{\left[n^2\left(4p_1^2+p_1+3p_2\right)+2n\left(11p_1^2+4p_1+12p_2\right)+3\left(6p_1^2+5p_1+15p_2\right)\right] k_1 k_2^{4p_1-1}}{384(n+1)^3(n+3)(n+5)(n+7)} x^8 \\ &- \frac{\left[n^2\left(p_1^3+4p_2p_1+p_3\right)+2n\left(p_1^3+11p_2p_1+4p_3\right)+p_1^3+18p_2p_1+15p_3\right] k_1^3 k_2^{4p_1-3}}{2\cdot 4\cdot 6\cdot 8(n+1)^3(n+3)(n+5)(n+7)} x^8 \\ &- P_5(n) x^{10} + \cdots , \end{aligned}$$

where $P_5(n)$ *is given by*

$$\begin{aligned} P_5 =& \frac{n^4\left(p_1^4+11p_2p_1^2+7p_3p_1+4p_2^2+p_4\right) k_1^4 k_2^{5p_1-4}}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{n^3\left(3p_1^4+64p_2p_1^2+54p_3p_1+28p_2^2+9p_4\right) k_1^4 k_2^{5p_1-4}}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{n^2\left(6p_1^4+233p_2p_1^2+283p_3p_1+128p_2^2+58p_4\right) k_1^4 k_2^{5p_1-4}}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{n\left(5p_1^4+296p_2p_1^2+570p_3p_1+228p_2^2+159p_4\right) k_1^4 k_2^{5p_1-4}}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{\left(p_1^4+81p_2p_1^2+225p_3p_1+84p_2^2+105p_4\right) k_1^4 k_2^{5p_1-4}}{1280(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &- \frac{n^4\left(11p_1^3+7p_1^2+29p_2p_1+4p_2+6p_3\right) k_1^2 k_2^{5p_1-2}}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &- \frac{n^3\left(32p_1^3+27p_1^2+109p_2p_1+18p_2+27p_3\right) k_1^2 k_2^{5p_1-2}}{960(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &- \frac{n^2\left(233p_1^3+283p_1^2+1105p_2p_1+232p_2+348p_3\right) k_1^2 k_2^{5p_1-2}}{1920(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &- \frac{n\left(148p_1^3+285p_1^2+1083p_2p_1+318p_2+477p_3\right) k_1^2 k_2^{5p_1-2}}{960(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &- \frac{3\left(27p_1^3+75p_1^2+281p_2p_1+140p_2+210p_3\right) k_1^2 k_2^{5p_1-2}}{1280(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} \\ &+ \frac{\left[n\left(4p_1^2+p_1+3p_2\right)+4p_1^2+5p_1+15p_2\right] k_2^{5p_1}}{3840(n+1)^4(n+5)(n+9)}; \end{aligned} \tag{3.14}$$

with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$, $k_1 = \sin 1, k_2 = \cos 1$.

Proof. One sees in this case that the nonlinear function is given by $f(y) = \cos^m(y(x))$ with $A = 1$. Upon setting $a_0 = h(1) = \cos 1 = k_2$, $a_1 = h'(1) = -k_1 = -\sin 1$, $a_2 = h''(1) = -k_2$, $a_3 = h'''(1) = k_1$, $a_4 = h^{(4)}(1) = k_2$; with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$ gives the expansion coefficients $P_\ell(n), \ell = 1, 2, 3, 4, 5, \ldots$ as required. □

We also in this case give some special cases of Corollary 3.3 for more explicit expression of these expansion coefficients. These special cases are also given here as examples.

Example 3.5 ($m = 1$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \cos(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.15}$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{k_2 x^2}{2(n+1)} - \frac{k_1 k_2 x^4}{8(n+1)(n+3)} - \frac{(n+1)k_1^2 k_2 - (n+3)k_2^3}{48(n+1)^2(n+3)(n+5)}x^6 \\ &+ \frac{\left(5n^2+30n+33\right)k_1 k_2^3 - \left(n^2+2n+1\right)k_1^3 k_2}{384(n+1)^3(n+3)(n+5)(n+7)}x^8 \\ &- \frac{(n+3)^2(n+7)(5n+9)k_2^5 + \left(n^4+6n^3+12n^2+10n+3\right)k_1^4 k_2}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10} \\ &+ \frac{\left(18n^4+236n^3+1032n^2+1732n+918\right)k_1^2 k_2^3}{3840(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10} + \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem ([2])

$$y''(x) + \frac{2}{x}y'(x) + \cos(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1,$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{k_2}{6}x^2 - \frac{k_1 k_2}{120}x^4 - \left(\frac{k_1^2 k_2}{5040} - \frac{k_2^3}{3024}\right)x^6 + \left(\frac{113 k_1 k_2^3}{3265920} - \frac{k_1^3 k_2}{362880}\right)x^8 \\ &- \left(\frac{19 k_2^5}{23950080} - \frac{1781 k_1^2 k_2^3}{898128000} + \frac{k_1^4 k_2}{39916800}\right)x^{10} + \cdots . \end{aligned}$$

Example 3.6 ($m = 2$). Consider the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \cos^2(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1. \tag{3.16}$$

The solution of this problem is given by

$$\begin{aligned} y(x) =& 1 - \frac{k_2^2 x^2}{2(n+1)} - \frac{k_1 k_2^3 x^4}{4(n+1)(n+3)} - \frac{(3n+5)k_1^2 k_2^4 - (n+3)k_2^6}{24(n+1)^2(n+3)(n+5)} x^6 \\ &+ \frac{\left(3n^2+19n+24\right) k_1 k_2^7 - (n+1)(3n+10)k_1^3 k_2^5}{48(n+1)^3(n+3)(n+5)(n+7)} x^8 \\ &- \frac{(n+3)^2(n+7)(3n+7)k_2^{10} + 3(n+1)(n+5)\left(5n^2+26n+25\right) k_1^4 k_2^6}{480(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} x^{10} \\ &+ \frac{\left(15n^4+209n^3+985n^2+1823n+1080\right) k_1^2 k_2^8}{240(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} x^{10} + \cdots \end{aligned}$$

.

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x) + \frac{2}{x} y'(x) + \cos^2(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \le 1,$$

admits the series

$$y(x) = 1 - \frac{k_2^2}{6} x^2 - \frac{k_1 k_2^3}{60} x^4 - \left(\frac{11 k_1^2 k_2^4}{7560} - \frac{k_2^6}{1512} \right) x^6 + \left(\frac{37 k_1 k_2^7}{204120} - \frac{k_1^3 k_2^5}{8505} \right) x^8 + \cdots .$$

Example 3.7 ($m = 3$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x} y'(x) + \cos^3(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \le 1, \tag{3.17}$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{k_2^3 x^2}{2(n+1)} - \frac{k_1 k_2^5 x^4}{8(n+1)(n+3)} - \frac{(5n+9)k_1^2 k_2^7 - 3(n+3)k_2^9}{16(n+1)^2(n+3)(n+5)} x^6 \\ &+ \frac{\left(19n^2+122n+159\right) k_1 k_2^{11} - \left(35n^2+166n+147\right) k_1^3 k_2^9}{128(n+1)^3(n+3)(n+5)(n+7)} x^8 \\ &- \frac{3\left(105n^3+1157n^2+3903n+3843\right) k_1^4 k_2^{11}}{1280(n+1)^3(n+3)^2(n+5)(n+7)(n+9)} x^{10} \\ &+ \frac{\left(157n^4+2226n^3+10736n^2+20550n+12843\right) k_1^2 k_2^{13}}{640(n+1)^4(n+3)^2(n+5)(n+7)(n+9)} x^{10} \\ &- \frac{(19n+47)k_2^{15}}{1280(n+1)^4(n+5)(n+9)} x^{10} + \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x) + \frac{2}{x} y'(x) + \cos^3(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \le 1,$$

is given by

$$y(x) = 1 - \frac{k_2^3}{6}x^2 - \frac{1}{40}k_1k_2^5x^4 - \left(\frac{19k_1^2k_2^7}{5040} - \frac{k_2^9}{1008}\right)x^6 + \left(\frac{479k_1k_2^{11}}{1088640} - \frac{619k_1^3k_2^9}{1088640}\right)x^8 + \cdots .$$

Example 3.8 ($m = 4$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \cos^4(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.18}$$

admits the series

$$\begin{aligned} y(x) = & 1 - \frac{k_2^4x^2}{2(n+1)} - \frac{k_1k_2^7x^4}{2(n+1)(n+3)} - \frac{(7n+13)k_1^2k_2^{10} - (n+3)k_2^{12}}{12(n+1)^2(n+3)(n+5)}x^6 \\ & + \frac{\left(13n^2+84n+111\right)k_1k_2^{15} - \left(35n^2+172n+161\right)k_1^3k_2^{13}}{48(n+1)^3(n+3)(n+5)(n+7)}x^8 \\ & - \frac{\left(455n^4+5622n^3+23316n^2+37370n+19509\right)k_1^4k_2^{16}}{480(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10} \\ & + \frac{\left(75n^4+1072n^3+5226n^2+10160n+6507\right)k_1^2k_2^{18}}{120(n+1)^4(n+3)^2(n+5)(n+7)(n+9)}x^{10} \\ & - \frac{(13n+33)k_2^{20}}{480(n+1)^4(n+5)(n+9)}x^{10} + \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x) + \frac{2}{x}y'(x) + \cos^4(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1,$$

is given by

$$y(x) = 1 - \frac{k_2^4}{6}x^2 - \frac{1}{30}k_1k_2^7x^4 - \left(\frac{1}{140}k_1^2k_2^{10} - \frac{k_2^{12}}{756}\right)x^6 + \left(\frac{331k_1k_2^{15}}{408240} - \frac{43k_1^3k_2^{13}}{27216}\right)x^8 + \cdots .$$

The series solutions of the Lane-Emden problems (3.15), (3.16), (3.17), (3.18) for $n = 0, 1/2, 1, 3/2, 2$ are illustrated in Table 2.

