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THE VALUATION OF TIMER POWER OPTIONS WITH STOCHASTIC VOLATILITY

MIJIN HA[1], DONGHYUN KIM[1], SERYOONG AHN[2], AND JI-HUN YOON[1†]

[1]DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, SOUTH KOREA
Email address: †yssci99@pusan.ac.kr

[2]DIVISION OF BUSINESS ADMINISTRATION, PUKYONG NATIONAL UNIVERSITY, SOUTH KOREA
Email address: sahn@pknu.ac.kr

ABSTRACT. Timer options are one of the contingent claims that, for given the variance budget, its payoff depends on a random maturity in terms of the realized variance unlike the standard European vanilla option with a fixed time maturity. Since it was first launched by Société Générale Corporate and Investment Banking in 2007, the valuation of the timer options under several stochastic environment for the volatility has been conducted by many researches. In this study, we propose the pricing of timer power options combined with standard timer options and the index of the power to the underlying asset for the investors to actualize lower risks and higher returns at the same time under the uncertain markets. By using the asymptotic analysis, we obtain the first-order approximation of timer power options. Moreover, we demonstrate that our solution has been derived accurately by comparing it with the solution from the Monte-Carlo method. Finally, we analyze the impact of the stochastic volatility with regards to various parameters on the timer power options numerically.

1. INTRODUCTION

As financial markets have been growing and developing, the financial institutions and investors as well as the market speculators have been more interested in realizing higher profits under the risk of the unpredictable market such as the global financial crisis from the U.S., the COVID–19 pandemic and the Russia–Ukraine conflict. Hence, it enables many researchers to pay attention to the economical market modelling to predict the price behavior of diverse derivatives in the financial market. For example, since the dynamics of the risky asset price, called by geometric Brownian motion(GBM) has been proposed by Black-Scholes [1], stochastic volatility(SV) models have been introduced and developed by many researchers, capturing and reflecting the empirical evidence observed in the financial market (see Hull and White [2], Heston [3], Fouque et al. [4]). Overcoming the disadvantages of Black-Scholes model and verifying that the implied volatility of equity options exhibits a smile or skew phenomenon,

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† Corresponding author.

the stochastic volatility (SV) models have become helpful for the pricing and hedging of the derivative pricing for many years as the existence of a nonflat implied volatility surface has become more noticeable. However, in the real market, the risk premium which arises from the price uncertainty of the underlying asset enables the level of implied volatility to tend to get greater than the realized volatility, showing a higher implied volatility in the financial securities hints that the option would be overvalued.

In this regard, in April 2007, Société Générale Corporate and Investment Banking (SG CIB) has first launched one of innovative financial derivatives, called as the *"timer options"* that have a random maturity relying on the realized variance under the variance budget. A *timer option* is a new form of financial securities which enables investors to determine the volatility level to exercise their option with a random maturity in contrast with a standard European vanilla option. According to Li [5] and Sawyer [6], if the volatility is high, the timer options are exercised at an early stage. Meanwhile, it takes longer time for the timer options to reach its maturity if the volatility is low. Based on this situation, if the sudden changes in the financial market, such as the global financial crisis in 2007-2008, take place, then leads to the drastic changes of the volatility directly, followed by the results that, in case of the timer put option, it makes the option get exercised rapidly. It implies that the portfolio managers can hedge the unexpected risks more easily and effectively. In addition, referring to Bernard and Cui [7], the contract for timer options can serve as hedging or replication techniques for the variance swap or the volatility swaps. Recently, lots of studies concerning the pricing of the timer options have been conducted by many researchers. For example, Carr and Lee [8] found the robust model-free hedges and price bound options on the realized variance of an underlying asset price including the timer style options. Bernard and Cui [7] first found how the problems of the timer options under the stochastic volatility are resolved and the analytic pricing formula is derived by an efficient almost exact Monte Carlo approach. Zheng and Zeng [9] analyzed the pricing formula for the timer options under the 3/2 model by utilizing a closed-form partial transform. Morevoer, Li [5] presented the joint probability density function for the first-passage time that the realized variance reaches the variance budget at the first time to find the analytic pricing formula of the timer options under the Heston model.