3.3. **The Special Case $f(y(x)) = \sinh^m y(x), m = 1, 2, 3, \ldots$.** We consider the case of the hyperbolic sine function. This case is given as a corollary followed by a series of examples.

Corollary 3.4. *For $0 < x \leq 1$, $m \in \mathbb{N}_0$, real $n \geq 0$, the Lane-Emden type problem*

$$y''(x) + \frac{n}{x}y'(x) + \sinh^m(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1,$$

TABLE 2. Description of the expansion coefficients of the series solutions of the problems (3.15), (3.16), (3.17), (3.18) for $n = 0, 1/2, 1, 3/2, 2$

$P_\ell(n)/f(y)$	$P_1(0)$	$P_2(0)$	$P_3(0)$	$P_4(0)$
$\cos y$	$\frac{k_2}{2}$	$-\frac{1}{24}k_1k_2$	$\frac{1}{720}k_2\left(k_1^2-3k_2^2\right)$	$-\frac{k_1k_2\left(k_1^2-33k_2^2\right)}{40320}$
$\cos^2 y$	$\frac{k_2^2}{2}$	$-\frac{1}{12}k_1k_2^3$	$\frac{1}{360}k_2^4\left(5k_1^2-3k_2^2\right)$	$-\frac{k_1k_2^5\left(5k_1^2-12k_2^2\right)}{2520}$
$\cos^3 y$	$\frac{k_2^3}{2}$	$-\frac{1}{8}k_1k_2^5$	$\frac{1}{80}k_2^7\left(3k_1^2-k_2^2\right)$	$-\frac{k_1k_2^9\left(49k_1^2-53k_2^2\right)}{4480}$
$\cos^4 y$	$\frac{k_2^4}{2}$	$-\frac{1}{6}k_1k_2^7$	$\frac{1}{180}k_2^{10}\left(13k_1^2-3k_2^2\right)$	$-\frac{k_1k_2^{13}\left(161k_1^2-111k_2^2\right)}{5040}$
$P_\ell(n)\ /f(y)$	$P_1\left(\frac{1}{2}\right)$	$P_2\left(\frac{1}{2}\right)$	$P_3\left(\frac{1}{2}\right)$	$P_4\left(\frac{1}{2}\right)$
$\cos y$	$\frac{k_2}{3}$	$-\frac{1}{42}k_1k_2$	$\frac{k_2\left(3k_1^2-7k_2^2\right)}{4158}$	$-\frac{k_1k_2\left(9k_1^2-197k_2^2\right)}{748440}$
$\cos^2 y$	$\frac{k_2^2}{3}$	$-\frac{1}{21}k_1k_2^3$	$\frac{k_2^4\left(13k_1^2-7k_2^2\right)}{2079}$	$-\frac{k_1k_2^5\left(69k_1^2-137k_2^2\right)}{93555}$
$\cos^3 y$	$\frac{k_2^3}{3}$	$-\frac{1}{14}k_1k_2^5$	$\frac{k_2^7\left(23k_1^2-7k_2^2\right)}{1386}$	$-\frac{k_1k_2^9\left(955k_1^2-899k_2^2\right)}{249480}$
$\cos^4 y$	$\frac{k_2^4}{3}$	$-\frac{2}{21}k_1k_2^7$	$\frac{2k_2^{10}\left(33k_1^2-7k_2^2\right)}{2079}$	$-\frac{k_1k_2^{13}\left(1023k_1^2-625k_2^2\right)}{93555}$
$P\ell(n)/f(y)$	$P_1(1)$	$P_2(1)$	$P_3(1)$	$P_4(1)$
$\cos y$	$\frac{k_2}{4}$	$-\frac{1}{64}k_1k_2$	$\frac{k_2\left(k_1^2-2k_2^2\right)}{2304}$	$-\frac{k_1k_2\left(k_1^2-17k_2^2\right)}{147456}$
$\cos^2 y$	$\frac{k_2^2}{4}$	$-\frac{1}{32}k_1k_2^3$	$\frac{1}{576}k_2^4\left(2k_1^2-k_2^2\right)$	$-\frac{k_1k_2^5\left(13k_1^2-23k_2^2\right)}{36864}$
$\cos^3 y$	$\frac{k_2^3}{4}$	$-\frac{3}{64}k_1k_2^5$	$\frac{1}{768}k_2^7\left(7k_1^2-2k_2^2\right)$	$-\frac{k_1k_2^9\left(29k_1^2-25k_2^2\right)}{16384}$
$\cos^4 y$	$\frac{k_2^4}{4}$	$-\frac{1}{16}k_1k_2^7$	$\frac{1}{288}k_2^{10}\left(5k_1^2-k_2^2\right)$	$-\frac{k_1k_2^{13}\left(23k_1^2-13k_2^2\right)}{4608}$
$P_\ell(n)/f(y)$	$P_1\left(\frac{3}{2}\right)$	$P_2\left(\frac{3}{2}\right)$	$P_3\left(\frac{3}{2}\right)$	$P_4\left(\frac{3}{2}\right)$
$\cos y$	$\frac{k_2}{5}$	$-\frac{1}{90}k_1k_2$	$\frac{k_2\left(5k_1^2-9k_2^2\right)}{17550}$	$-\frac{k_1k_2\left(25k_1^2-357k_2^2\right)}{5967000}$
$\cos^2 y$	$\frac{k_2^2}{5}$	$-\frac{1}{45}k_1k_2^3$	$\frac{k_2^4\left(19k_1^2-9k_2^2\right)}{8775}$	$-\frac{k_1k_2^5\left(145k_1^2-237k_2^2\right)}{745875}$
$\cos^3 y$	$\frac{k_2^3}{5}$	$-\frac{1}{30}k_1k_2^5$	$\frac{k_2^7\left(11k_1^2-3k_2^2\right)}{1950}$	$-\frac{k_1k_2^9\left(211k_1^2-171k_2^2\right)}{221000}$
$\cos^4 y$	$\frac{k_2^4}{5}$	$-\frac{2}{45}k_1k_2^7$	$\frac{2k_2^{10}\left(47k_1^2-9k_2^2\right)}{8775}$	$-\frac{k_1k_2^{13}\left(1991k_1^2-1065k_2^2\right)}{745875}$
$P_\ell(n)/f(y)$	$P_1(2)$	$P_2(2)$	$P_3(2)$	$P_4(2)$
$\cos y$	$\frac{k_2}{6}$	$-\frac{1}{120}k_1k_2$	$\frac{k_2\left(3k_1^2-5k_2^2\right)}{15120}$	$-\frac{k_1k_2\left(9k_1^2-113k_2^2\right)}{3265920}$
$\cos^2 y$	$\frac{k_2^2}{6}$	$-\frac{1}{60}k_1k_2^3$	$\frac{k_2^4\left(11k_1^2-5k_2^2\right)}{7560}$	$-\frac{k_1k_2^5\left(24k_1^2-37k_2^2\right)}{204120}$
$\cos^3 y$	$\frac{k_2^3}{6}$	$-\frac{1}{40}k_1k_2^5$	$\frac{k_2^7\left(19k_1^2-5k_2^2\right)}{5040}$	$-\frac{k_1k_2^9\left(619k_1^2-479k_2^2\right)}{1088640}$
$\cos^4 y$	$\frac{k_2^4}{6}$	$-\frac{1}{30}k_1k_2^7$	$\frac{k_2^{10}\left(27k_1^2-5k_2^2\right)}{3780}$	$-\frac{k_1k_2^{13}\left(645k_1^2-331k_2^2\right)}{408240}$

admits an analytical solution given by the series

$$
\begin{aligned}
y(x) =& 1 - \frac{\left(e^2-1\right)^{p_1} x^2}{2^{p_1+1}(n+1)e^{p_1}} + \frac{p_1\left(1+e^2\right)\left(e^2-1\right)^{2p_1-1}}{2^{2p_1+3}(n+1)(n+3)e^{2p_1}}x^4 \\
&- \frac{e^4\left[n(p_1^2+p_1+p_2)+p_1^2+3p_1+3p_2\right]\left(e^2-1\right)^{3p_1-2}}{3\cdot 2^{3p_1+4}(n+1)^2(n+3)(n+5)e^{3p_1}}x^6 \\
&- \frac{e^2\left[n(p_1^2-p_1+p_2)+p_1^2-3p_1+3p_2\right]\left(e^2-1\right)^{3p_1-2}}{3\cdot 2^{3p_1+3}(n+1)^2(n+3)(n+5)e^{3p_1}}x^6 \\
&- \frac{\left[n(p_1^2+p_1+p_2)+p_1^2+3p_1+3p_2\right]\left(e^2-1\right)^{3p_1-2}}{3\cdot 2^{3p_1+4}(n+1)^2(n+3)(n+5)e^{3p_1}}x^6 \\
&+ \frac{n^2e^4\left(p_1^3+4p_1^2+p_1+4p_2p_1+3p_2+p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{ne^4\left(p_1^3+11p_1^2+4p_1+11p_2p_1+12p_2+4p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{e^4\left(p_1^3+18p_1^2+18p_2p_1+15p_1+45p_2+15p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{n^2e^2\left(p_1^3-4p_1^2-p_1+4p_2p_1-3p_2+p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{ne^2\left(p_1^3-11p_1^2-4p_1+11p_2p_1-12p_2+4p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+5}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{e^2\left(p_1^3-18p_1^2+18p_2p_1-15p_1-45p_2+15p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{n^2\left(p_1^3+4p_1^2+p_1+4p_2p_1+3p_2+p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{n\left(p_1^3+11p_1^2+4p_1+11p_2p_1+12p_2+4p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{\left(p_1^3+18p_1^2+18p_2p_1+15p_1+45p_2+15p_3\right)\left(1+e^2\right)\left(e^2-1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 + \cdots;
\end{aligned}
$$

with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$.