Power options are one of the options in which the underlying asset price in the payoff function depends on an index of a positive integer to the underlying asset price process at the expiry date. According to Zhang et al. [10], the type of option can provide the flexibility and a substantial amount of leverage to investors compared with the standard European vanilla options. So, the derivatives attract much more attention in both financial engineering and economic fields as shown in [11]. For example, as Bankers Trust in Germany has issued the power options with a power of order 2, the power options have widely been studied in academia. Macovschi [12] considered the power options under the Heston stochastic volatility model and a pure jump Levy model. Kim et al. [13] found the semi-analytic pricing formula of the power options under the Heston model and presented numerical methods for deriving the value of power put options and capped power call options. Zhang et al. [10] investigated the price of the power options under the assumption that the underlying stock price follows an uncertain differential equation contrary to the Black-Scholes setting, and obtain the analytic solutions for the price of power options for Liu's uncertain stock model with the techniques of uncertain calculus based upon uncertainty theory.

In this study, combining the index to the underlying asset of the standard power options mentioned above into the timer options, **we propose timer power options (TPO) under the general stochastic volatility (SV)**.

Recently, SV models have widely been employed in the diverse problems of the option valuation since they have overcome a shortcoming of the existing Black-Scholes model and reflected the empirical evidences in the real financial market. In fact, as demonstrated in Choi et al. [14], it is well known that the assumption of constant volatility would not capture an exogenous stochastic phenomenon since the global financial crisis in 2007-2008. Furthermore, the flat implied volatility of the Black–Scholes model has a difficulty capturing the volatility smile or skewness, which may be observed in the real finance market. So, in this case, from the empirical results that the volatility of the underlying asset price is a stochastic process, it has become possible to describe the market dynamics more effectively and accurately. Fouque et al. [4] has studied the several types of options under the SV model incorporated by a fast mean-reverting process. Since then, there have been a lot of researches for the financial derivatives based on the SV model. For example, Wong and Chan [15] took account of the pricing of looback options or dynamic fund protections under a multiscale SV model. Chiarella et al. [16] dealt with the pricing problem of barrier options when the volatility of the underlying asset price is driven by the Heston model. Recently, Kim et al. [17] applied the SV model to external barrier options and then obtained the analytic pricing formula by using the method of the asymptotic analysis.

The main contributions of this paper are as follows: Firstly, we construct the market dynamics for the TPOs and derive the partial differential equations(PDEs) for the price of TPOs. Secondly, we obtain the approximation solutions for the given PDE by using asymptotic analysis, which is very helpful for us to deal with the TPO prices. Thirdly, we verify the pricing accuracy of the derived approximated solutions through the Monte-Carlo simulation. Finally, we provide the numerical implications of TPO prices and investigate some economical meanings in terms of several model parameters.

This paper is organized as follows. In Section 2, we provide the market model for the TPOs and derive the partial differential equations(PDEs) for the price of TPOs. In Section 3, we obtain the first-order approximation solution for the option price of the TPOs by using the asymptotic analysis. In Section 4, we verify the accuracy of our pricing formula for the TPOs through Monte-Carlo method and also present the numerical implications against some model parameters. Finally, Section 5 provides the concluding remarks.

2. MODEL FORMULATION

In this section, we present a mathematical model to deal with timer power option pricing. First, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, Ω is a set of outcomes, $\mathcal{F}$ is a $\sigma-$field of subsets of Ω, and $\mathbb{P}$ is a physical probability measure on $(\Omega, \mathcal{F})$. Now, assume that X_t is the price of the underlying asset at time t, and Y_t is the Ornstein-Uhlenbeck process driving the

SV embedded in the volatility of the underlying asset. Then, under the measure $\mathbb{P}$, the model dynamics of X_t and Y_t are given by the following stochastic differential equations:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + f(Y_t) X_t \mathrm{d}W_t^1, \ \ \mathrm{d}Y_t = \alpha(m - Y_t)\mathrm{d}t + \beta \mathrm{d}W_t^2, \tag{2.1}$$

where μ is a constant mean return rate of X_t, f is smooth function such that $0 < c_1 \leq f \leq c_2 < \infty$ for some constants c_1 and c_2, $f(Y_t)$ is denoted by the SV of the underlying asset value X_t and α and β are positive constants. Furthermore, W_t^1 and W_t^2 are the standard Brownian motions under the measure $\mathbb{P}$ with correlation structue $\mathrm{d}\left\langle W^1, W^2 \right\rangle_t = \rho \mathrm{d}t$ such that $|\rho| \leq 1$.