Proof. One sees in this case that the nonlinear function is given by $f(y) = \sinh^m(y(x))$ with $A = 1$. Now, setting $a_0 = h(1) = \sinh 1 = (e^1 - e^{-1})/2$, $a_1 = h'(1) = \cosh 1 = (e^1+e^{-1})/2$, $a_2 = h''(1) = (e^1-e^{-1})/2$, $a_3 = h'''(1) = (e^1+e^{-1})/2$, $a_4 = h^{(4)}(1) = (e^1-e^{-1})/2$; with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$, we obtain the required expansion coefficients $P_\ell(n), \ell = 1, 2, 3, 4, 5, \ldots$.

□

As earlier stated we give some special cases of Corollary 3.4 for clearer and more explicit expression of these expansion coefficients. These special cases are given as examples.

Example 3.9 ($m = 1$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \sinh(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.19}$$

admits the series formulation

$$\begin{aligned} y(x) =& 1 - \frac{\left(e^2-1\right)x^2}{4(n+1)e} + \frac{\left(e^4-1\right)x^4}{32(n+1)(n+3)e^2} - \frac{e^6\left(n+2\right) - e^4\left(n+4\right) + e^2\left(n+4\right) - n - 2}{192(n+1)^2(n+3)(n+5)e^3}x^6 \\ &+ \frac{e^8\left(3n^2+16n+17\right) - 4e^6\left(n^2+7n+8\right) + 4e^2\left(n^2+7ne^2+8\right) - 3n^2 - 16n - 17}{3072(n+1)^3(n+3)(n+5)(n+7)e^4}x^8 \\ &+ \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem ([2, 22, 60])

$$y''(x) + \frac{2}{x}y'(x) + \sinh(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1,$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{e^2-1}{12e}x^2 + \frac{e^4-1}{480e^2}x^4 - \frac{2e^6-3e^4+3e^2-2}{30240e^3}x^6 + \frac{61e^8-104e^6+104e^2-61}{26127360e^4}x^8 \\ &+ \cdots , \end{aligned}$$

which agrees with Wazwaz [2] (see also [22]).

Example 3.10 ($m = 2$). Consider the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \sinh^2(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \leq 1. \tag{3.20}$$

The solution of this problem is given by

$$\begin{aligned}
y(x) =& 1-\frac{e^4-2e^2+1}{8e^2(n+1)}x^2+\frac{e^8-2e^6+2e^2-1}{64e^4(n+1)(n+3)}x^4\\
&-\frac{e^{12}(n+2)-e^{10}(3n+7)+e^8(3n+10)-2e^6(n+5)}{384e^6(n+1)^2(n+3)(n+5)}x^6\\
&-\frac{e^4(3n+10)-e^2(3n+7)+n+2}{384e^6(n+1)^2(n+3)(n+5)}x^6\\
&+\frac{e^{16}\left(3n^2+16n+17\right)-2e^{14}\left(6n^2+35n+41\right)+2e^{12}\left(9n^2+60n+79\right)}{6144e^8(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&-\frac{e^{10}\left(6n^2+47n+69\right)-e^6\left(6n^2+47n+69\right)+e^4\left(9n^2+60n+79\right)}{3072e^8(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&+\frac{2e^2\left(6n^2+35n+41\right)-3n^2-16n-17}{6144e^8(n+1)^3(n+3)(n+5)(n+7)}x^8+\cdots.
\end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x)+\frac{2}{x}y'(x)+\sinh^2(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1,$$

is given by

$$\begin{aligned}
y(x) =& 1-\frac{e^4-2e^2+1}{24e^2}x^2+\frac{e^8-2e^6+2e^2-1}{960e^4}x^4\\
&-\frac{4e^{12}-13e^{10}+16e^8-14e^6+16e^4-13e^2+4}{120960e^6}x^6\\
&+\frac{61e^{16}-270e^{14}+470e^{12}-374e^{10}+374e^6-470e^4+270e^2-61}{52254720e^8}x^8+\cdots.
\end{aligned}$$

Example 3.11 ($m = 3$). The approximate solution of the Lane-Emden type problem

$$y''(x)+\frac{n}{x}y'(x)+\sinh^3(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1, \tag{3.21}$$

admits the series

$$\begin{aligned}y(x) =&1-\frac{e^6-3e^4+3e^2-1}{16e^3(n+1)}x^2+\frac{3e^{12}-12e^{10}+15e^8-15e^4+12e^2-3}{512e^6(n+1)(n+3)}x^4\\&-\frac{3e^{18}(n+2)-e^{16}(17n+36)+2e^{14}(19n+45)-42e^{12}(n+3)+14e^{10}(2n+9)}{4096e^9(n+1)^2(n+3)(n+5)}x^6\\&+\frac{14e^8(2n+9)-42e^6(n+3)+2e^4(19n+45)-e^2(17n+36)+3(n+2)}{4096e^9(n+1)^2(n+3)(n+5)}x^6\\&+\frac{9e^{24}\left(3n^2+16n+17\right)-4e^{22}\left(50n^2+277n+309\right)+4e^{20}\left(157n^2+920n+1095\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\&-\frac{36e^{18}\left(30n^2+191n+247\right)-3e^{16}\left(365n^2+2608n+3711\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\&-\frac{24e^{14}\left(26n^2+211n+327\right)-24e^{10}\left(26n^2+211n+327\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\&-\frac{3e^8\left(365n^2+2608n+3711\right)-36e^6\left(30n^2+191n+247\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\&-\frac{4e^4\left(157n^2+920n+1095\right)-4e^2\left(50n^2+277n+309\right)+9\left(3n^2+16n+17\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\&+\cdots.\end{aligned}$$

In particular, for $n=2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x)+\frac{2}{x}y'(x)+\sinh^3(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1,$$

is given by

$$\begin{aligned}y(x) =&1-\frac{e^6-3e^4+3e^2-1}{48e^3}x^2+\frac{3e^{12}-12e^{10}+15e^8-15e^4+12e^2-3}{7680e^6}x^4\\&-\frac{12e^{18}-70e^{16}+166e^{14}-210e^{12}+182e^{10}-182e^8+210e^6-166e^4+70e^2-12}{1290240e^9}x^6\\&+\frac{549e^{24}-4252e^{22}+14252e^{20}-26964e^{18}+31161e^{16}-20472e^{14}+20472e^{10}}{2229534720e^{12}}x^8\\&-\frac{31161e^8-26964e^6+14252e^4-4252e^2+549}{2229534720e^{12}}x^8+\cdots.\end{aligned}$$

Example 3.12 ($m=4$). The approximate solution of the Lane-Emden type problem

$$y''(x)+\frac{n}{x}y'(x)+\sinh^4(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1, \tag{3.22}$$

is given by

$$
\begin{aligned}
y(x) =& 1 - \frac{\left(e^2-1\right)^4 x^2}{32(n+1)e^4} + \frac{(e^2-1)^7\left(1+e^2\right)}{512e^8(n+1)(n+3)}x^4 \\
& - \frac{2e^4(n+2)+e^2(3n+5)+2n+4}{12288e^{12}(e^2-1)^{-10}(n+1)^2(n+3)(n+5)}x^6 \\
& + \frac{e^4\left(12n^2+64n+68\right)+e^2\left(11n^2+44n+25\right)+12n^2+64n+68}{786432e^{16}\left(1+e^2\right)^{-1}(e^2-1)^{-13}(n+1)^3(n+3)(n+5)(n+7)}x^8 + \cdots .
\end{aligned}
$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem

$$
y''(x) + \frac{2}{x}y'(x) + \sinh^4(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \le 1,
$$

admits the series

$$
\begin{aligned}
y(x) =& 1 - \frac{\left(e^2-1\right)^4}{96e^4}x^2 + \frac{(e^2-1)^7\left(1+e^2\right)}{7680e^8}x^4 - \frac{(e^2-1)^{10}\left(8e^4+11e^2+8\right)}{3870720e^{12}}x^6 \\
& + \frac{(e^2-1)^{13}\left(1+e^2\right)\left(244e^4+157e^2+244\right)}{6688604160e^{16}}x^8 + \cdots .
\end{aligned}
$$

Table 3 tabulates the series solutions of the Lane-Emden problems (3.19), (3.20), (3.21), (3.22) for $n = 0, 1/2, 1, 3/2, 2$.

3.4. **The Special Case** $f(y(x)) = \cosh^m y(x), m = 1, 2, 3, \ldots$**.** We consider the case of the hyperbolic cosine function. This case is given as a corollary followed by a series of examples.

Corollary 3.5. *For* $0 < x \le 1$*,* $m \in \mathbb{N}_0$*, real* $n \ge 0$*, the Lane-Emden type problem*

$$
y''(x) + \frac{n}{x}y'(x) + \cosh^m(y(x)) = 0, \quad y(0) = 1, \;\; y'(0) = 0, \quad 0 < x \le 1,
$$

TABLE 3. Illustration of the expansion coefficients of the series solutions of the problems (3.19), (3.20), (3.21), (3.22) for $n = 0, 1/2, 1, 3/2, 2$

$P_\ell(n)/f(y)$	$P_1(0)$	$P_2(0)$	$P_3(0)$	$P_4(0)$
$\sinh y$	$\frac{e^2-1}{4e}$	$\frac{e^4-1}{96e^2}$	$\frac{(e^2-1)(1-e^2+e^4)}{1440e^3}$	$\frac{(e^4-1)(17-32e^2+17e^4)}{322560e^4}$
$\sinh^2 y$	$\frac{(e^2-1)^2}{8e^2}$	$\frac{(e^2-1)^3(1+e^2)}{192e^4}$	$\frac{(e^2-1)^4(2+e^2+2e^4)}{5760e^6}$	$\frac{(e^2-1)^5(1+e^2)(17-14e^2+17e^4)}{645120e^8}$
$\sinh^3 y$	$\frac{(e^2-1)^3}{16e^3}$	$\frac{(e^2-1)^5(1+e^2)}{512e^6}$	$\frac{(e^2-1)^7(1-e+e^2)(1+e+e^2)}{10240e^9}$	$\frac{(e^2-1)^9(1+e^2)(51-4e^2+51e^4)}{9175040e^{12}}$
$\sinh^4 y$	$\frac{(e^2-1)^4}{32e^4}$	$\frac{(e^2-1)^7(1+e^2)}{1536e^8}$	$\frac{(e^2-1)^{10}(4+5e^2+4e^4)}{184320e^{12}}$	$\frac{(e^2-1)^{13}(1+e^2)(68+25e^2+68e^4)}{82575360e^{16}}$
$P_\ell(n)\,/f(y)$	**$P_1\left(\frac{1}{2}\right)$**	**$P_2\left(\frac{1}{2}\right)$**	**$P_3\left(\frac{1}{2}\right)$**	**$P_4\left(\frac{1}{2}\right)$**
$\sinh y$	$\frac{e^2-1}{6e}$	$\frac{e^4-1}{168e^2}$	$\frac{(e^2-1)(5-4e^2+5e^4)}{16632e^3}$	$\frac{(e^4-1)(103-188e^2+103e^4)}{5987520e^4}$
$\sinh^2 y$	$\frac{(e^2-1)^2}{12e^2}$	$\frac{(e^2-1)^3(1+e^2)}{336e^4}$	$\frac{(e^2-1)^4(5+3e^2+5e^4)}{33264e^6}$	$\frac{(e^2-1)^5(1+e^2)(103-68e^2+103e^4)}{11975040e^8}$
$\sinh^3 y$	$\frac{(e^2-1)^3}{24e^3}$	$\frac{(e^2-1)^5(1+e^2)}{896e^6}$	$\frac{(e^2-1)^7(15+16e^2+15e^4)}{354816e^9}$	$\frac{(e^2-1)^9(1+e^2)(927+56e^2+927e^4)}{510935040e^{12}}$
$\sinh^4 y$	$\frac{(e^2-1)^4}{48e^4}$	$\frac{(e^2-1)^7(1+e^2)}{2688e^8}$	$\frac{(e^2-1)^{10}(10+13e^2+10e^4)}{1064448e^{12}}$	$\frac{(e^2-1)^{13}(1+e^2)(412+199e^2+412e^4)}{1532805120e^{16}}$
$P\ell(n)/f(y)$	**$P_1(1)$**	**$P_2(1)$**	**$P_3(1)$**	**$P_4(1)$**
$\sinh y$	$\frac{e^2-1}{8e}$	$\frac{e^4-1}{256e^2}$	$\frac{(e^2-1)(3-2e^2+3e^4)}{18432e^3}$	$\frac{(e^4-1)(9-16e^2+9e^4)}{1179648e^4}$
$\sinh^2 y$	$\frac{(e^2-1)^2}{16e^2}$	$\frac{(e^2-1)^3(1+e^2)}{512e^4}$	$\frac{(e^2-1)^4(3+2e^2+3e^4)}{36864e^6}$	$\frac{(e^2-1)^5(1+e^2)(9-5e^2+9e^4)}{2359296e^8}$
$\sinh^3 y$	$\frac{(e^2-1)^3}{32e^3}$	$\frac{3(e^2-1)^5(1+e^2)}{4096e^6}$	$\frac{(e^2-1)^7(9+10e^2+9e^4)}{393216e^9}$	$\frac{(e^2-1)^9(1+e^2)(27+4e^2+27e^4)}{33554432e^{12}}$
$\sinh^4 y$	$\frac{(e^2-1)^4}{64e^4}$	$\frac{(e^2-1)^7(1+e^2)}{4096e^8}$	$\frac{(e^2-1)^{10}(3+4e^2+3e^4)}{589824e^{12}}$	$\frac{(e^2-1)^{13}(1+e^2)(9+5e^2+9e^4)}{75497472e^{16}}$
$P_\ell(n)/f(y)$	**$P_1\left(\frac{3}{2}\right)$**	**$P_2\left(\frac{3}{2}\right)$**	**$P_3\left(\frac{3}{2}\right)$**	**$P_4\left(\frac{3}{2}\right)$**
$\sinh y$	$\frac{e^2-1}{10e}$	$\frac{e^4-1}{360e^2}$	$\frac{(e^2-1)(7-4e^2+7e^4)}{70200e^3}$	$\frac{(e^4-1)(191-332e^2+191e^4)}{47736000e^4}$
$\sinh^2 y$	$\frac{(e^2-1)^2}{20e^2}$	$\frac{(e^2-1)^3(1+e^2)}{720e^4}$	$\frac{(e^2-1)^4(7+5e^2+7e^4)}{140400e^6}$	$\frac{(e^2-1)^5(1+e^2)(191-92e^2+191e^4)}{95472000e^8}$
$\sinh^3 y$	$\frac{(e^2-1)^3}{40e^3}$	$\frac{(e^2-1)^5(1+e^2)}{1920e^6}$	$\frac{(e^2-1)^7(7+8e^2+7e^4)}{499200e^9}$	$\frac{(e^2-1)^9(1+e^2)(191+40e^2+191e^4)}{452608000e^{12}}$
$\sinh^4 y$	$\frac{(e^2-1)^4}{80e^4}$	$\frac{(e^2-1)^7(1+e^2)}{5760e^8}$	$\frac{(e^2-1)^{10}(14+19e^2+14e^4)}{4492800e^{12}}$	$\frac{(e^2-1)^{13}(1+e^2)(764+463e^2+764e^4)}{12220416000e^{16}}$
$P_\ell(n)/f(y)$	**$P_1(2)$**	**$P_2(2)$**	**$P_3(2)$**	**$P_4(2)$**
$\sinh y$	$\frac{e^2-1}{12e}$	$\frac{e^4-1}{480e^2}$	$\frac{(e^2-1)(2-e^2+2e^4)}{30240e^3}$	$\frac{(e^4-1)(61-104e^2+61e^4)}{26127360e^4}$
$\sinh^2 y$	$\frac{(e^2-1)^2}{24e^2}$	$\frac{(e^2-1)^3(1+e^2)}{960e^4}$	$\frac{(e^2-1)^4(4+3e^2+4e^4)}{120960e^6}$	$\frac{(e^2-1)^5(1+e^2)(61-26e^2+61e^4)}{52254720e^8}$
$\sinh^3 y$	$\frac{(e^2-1)^3}{48e^3}$	$\frac{(e^2-1)^5(1+e^2)}{2560e^6}$	$\frac{(e^2-1)^7(6+7e^2+6e^4)}{645120e^9}$	$\frac{(e^2-1)^9(1+e^2)(549+140e^2+549e^4)}{2229534720e^{12}}$
$\sinh^4 y$	$\frac{(e^2-1)^4}{96e^4}$	$\frac{(e^2-1)^7(1+e^2)}{7680e^8}$	$\frac{(e^2-1)^{10}(8+11e^2+8e^4)}{3870720e^{12}}$	$\frac{(e^2-1)^{13}(1+e^2)(244+157e^2+244e^4)}{6688604160e^{16}}$

TABLE 4. Illustration of the expansion coefficients of the series solutions of the problems (3.23), (3.24), (3.25), (3.26) for $n = 0, 1/2, 1, 3/2, 2$