In the real financial market, it can be observed that the volatility of stock prices has a tendency to turn back to the specific mean, as described in Fouque et al. [4]. Also, from the empirical studies based on Standard and Poor's 500 index data, the fluctuation size of the volatility looks like one from a fast mean-reverting stochastic process throughout the lifetime of financial contracts. In this regard, based on the above empirical results, we assume that the SV of X_t is a fast mean-reverting process whose the characteristic time to return to the mean level of its long-run distribution is given by the parameter α and its invariant distribution is normal with $\mathcal{N}(m, u^2)$, where $u = \beta/\sqrt{2\alpha}$ and the parameter u is the variance of the invariant distribution of Y_t. In addition, the probability density function of the invariant distribution of Y_t is described by $\frac{1}{\sqrt{2\pi u^2}} \exp\left(-\frac{(y-m)^2}{2u^2}\right)$. Moreover, the parameter α is called as the rate of the mean reversion. If α is sufficiently large, the process Y_t in (2.1) revert to the long-run mean level m regardless of the time, and the volatility tends to get closer to $f(m)$ asymptotically.

By utilizing the Girsanov theorem, under an equivalent martingale measure $\widetilde{\mathbb{P}}$, the model dynamics (2.1) can be transformed into

$$\begin{aligned}
\mathrm{d}X_t &= r X_t \mathrm{d}t + f(Y_t) X_t \mathrm{d}\widetilde{W}_t^1, \\
\mathrm{d}Y_t &= \left(\frac{1}{\epsilon}(m - Y_t) - \frac{u\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t)\right)\mathrm{d}t + \frac{u\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}\widetilde{W}_t^2,
\end{aligned}$$

where r is a risk-free interest rate, ϵ is the rate of the mean reversion with $0 < \epsilon \ll 1$, m is the long-run mean level, $u(\sim \mathcal{O}(1))$ is the standard deviation of the ergodic process Y_t and $\widetilde{W}_t^1$ and $\widetilde{W}_t^2$ are the transformed standard Brownian motions satisfying $\mathrm{d}\left\langle \widetilde{W}^1, \widetilde{W}^2 \right\rangle_t = \rho \mathrm{d}t$. Here, Λ is the market price of volatility risk.

Unlike the standard European vanilla options, the investors of the timer style options specify the variance budget $\mathbb{V}(> 0)$ considering the accumulated realized variance to exercise the option. The random maturity is determined by the first time passage $\tau_{\mathbb{V}}$, such that $\tau_{\mathbb{V}}$ is given by

$$\tau_{\mathbb{V}} := \inf\left\{t > 0 : V_t = \mathbb{V}\right\}, \tag{2.2}$$

where a continuous-time version of the cumulative realized variance process V_t is defined by $V_t := \int_0^t f^2(Y_s)\mathrm{d}s$. It implies that the stopping occurs when the process V_t first hits the given variance budget $\mathbb{V}$. Timer options can be regarded as a European call option with SV depending on the random maturity under the fixed variance budget instead of considering only the time to maturity as the expirtation date.

Now, we present the payoff function of the timer power options (TPOs) combined with standard timer options and the index of the power to the underlying asset as follows:

$$h(X_{\tau_{\mathbb{V}}}) = (X^c_{\tau_{\mathbb{V}}} - K)^+, \tag{2.3}$$

where K is the predetermined strike price and $c \in \mathbb{N}$ is a constant index of the power. Then, under the risk-neutral measure $\widetilde{\mathbb{P}}$, the no-arbitrage price of TPOs with payoff function (2.3) at a random maturity (2.2) is described by

$$\begin{aligned} P(t \wedge \tau_{\mathbb{V}}, x, y, v) &= \widetilde{\mathbb{E}}\left[e^{-r(\tau_{\mathbb{V}} - t\wedge\tau_{\mathbb{V}})} h(X_{\tau_{\mathbb{V}}}) \middle| X_{t\wedge\tau_{\mathbb{V}}} = x, Y_{t\wedge\tau_{\mathbb{V}}} = y, V_{t\wedge\tau_{\mathbb{V}}} = v \right] \\ &= \widetilde{\mathbb{E}}\left[e^{-r\tau_{\mathbb{V}-v}} (X^c_{\tau_{\mathbb{V}-v}} - K)^+ \mid X_0 = x, Y_0 = y \right]. \end{aligned} \tag{2.4}$$