$P_\ell(n)/f(y)$	$P_1(0)$	$P_2(0)$	$P_3(0)$	$P_4(0)$
$\cosh y$	$\frac{1+e^2}{4e}$	$\frac{e^4-1}{96e^2}$	$\frac{(1+e^2)(1+e^2+e^4)}{1440e^3}$	$\frac{(e^4-1)(17+32e^2+17e^4)}{322560e^4}$
$\cosh^2 y$	$\frac{(1+e^2)^2}{8e^2}$	$\frac{(e^2-1)(1+e^2)^3}{192e^4}$	$\frac{(1+e^2)^4(2-e^2+2e^4)}{5760e^6}$	$\frac{(e^2-1)(1+e^2)^5(17+14e^2+17e^4)}{645120e^8}$
$\cosh^3 y$	$\frac{(1+e^2)^3}{16e^3}$	$\frac{(e^2-1)(1+e^2)^5}{512e^6}$	$\frac{(1+e^2)^7(1-e^2+e^4)}{10240e^9}$	$\frac{(e^2-1)(1+e^2)^9(51+4e^2+51e^4)}{9175040e^{12}}$
$\cosh^4 y$	$\frac{(1+e^2)^4}{32e^4}$	$\frac{(e^2-1)(1+e^2)^7}{1536e^8}$	$\frac{(1+e^2)^{10}(4-5e^2+4e^4)}{184320e^{12}}$	$\frac{(e^2-1)(1+e^2)^{13}(68-25e^2+68e^4)}{82575360e^{16}}$
$P_\ell(n)\,/f(y)$	$P_1\left(\frac{1}{2}\right)$	$P_2\left(\frac{1}{2}\right)$	$P_3\left(\frac{1}{2}\right)$	$P_4\left(\frac{1}{2}\right)$
$\cosh y$	$\frac{1+e^2}{6e}$	$\frac{e^4-1}{168e^2}$	$\frac{(1+e^2)(5+4e^2+5e^4)}{16632e^3}$	$\frac{(e^4-1)(103+188e^2+103e^4)}{5987520e^4}$
$\cosh^2 y$	$\frac{(1+e^2)^2}{12e^2}$	$\frac{(e^2-1)(1+e^2)^3}{336e^4}$	$\frac{(1+e^2)^4(5-3e^2+5e^4)}{33264e^6}$	$\frac{(e^2-1)(1+e^2)^5(103+68e^2+103e^4)}{11975040e^8}$
$\cosh^3 y$	$\frac{(1+e^2)^3}{24e^3}$	$\frac{(e^2-1)(1+e^2)^5}{896e^6}$	$\frac{(1+e^2)^7(15-16e^2+15e^4)}{354816e^9}$	$\frac{(e^2-1)(1+e^2)^9(927-56e^2+927e^4)}{510935040e^{12}}$
$\cosh^4 y$	$\frac{(1+e^2)^4}{48e^4}$	$\frac{(e^2-1)(1+e^2)^7}{2688e^8}$	$\frac{(1+e^2)^{10}(10-13e^2+10e^4)}{1064448e^{12}}$	$\frac{(e^2-1)(1+e^2)^{13}(412-199e^2+412e^4)}{1532805120e^{16}}$
$P_\ell(n)/f(y)$	$P_1(1)$	$P_2(1)$	$P_3(1)$	$P_4(1)$
$\cosh y$	$\frac{1+e^2}{8e}$	$\frac{e^4-1}{256e^2}$	$\frac{(1+e^2)(3+2e^2+3e^4)}{18432e^3}$	$\frac{(e^4-1)(9+16e^2+9e^4)}{1179648e^4}$
$\cosh^2 y$	$\frac{(1+e^2)^2}{16e^2}$	$\frac{(e^2-1)(1+e^2)^3}{512e^4}$	$\frac{(1+e^2)^4(3-2e^2+3e^4)}{36864e^6}$	$\frac{(e^2-1)(1+e^2)^5(9+5e^2+9e^4)}{2359296e^8}$
$\cosh^3 y$	$\frac{(1+e^2)^3}{32e^3}$	$\frac{3(e^2-1)(1+e^2)^5}{4096e^6}$	$\frac{(1+e^2)^7(9-10e^2+9e^4)}{393216e^9}$	$\frac{(e^2-1)(1+e^2)^9(27-4e^2+27e^4)}{33554432e^{12}}$
$\cosh^4 y$	$\frac{(1+e^2)^4}{64e^4}$	$\frac{(e^2-1)(1+e^2)^7}{4096e^8}$	$\frac{(1+e^2)^{10}(3-4e^2+3e^4)}{589824e^{12}}$	$\frac{(e^2-1)(1+e^2)^{13}(9-5e^2+9e^4)}{75497472e^{16}}$
$P_\ell(n)/f(y)$	$P_1\left(\frac{3}{2}\right)$	$P_2\left(\frac{3}{2}\right)$	$P_3\left(\frac{3}{2}\right)$	$P_4\left(\frac{3}{2}\right)$
$\cosh y$	$\frac{1+e^2}{10e}$	$\frac{e^4-1}{360e^2}$	$\frac{(1+e^2)(7+4e^2+7e^4)}{70200e^3}$	$\frac{(e^4-1)(191+332e^2+191e^4)}{47736000e^4}$
$\cosh^2 y$	$\frac{(1+e^2)^2}{20e^2}$	$\frac{(e^2-1)(1+e^2)^3}{720e^4}$	$\frac{(1+e^2)^4(7-5e^2+7e^4)}{140400e^6}$	$\frac{(e^2-1)(1+e^2)^5(191+92e^2+191e^4)}{95472000e^8}$
$\cosh^3 y$	$\frac{(1+e^2)^3}{40e^3}$	$\frac{(e^2-1)(1+e^2)^5}{1920e^6}$	$\frac{(1+e^2)^7(7-8e^2+7e^4)}{499200e^9}$	$\frac{(e^2-1)(1+e^2)^9(191-40e^2+191e^4)}{452608000e^{12}}$
$\cosh^4 y$	$\frac{(1+e^2)^4}{80e^4}$	$\frac{(e^2-1)(1+e^2)^7}{5760e^8}$	$\frac{(1+e^2)^{10}(14-19e^2+14e^4)}{4492800e^{12}}$	$\frac{(e^2-1)(1+e^2)^{13}(764-463e^2+764e^4)}{12220416000e^{16}}$
$P_\ell(n)/f(y)$	$P_1(2)$	$P_2(2)$	$P_3(2)$	$P_4(2)$
$\cosh y$	$\frac{1+e^2}{12e}$	$\frac{e^4-1}{480e^2}$	$\frac{(1+e^2)(2+e^2+2e^4)}{30240e^3}$	$\frac{(e^4-1)(61+104e^2+61e^4)}{26127360e^4}$
$\cosh^2 y$	$\frac{(1+e^2)^2}{24e^2}$	$\frac{(e^2-1)(1+e^2)^3}{960e^4}$	$\frac{(1+e^2)^4(4-3e^2+4e^4)}{120960e^6}$	$\frac{(e^2-1)(1+e^2)^5(61+26e^2+61e^4)}{52254720e^8}$
$\cosh^3 y$	$\frac{(1+e^2)^3}{48e^3}$	$\frac{(e^2-1)(1+e^2)^5}{2560e^6}$	$\frac{(1+e^2)^7(6-7e^2+6e^4)}{645120e^9}$	$\frac{(e^2-1)(1+e^2)^9(549-140e^2+549e^4)}{2229534720e^{12}}$
$\cosh^4 y$	$\frac{(1+e^2)^4}{96e^4}$	$\frac{(e^2-1)(1+e^2)^7}{7680e^8}$	$\frac{(1+e^2)^{10}(8-11e^2+8e^4)}{3870720e^{12}}$	$\frac{(e^2-1)(1+e^2)^{13}(244-157e^2+244e^4)}{6688604160e^{16}}$

admits an analytical solution given by the series

$$
\begin{aligned}
y(x) =& 1 - \frac{\left(e^2+1\right)^{p_1} x^2}{2^{p_1+1}(n+1)e^{p_1}} + \frac{p_1(e^2-1)\left(e^2+1\right)^{2p_1-1}}{2^{2p_1+3}(n+1)(n+3)e^{2p_1}}x^4 \\
&- \frac{e^4\left[n(p_1^2+p_1+p_2)+p_1^2+3p_1+3p_2\right]\left(e^2+1\right)^{3p_1-2}}{3\cdot 2^{3p_1+4}(n+1)^2(n+3)(n+5)e^{3p_1}}x^6 \\
&+ \frac{e^2\left[n(p_1^2-p_1+p_2)+p_1^2-3p_1+3p_2\right]\left(e^2+1\right)^{3p_1-2}}{3\cdot 2^{3p_1+3}(n+1)^2(n+3)(n+5)e^{3p_1}}x^6 \\
&- \frac{\left[n(p_1^2+p_1+p_2)+p_1^2+3p_1+3p_2\right]\left(e^2+1\right)^{3p_1-2}}{3\cdot 2^{3p_1+4}(n+1)^2(n+3)(n+5)e^{3p_1}}x^6 \\
&+ \frac{n^2e^4\left(p_1^3+4p_1^2+p_1+4p_2p_1+3p_2+p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{ne^4\left(p_1^3+11p_1^2+4p_1+11p_2p_1+12p_2+4p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{e^4\left(p_1^3+18p_1^2+18p_2p_1+15p_1+45p_2+15p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&- \frac{n^2e^2\left(p_1^3-4p_1^2-p_1+4p_2p_1-3p_2+p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}} \\
&- \frac{ne^2\left(p_1^3-11p_1^2-4p_1+11p_2p_1-12p_2+4p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+5}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&- \frac{e^2\left(p_1^3-18p_1^2+18p_2p_1-15p_1-45p_2+15p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{n^2\left(p_1^3+4p_1^2+p_1+4p_2p_1+3p_2+p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{n\left(p_1^3+11p_1^2+4p_1+11p_2p_1+12p_2+4p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+6}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 \\
&+ \frac{\left(p_1^3+18p_1^2+18p_2p_1+15p_1+45p_2+15p_3\right)(e^2-1)\left(e^2+1\right)^{4p_1-3}}{3\cdot 2^{4p_1+7}(n+1)^3(n+3)(n+5)(n+7)e^{4p_1}}x^8 + \cdots .
\end{aligned}
$$

with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$.

Proof. One sees in this case that the nonlinear function is given by $f(y) = \cosh^m(y(x))$ with $A = 1$. Now, setting $a_0 = h(1) = \cosh 1 = (e^1+e^{-1})/2$, $a_1 = h'(1) = \sinh 1 = (e^1-e^{-1})/2$, $a_2 = h''(1) = (e^1+e^{-1})/2$, $a_3 = h'''(1) = (e^1-e^{-1})/2$, $a_4 = h^{(4)}(1) = (e^1+e^{-1})/2$; with $p_1 = m$, $p_2 = m(m-1)$, $p_3 = m(m-1)(m-2)$, $p_4 = m(m-1)(m-2)(m-3)$, we obtain the following set of expansion coefficients $P_\ell(n), \ell = 1,2,3,4,5,\dots$ as required.

□

Next we give some special cases of Corollary 3.5 for clearer and more explicit expression of these expansion coefficients. These special cases are given as examples.

Example 3.13 ($m = 1$). The approximate solution of the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \cosh(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1, \tag{3.23}$$

is given by

$$\begin{aligned} y(x) =& 1 - \frac{\left(1+e^2\right)x^2}{4e(n+1)} + \frac{\left(e^4-1\right)x^4}{32e^2(n+1)(n+3)} - \frac{e^6(n+2)+e^4(n+4)+e^2(n+4)+n+2}{192e^3(n+1)^2(n+3)(n+5)}x^6 \\ &+ \frac{4e^6\left(n^2+7n+8\right)-4e^2\left(n^2+7n+8\right)+e^8\left(3n^2+16n+17\right)-3n^2-16n-17}{3072e^4(n+1)^3(n+3)(n+5)(n+7)}x^8 \\ &+ \cdots . \end{aligned}$$

In particular, for $n = 2$, the approximate solution of the classical Lane-Emden type problem ([2, 22, 60])

$$y''(x) + \frac{2}{x}y'(x) + \cosh(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1,$$

admits the series formulation

$$y(x) = 1 - \frac{e^2+1}{12e}x^2 + \frac{e^4-1}{480e^2}x^4 - \frac{2e^6+3e^4+3e^2+2}{30240e^3}x^6 + \frac{61e^8+104e^6-104e^2-61}{26127360e^4}x^8 + \cdots ,$$

which agrees with Wazwaz [2] (see also [22]).

Example 3.14 ($m = 2$). Consider the Lane-Emden type problem

$$y''(x) + \frac{n}{x}y'(x) + \cosh^2(y(x)) = 0, \quad y(0) = 1, \; y'(0) = 0, \quad 0 < x \leq 1. \tag{3.24}$$

The solution of this problem is given by

$$\begin{aligned}y(x) =&1-\frac{1+2e^2+e^4}{8e^2(n+1)}x^2+\frac{-1-2e^2+2e^6+e^8}{64e^4(n+1)(n+3)}x^4\\&-\frac{e^{12}(n+2)+e^{10}(3n+7)+e^8(3n+10)+2e^6(n+5)}{384e^6(n+1)^2(n+3)(n+5)}x^6\\&-\frac{e^4(3n+10)+e^2(3n+7)+n+2}{384e^6(n+1)^2(n+3)(n+5)}x^6\\&+\frac{e^{16}\left(3n^2+16n+17\right)+2e^{14}\left(6n^2+35n+41\right)+2e^{12}\left(9n^2+60n+79\right)}{6144e^8(n+1)^3(n+3)(n+5)(n+7)}x^8\\&+\frac{2e^{10}\left(6n^2+47n+69\right)-2e^6\left(6n^2+47n+69\right)-2e^4\left(9n^2+60n+79\right)}{6144e^8(n+1)^3(n+3)(n+5)(n+7)}x^8\\&-\frac{2e^2\left(6n^2+35n+41\right)+3n^2+16n+17}{6144e^8(n+1)^3(n+3)(n+5)(n+7)}x^8+\cdots.\end{aligned}$$

In particular, for $n=2$, the approximate solution of the classical Lane-Emden type problem

$$y''(x)+\frac{2}{x}y'(x)+\cosh^2(y(x))=0,\quad y(0)=1,\;\; y'(0)=0,\quad 0<x\le 1,$$

is given by

$$\begin{aligned}y(x) =&1-\frac{e^4+2e^2+1}{24e^2}x^2+\frac{e^8+2e^6-2e^2-1}{960e^4}x^4\\&-\frac{4e^{12}+13e^{10}+16e^8+14e^6+16e^4+13e^2+4}{120960e^6}x^6\\&+\frac{61e^{16}+270e^{14}+470e^{12}+374e^{10}-374e^6-470e^4-270e^2-61}{52254720e^8}x^8+\cdots.\end{aligned}$$

Example 3.15 ($m=3$). The approximate solution of the Lane-Emden type problem

$$y''(x)+\frac{n}{x}y'(x)+\cosh^3(y(x))=0,\quad y(0)=1,\;\; y'(0)=0,\quad 0<x\le 1, \tag{3.25}$$

admits the series

$$
\begin{aligned}
y(x) =&1-\frac{e^6+3e^4+3e^2+1}{16e^3(n+1)}x^2+\frac{3e^{12}+12e^{10}+15e^8-15e^4-12e^2-3}{512e^6(n+1)(n+3)}x^4\\
&-\frac{3e^{18}(n+2)+e^{16}(17n+36)+2e^{14}(19n+45)+42e^{12}(n+3)+14e^{10}(2n+9)}{4096e^9(n+1)^2(n+3)(n+5)}x^6\\
&-\frac{14e^{8}(2n+9)+42e^{6}(n+3)+2e^{4}(19n+45)+e^{2}(17n+36)+3(n+2)}{4096e^9(n+1)^2(n+3)(n+5)}x^6\\
&+\frac{9e^{24}\left(3n^2+16n+17\right)+4e^{22}\left(50n^2+277n+309\right)+4e^{20}\left(157n^2+920n+1095\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&+\frac{36e^{18}\left(30n^2+191n+247\right)+3e^{16}\left(365n^2+2608n+3711\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&+\frac{24e^{14}\left(26n^2+211n+327\right)-24e^{10}\left(26n^2+211n+327\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&-\frac{3e^{8}\left(365n^2+2608n+3711\right)+36e^{6}\left(30n^2+191n+247\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&-\frac{4e^{4}\left(157n^2+920n+1095\right)+4e^{2}\left(50n^2+277n+309\right)+9\left(3n^2+16n+17\right)}{262144e^{12}(n+1)^3(n+3)(n+5)(n+7)}x^8\\
&+\cdots .
\end{aligned}
$$

In particular, for $n=2$, the approximate solution of the classical Lane-Emden type problem

$$
y''(x)+\frac{2}{x}y'(x)+\cosh^3(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1,
$$

is given by

$$
\begin{aligned}
y(x) =&1-\frac{e^6+3e^4+3e^2+1}{48e^3}x^2+\frac{3e^{12}+12e^{10}+15e^8-15e^4-12e^2-3}{7680e^6}x^4\\
&-\frac{12e^{18}+70e^{16}+166e^{14}+210e^{12}+182e^{10}+182e^8+210e^6+166e^4+70e^2+12}{1290240e^9}x^6\\
&+\frac{+549e^{24}+4252e^{22}+14252e^{20}+26964e^{18}+31161e^{16}+20472e^{14}-20472e^{10}}{2229534720e^{12}}x^8\\
&-\frac{31161e^{8}+26964e^{6}+14252e^{4}+4252e^{2}+549}{2229534720e^{12}}x^8+\cdots .
\end{aligned}
$$

Example 3.16 ($m=4$). The approximate solution of the Lane-Emden type problem

$$
y''(x)+\frac{n}{x}y'(x)+\cosh^4(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\leq 1, \tag{3.26}
$$

is given by

$$
\begin{aligned}
y(x) =& 1-\frac{\left(1+e^2\right)^4}{32e^4(n+1)}x^2+\frac{(e^2-1)\left(1+e^2\right)^7}{512e^8(n+1)(n+3)}x^4\\
&-\frac{2e^4(n+2)-e^2(3n+5)+2n+4}{12288e^{12}\left(1+e^2\right)^{-10}(n+1)^2(n+3)(n+5)}x^6\\
&+\frac{e^4\left(12n^2+64n+68\right)-e^2\left(11n^2+44n+25\right)+12n^2+64n+68}{786432e^{16}(e^2-1)^{-1}\left(1+e^2\right)^{-13}(n+1)^3(n+3)(n+5)(n+7)}x^8+\cdots.
\end{aligned}
$$

In particular, for $n=2$, the approximate solution of the classical Lane-Emden type problem

$$
y''(x)+\frac{2}{x}y'(x)+\cosh^4(y(x))=0,\quad y(0)=1,\ \ y'(0)=0,\quad 0<x\le 1,
$$

admits the series

$$
\begin{aligned}
y(x) =& 1-\frac{\left(1+e^2\right)^4}{96e^4}x^2+\frac{(e^2-1)x^4}{7680e^8\left(1+e^2\right)^{-7}}-\frac{\left(8e^4-11e^2+8\right)x^6}{3870720e^{12}\left(1+e^2\right)^{-10}}\\
&+\frac{(e^2-1)\left(244e^4-157e^2+244\right)}{6688604160e^{16}\left(1+e^2\right)^{-13}}x^8+\cdots.
\end{aligned}
$$

Table 4 presents the series solutions of the Lane-Emden problems (3.23), (3.24), (3.25), (3.26) for $n=0,1/2,1,3/2,2$.