Next, by applying the well–known Feynman–Kac theorem to the expectation form (2.4) and using the property that the price of a timer style option is independent of time variable t (as shown in Li [5]), we have the following time–invariant PDE:

$$\begin{aligned} \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right) P(x, y, v) = 0, \quad v < \mathbb{V} \\ P(x, y, \mathbb{V}) = (x^c - K)^+, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \mathcal{L}_0 &:= (m - y)\frac{\partial}{\partial y} + u^2 \frac{\partial^2}{\partial y^2}, \\ \mathcal{L}_1 &:= -\, u\sqrt{2}\Lambda(y)\frac{\partial}{\partial y} + u\sqrt{2}\rho f(y) x \frac{\partial^2}{\partial x \partial y}, \\ \mathcal{L}_2 &:= r\left(x\frac{\partial}{\partial x} - I \right) + f^2(y)\frac{\partial}{\partial v} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2}{\partial x^2}. \end{aligned}$$

Here, I is the identity operator.

3. Asymptotic analysis

In this section, we derive an approximate option price by using the asymptotic analysis, as described by Fouque et al. [4]. First, we asymptotically expand P in terms of the small parameter $\sqrt{\epsilon}$ as follows:

$$P(x, y, v) = \sum_{n=0}^{\infty} \epsilon^{\frac{n}{2}} P_n(x, y, v) \;\text{ for } 0 < \epsilon \ll 1, \tag{3.1}$$

where $P_0, P_1, \cdots$ are functions such that $P_0(x, y, \mathbb{V}) = (x^c - K)^+$ and $P_i(x, y, \mathbb{V}) = 0$ if $i \geq 1$. Here, we are mainly focused on the first two terms $P_0 + \sqrt{\epsilon}P_1$. Next, by substituting

(3.1) into (2.5), we have

$$\frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) = \mathcal{O}(\epsilon),$$

which is satisfied for any $\epsilon > 0$. Then, we can deduce a hierarchy of differential equations as follows:

$$\begin{aligned}
&\mathcal{L}_0 P_0 = 0,\\
&\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0,\\
&\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0,\\
&\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0,\\
&\cdots .
\end{aligned} \tag{3.2}$$

Now, we present the growth condition and the centering condition (or solvability condition), which play an important role in the procedure of this section.

Lemma 3.1. *(Growth condition) If P_0 and P_1 do not grow as much as $\frac{\partial P_0}{\partial y} \sim e^{\frac{y^2}{2}}$ and $\frac{\partial P_1}{\partial y} \sim e^{\frac{y^2}{2}}$ as $y \to \infty$, then $P_0(x, y, v)$ and $P_1(x, y, v)$ are independent of unobservable variable y.*

Proof. This proof is similar to that of Lemma 3.1 in Kim et al. [17]. From the ordinary differential equation $\mathcal{L}_0 P_0 = 0$ presented in (3.2), we obtain

$$P_0(x, y, v) = K_1(x, v)\int_0^y e^{\frac{(m-\gamma)^2}{2u^2}}\,\mathrm{d}\gamma + K_2(x, v)$$

for some functions $K_1(x, v)$ and $K_2(x, v)$, which are independent of y. Then, using the assumption of the growth condition on P_0, $K_1(x, v) = 0$ holds, and then we have $P_0 = P_0(x, v)$, which does not depend on y. Next, the second equation $\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$ in (3.2) leads to $\mathcal{L}_0 P_1 = 0$. Similar to the above case of $\mathcal{L}_0 P_0 = 0$, by using the growth condition, therefore, $P_1 = P_1(x, v)$ is independent of y. This completes the proof. □

Lemma 3.2. *(Centering condition) If we consider the equation $\mathcal{L}_0\chi(y) + g(y) = 0$, which is the Poisson equation for χ, with a solution, then the centering condition $\langle g\rangle = 0$ must be satisfied. Here, $\langle\cdot\rangle$ is denoted by an expectation under the invariant distribution $\mathcal{N}(m, u^2)$ of Y, represented by $\langle g\rangle = \frac{1}{\sqrt{2\pi u}}\int_{-\infty}^{\infty} g(y)\exp\left(-\frac{(y-m)^2}{2u^2}\right)\mathrm{d}y$ for any function g.*