4. Nested Second Derivative Two-Step Runge-Kutta Method for Solving Lane-Emden Type Problem

This section presents a numerical method for finding the solution of the Lane-Emden problem (3.1). Here our interest is to implement an eighth order nested second derivative two-step Runge-Kutta method, NSDTSRKM ([59]) on the Lane-Emden equation (3.1). The NSDTSRKM is an extension of the two-step Runge-Kutta method (TSRKM), where the first and second derivatives are computed. The eighth order NSDTSRK scheme ([59]) for the Lane-Emden problem (3.1) is given by

$$\begin{cases} Y_1^{[n]} = u_{11}y_{n-1} + u_{12}y_{n-2} + h\left(a_{11}f(Y_1^{[n]}) + u_{13}(c_1)f(Y_1^{[n-1]}) + u_{14}(c_1)f(Y_2^{[n-1]})\right. \\ \qquad\qquad \left.+u_{15}f(Y_3^{[n-1]})\right) + h^2\left(\bar{a}_{11}g(Y_1^{[n]}) + \bar{a}_{13}g(Y_3^{[n]})\right), \\ Y_2^{[n]} = u_{21}y_{n-1} + u_{22}y_{n-2} + h\left(a_{21}f(Y_1^{[n]}) + a_{22}f(Y_2^{[n]}) + u_{23}f(Y_1^{[n-1]})\right. \\ \qquad\qquad \left.+u_{24}f(Y_2^{[n-1]}) + u_{25}f(Y_3^{[n-1]})\right) + h^2\left(\bar{a}_{22}g(Y_2^{[n]}) + \bar{a}_{23}g(Y_3^{[n]})\right), \\ Y_3^{[n]} = u_{31}y_{n-1} + u_{32}y_{n-2} + h\left(a_{31}f(Y_1^{[n]}) + a_{32}f(Y_2^{[n]}) + a_{33}f(Y_3^{[n]})\right. \\ \qquad\qquad \left.+u_{33}f(Y_1^{[n-1]}) + u_{34}f(Y_2^{[n-1]}) + u_{35}f(Y_3^{[n-1]})\right) + h^2\bar{a}_{33}g(Y_3^{[n]}), \\ y_n = v_1y_{n-1} + v_2(1)y_{n-2} + h\left(b_1f(Y_1^{[n]}) + b_2f(Y_2^{[n]}) + b_3(1)f(Y_3^{[n]})\right. \\ \qquad\qquad \left.+v_3f(Y_1^{[n-1]}) + v_4f(Y_2^{[n-1]}) + v_5f(Y_3^{[n-1]})\right) + h^2\bar{b}_3g(Y_3^{[n]}). \end{cases} \tag{4.1}$$

Here h is the step size; $c_i,\ i = 1,2,3$ are abscissae values; the stages $Y_i^{[n]},\ i = 1,2,3$ are the numerical approximations to $y(x_n + c_ih),\ i = 1,2,3$; the derivatives $f(Y_i^{[n]})$ and $g(Y_i^{[n]})\ i = 1,2,3$ are the numerical approximations to the first and second derivatives of $y'(x_n + c_ih)$ and $y''(x_n + c_ih)\ i = 1,2,3$ respectively; and $u_{ij}, a_{ij}, \bar{a}_{ij}, b_i, \bar{b}_i, v_i$ are real coefficients determined by the order conditions stated in [59]. On derivation, and setting the abscissae values $c = [c_1, c_2, c_3]^T = \left[\frac{3}{4}, \frac{1}{50}, 1\right]^T$ in equation (4.1) gives the eighth order NSDTSRK method in compact form as

$$\left[\begin{array}{c} Y^{[n]} \\ \hline y^{[n]} \end{array}\right] = \left[\begin{array}{c|c|c} A & \bar{A} & U \\ \hline b^T & \bar{b}^T & V \end{array}\right] \left[\begin{array}{c} hf(Y^{[n]}) \\ h^2g(Y^{[n]}) \\ \hline y^{[n-1]} \end{array}\right], \tag{4.2}$$

where

$$hf(Y^{[n]}) = \left(hf(Y_1^{[n]}), hf(Y_2^{[n]}), hf(Y_3^{[n]})\right)^T, \quad h^2g(Y^{[n]}) = \left(h^2g(Y_1^{[n]}), h^2g(Y_2^{[n]}), h^2g(Y_3^{[n]})\right)^T,$$

$$Y^{[n]} = \left(Y_1^{[n]}, Y_2^{[n]}, Y_3^{[n]}\right)^T, \quad y^{[n]} = \left(y_n, y_{n-1}, hf(Y_1^{[n]}), hf(Y_2^{[n]}), hf(Y_3^{[n]})\right)^T,$$

$$y^{[n-1]} = \left(y_{n-1}, y_{n-2}, hf(Y_1^{[n-1]}), hf(Y_2^{[n-1]}), hf(Y_3^{[n-1]})\right)^T,$$

$$A = \left[\begin{array}{ccc} \frac{369515422034941}{921123745020992} & 0 & 0 \\ \frac{260287970555371446}{4013113355862263495 1171875} & \frac{793396588496998386181}{6012710877684963571 8750} & 0 \\ \frac{39396816896}{72382166325} & \frac{2793575000000}{361643059449} & -\frac{1302191806562}{306529647991875} \end{array}\right],$$

TABLE 5. Comparison between the present methods for the problem (3.1) with $m = 1, 2, 3, 4; n = 2$. We use the notation ADM/NSDTSRKM/(|error|)

$x/f(y)$	0.1	0.2	0.5	1.0
$\sin y$	0.9986/0.9856(0.013)	0.9944/0.9684(0.026)	0.9652/0.9441(0.0211)	0.8634/0.8924(0.029)
$\sin^2 y$	0.9988/0.9893(0.0095)	0.9952/0.9787(0.0165)	0.9708/0.9441(0.0267)	0.8873/0.8950(0.0077)
$\sin^3 y$	0.9990/0.9893(0.0097)	0.9960/0.9791(0.0169)	0.9755/0.9479(0.0276)	0.9062/0.9148(0.0086)
$\sin^4 y$	0.9991/0.9893(0.0098)	0.9966/0.9792(0.0174)	0.9794/0.9491(0.0303)	0.9216/0.9166(0.005)
$x/f(y)$	0.1	0.2	0.5	1.0
$\cos y$	0.9991/0.9782(0.0209)	0.9964/0.9598(0.0366)	0.9773/0.8909(0.0864)	0.9061/0.8253(0.0808)
$\cos^2 y$	0.9995/0.9893(0.0102)	0.9981/0.9789(0.0192)	0.9877/0.9458(0.0419)	0.9491/0.9088(0.0403)
$\cos^3 y$	0.9997/0.9931(0.0066)	0.9989/0.9893(0.0096)	0.9934/0.9668(0.0266)	0.9727/0.9490(0.0237)
$\cos^4 y$	0.9999/0.9969(0.003)	0.9994/0.9929(0.0065)	0.9964/0.9808(0.0156)	0.9854/0.9698(0.0156)
$x/f(y)$	0.1	0.2	0.5	1.0
$\sinh y$	0.9980/0.9893(0.0087)	0.9921/0.9789(0.0132)	0.9519/0.9469(0.005)	0.8182/0.8326(0.0144)
$\sinh^2 y$	0.9977/0.9784(0.0193)	0.9908/0.9594(0.0314)	0.9449/0.8941(0.0508)	0.8050/0.8341(0.0291)
$\sinh^3 y$	0.9973/0.9789(0.0184)	0.9893/0.9680(0.0213)	0.9373/0.9214(0.0159)	0.7968/0.8771(0.0803)
$\sinh^4 y$	0.9968/0.9856(0.0112)	0.9875/0.9691(0.0184)	0.9292/0.9239(0.0053)	0.8081/0.8196(0.0115)
$x/f(y)$	0.1	0.2	0.5	1.0
$\cosh y$	0.9974/0.9943(0.0031)	0.9897/0.9872(0.0025)	0.9366/0.9615(0.0249)	0.7565/0.7154(0.0411)
$\cosh^2 y$	0.9960/0.9915(0.0045)	0.9842/0.9782(0.006)	0.9050/0.9251(0.0201)	0.6609/0.8010(0.1401)
$\cosh^3 y$	0.9940/0.9756(0.0184)	0.9759/0.9354(0.0405)	0.8609/0.9002(0.0393)	0.5938/0.8215(0.2277)
$\cosh^4 y$	0.9906/0.9994(0.0088)	0.9634/0.9902(0.0268)	0.8045/0.9247(0.1202)	1.2341/0.8576(0.4065)

$$\bar{A} = \begin{bmatrix} -\frac{343527961623}{5324414711104} & 0 & \frac{1744133391}{246215709184} \\ 0 & -\frac{191721539677633853}{298957719945776981250 0} & -\frac{13933337769513}{6825518720223218750000 0} \\ 0 & 0 & \frac{104612678}{12637792125} \end{bmatrix},$$

$$U = \begin{bmatrix} \frac{135530954111}{123107854592} & -\frac{12423099519}{123107854592} & -\frac{621160321005}{2246718346304} & -\frac{328581298828125}{16810508346633104} & \frac{66964660049}{123107854592} \\ \frac{9949736275821240 3693}{9949735743765625000 0} & -\frac{5320556153693}{99497357437656250000} & -\frac{244520012691449486}{257393445106376865234375} & -\frac{356936831531937}{43099737958849514796875} & \frac{2322257926947288562757}{34127593601116093750000} \\ \frac{51472}{54585} & \frac{3113}{54585} & \frac{32273434624}{94138655625} & \frac{1231525000000}{126405319983327} & -\frac{2123221144}{280839825} \end{bmatrix},$$

$$b^T = \begin{bmatrix} \frac{39396816896}{72382166325} & \frac{2793575000000}{361643059449} & -\frac{1302191806562}{306529647991875} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \bar{b}^T = \begin{bmatrix} 0 & 0 & \frac{104612678}{12637792125} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} \frac{51472}{54585} & \frac{3113}{54585} & \frac{32273434624}{94138655625} & \frac{1231525000000}{126405319983327} & -\frac{2123221144}{280839825} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The NSDTSRK method (4.2) has a region of absolute stability in the interval $-\infty$ to 0 making it suitable to solve problems having rapidly and slowly decaying transients in their solution. The NSDTSRKM is implemented on equation (3.1) and the results are presented in the next section.

Table 5 presents the comparison between ADM and NSDTSRKM.

5. NUMERICAL RESULTS AND DISCUSSION

This section discusses the comparison of numerical solutions of the Lane-Emden problem (3.1) using the ADM and the NSDTSRKM. Table 5 shows the tabular illustration while the graphical description of results are presented in Figures 1-4. The comparisons of the analytical results using the ADM and the numerical solutions using the NSDTSRK algorithm are presented for the nonlinearities $f(y) = \sin^m y, \cos^m y, \sinh^m y, \cosh^m y$ with the special values $m = 1, 2, 3, 4; n = 2$ in Table 5 and Figures 1 - 4. It can be seen in Table 5 and Figures 1 - 4 that a level of agreements exists between the results obtained from the ADM and those using the NSDTSRK algorithm.

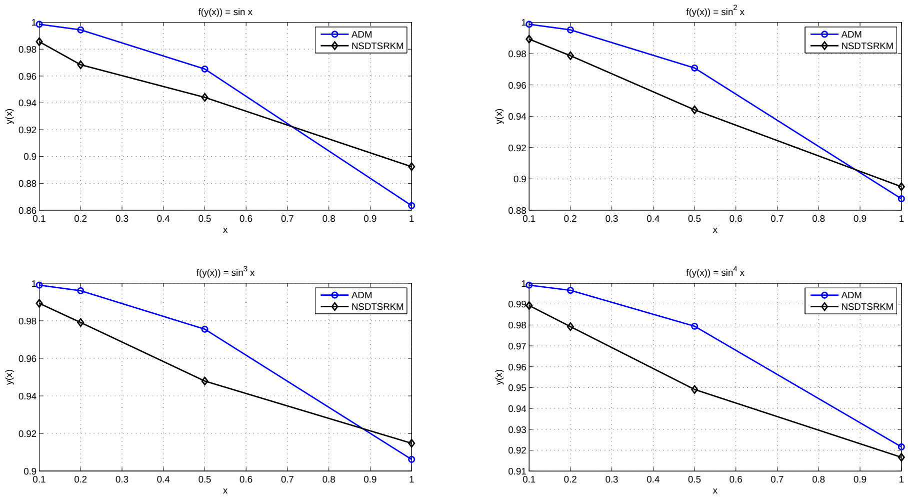

FIGURE 1. Comparison of solutions of the problem (3.1) using the present methods with $h^m(y) = \sin^m y, m = 1, 2, 3, 4; n = 2$.

Figure 1 shows the graphs of the approximate analytical solutions $y(x)$ using the ADM and the numerical solutions using the NSDTSRK algorithm for the Lane-Emden problem 3.1 associated with the nonlinear terms $f(y) = \sin^m y$, $m = 1, 2, 3, 4; n = 2$. Figure 2 presents the graphical illustration of the approximate analytical solutions $y(x)$ using the ADM and the numerical solutions using the NSDTSRK algorithm for the Lane-Emden problem (3.1) associated with the nonlinear terms $f(y) = \cos^m y$, $m = 1, 2, 3, 4; n = 2$. Figure 3 demonstrates the graphical description of the approximate analytical solutions $y(x)$ using the ADM and the numerical solutions using the NSDTSRKM for the Lane-Emden problem (3.1) associated with the nonlinear terms $f(y) = \sinh^m y$, $m = 1, 2, 3, 4; n = 2$; while in Figure 4, we present the graphical description of the approximate analytical solutions $y(x)$ using the ADM and the numerical solutions using the NSDTSRK algorithm for the Lane-Emden problem (3.1) associated with the nonlinear terms $f(y) = \cosh^m y$, $m = 1, 2, 3, 4; n = 2$. Of all these figures, it is observed that the solutions using the present methods agree approximately.

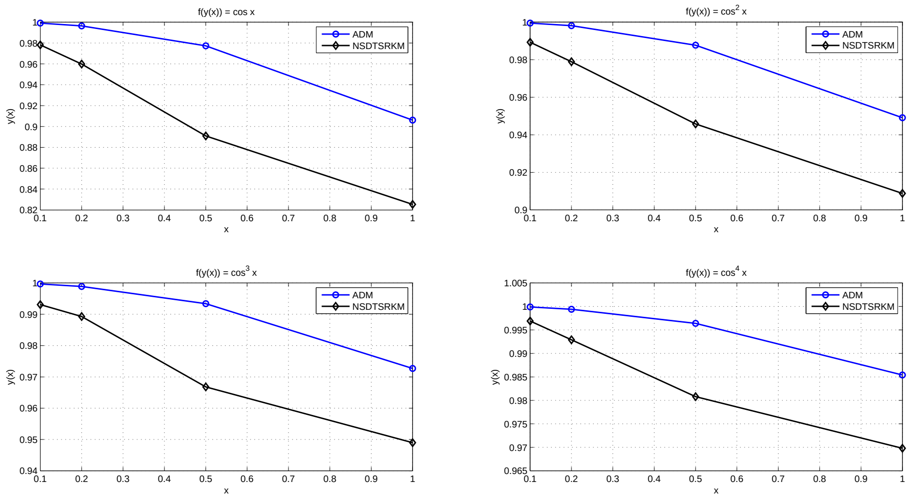

FIGURE 2. Comparison of solutions of the problem (3.1) using the present methods with $h^m(y) = \cos^m y, m = 1, 2, 3, 4; n = 2$.

6. CONCLUDING REMARKS

In this paper, we have used the Adomian decomposition method and the eighth-order nested second derivative two-step Runge-Kutta method to respectively find analytical solutions and numerical solutions of a class of generalised Lane-Emden equations whose nonlinear terms $f(y)$ are expressible as $f(y(x)) = h^m(y(x))$, for $m \geq 0$; where $h(y)$ is a continuous real-valued function and m is the integer power of $h(y)$. In each of these methods, a unified result was given for any continuous real-valued function $h(y)$. We then applied the results to the special cases for which the functions $h(y(x))$ are the trigonometric functions $\sin y(x)$, $\cos y(x)$: and the hyperbolic functions $\sinh y(x), \cosh y(x)$. In all these cases, the explicit analytical solutions were given in series representations. Several special cases of the main results were presented. The results of the special cases of these problems agree with those presented in [2, 3, 22, 36, 47, 60]. Tables of solutions of notable special Lane-Emden type equations were presented. Tabular and graphical illustrations of the results using both methods were given for comparison purposes.

The unified results obtained in this paper can therefore be applied to other similar cases in applied sciences where the models are given as strongly nonlinear ordinary differential equations of Lane-Emden type with any continuous real-valued nonlinear function $h(y)$, even when

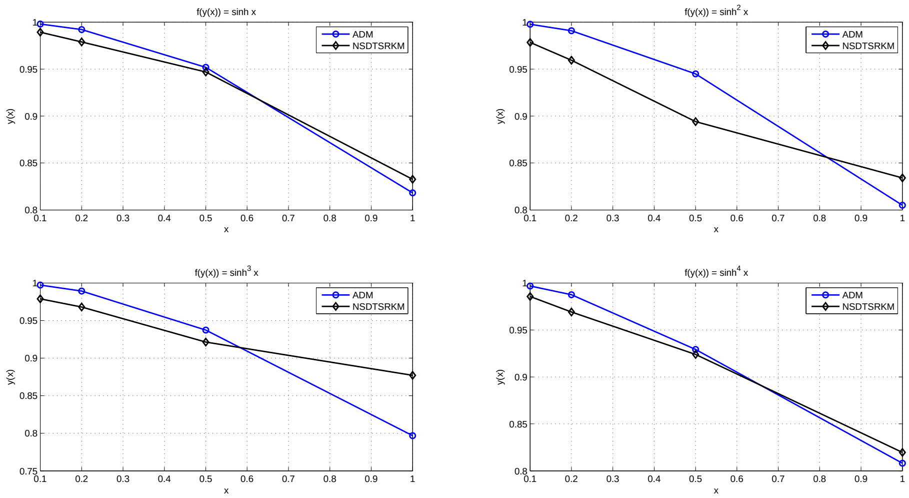

FIGURE 3. Comparison of solutions of the problem (3.1) using the present methods with $h^m(y) = \sinh^m y, m = 1, 2, 3, 4; n = 2$.

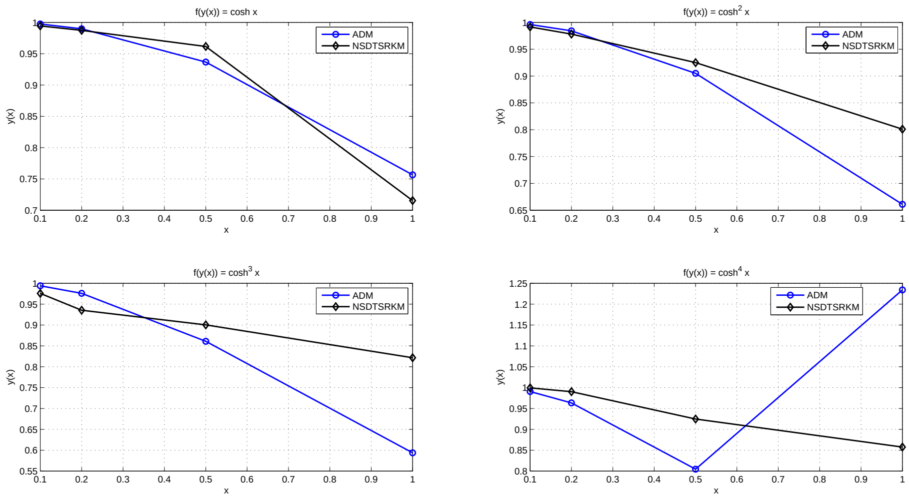

FIGURE 4. Comparison of solutions of the problem (3.1) using the present methods with $h^m(y) = \cosh^m y, m = 1, 2, 3, 4; n = 2$.

the functions $h(y)$ are described by several other special functions.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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