Proof. Refer to Fouque et al. [4]. □

Theorem 3.1. *Under the growth condition on P_0 presented in Lemma 3.1, the leading-order term P_0 satisfies the following homogeneous problem:*

$$\begin{aligned}
\mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f)P_0(x, v) &= 0, \qquad v < \mathbb{V},\\
P_0(x, \mathbb{V}) &= (x^c - K)^+,
\end{aligned} \tag{3.3}$$

where $\widetilde{\sigma}_f := \sqrt{\langle f^2 \rangle}$ *is the effective volatility and*

$$\mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f) := r\left(x\frac{\partial}{\partial x} - I\right) + \widetilde{\sigma}_f^2\frac{\partial}{\partial v} + \frac{\widetilde{\sigma}_f^2}{2}x^2\frac{\partial^2}{\partial x^2}.$$

Proof. If we define $\widetilde{\sigma}_f = \sqrt{\langle f^2 \rangle}$ as the effective volatility, then the expected differential operator $\langle \mathcal{L}_2 \rangle := \mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f)$ is expressed by $\mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f) = r\left(x\frac{\partial}{\partial x} - \cdot\right) + \widetilde{\sigma}_f^2\frac{\partial}{\partial v} + \frac{\widetilde{\sigma}_f^2}{2}x^2\frac{\partial^2}{\partial x^2}$. Now, based on the growth condition and the centering condition given by Lemma 3.1 and Lemma 3.2, respectively, the Poisson equation for P_2 in (3.2) yields $\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0$ because the leading order term P_0 does not depend on y. Therefore, we obtain a homogeneous PDE $\mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f)P_0 = 0$ for $v < \mathbb{V}$ and the boundary condition $P_0(x, \mathbb{V}) = (x^c - K)^+$. □

Theorem 3.2. *Under the growth condition on* P_1 *given in Lemma 3.1, the correction order term* $P_1(x, v)$ *satisfies the following non-homogeneous problem:*

$$\begin{aligned} \mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f)P_1(x, v) &= \mathcal{H}, \quad v < \mathbb{V}, \\ P_1(x, \mathbb{V}) &= 0, \end{aligned} \tag{3.4}$$

where $\mathcal{H} := -u\sqrt{2}\langle \Lambda\psi' \rangle \left(\partial_v P_0 + \frac{x^2}{2}\partial_{xx}P_0\right) + u\sqrt{2}\rho\langle f\psi' \rangle \left(x\partial_{xv}P_0 + x^2\partial_{xx}P_0 + \frac{x^3}{2}\partial_{xxx}P_0\right)$ *and* $\psi(y)$ *solves the Poisson equation* $\mathcal{L}_0\psi(y) = f^2(y) - \widetilde{\sigma}_f^2$.

Proof. From the $\langle \mathcal{L}_2 P_0 \rangle = 0$, we have

$$\begin{aligned} \mathcal{L}_2 P_0 &= \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle \\ &= (f^2(y) - \widetilde{\sigma}_f^2)\frac{\partial P_0}{\partial v} + \frac{1}{2}\left(f^2(y) - \widetilde{\sigma}_f^2\right)x^2\frac{\partial^2 P_0}{\partial x^2}. \end{aligned}$$

Now, by utilizing the centering condition of the Poisson equation for P_2, P_2 becomes

$$\begin{aligned} P_2 &= -\mathcal{L}_0^{-1}(\mathcal{L}_2 P_0) \\ &= -\mathcal{L}_0^{-1}\left[(f^2(y) - \widetilde{\sigma}_f^2)\frac{\partial P_0}{\partial v} + \frac{1}{2}\left(f^2(y) - \widetilde{\sigma}_f^2\right)x^2\frac{\partial^2 P_0}{\partial x^2}\right] \\ &= -(\psi + k)\frac{\partial P_0}{\partial v} - \frac{1}{2}(\psi + k)x^2\frac{\partial^2 P_0}{\partial x^2}, \end{aligned}$$

where k is the variable independent of y and ψ is a solution of the Poisson equation $\mathcal{L}_0\psi(y) = f^2(y) - \widetilde{\sigma}_f^2$. Next, let us consider a Poisson equation for P_3, represented by $\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$ and by using the centering condition for P_3, we obtain $\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0$, which leads to

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_1 &= -\langle \mathcal{L}_1 P_2 \rangle \\ &= \langle \mathcal{L}_1 \psi \rangle \frac{\partial P_0}{\partial v} + \frac{1}{2}\langle \mathcal{L}_1 \psi \rangle x^2\frac{\partial^2 P_0}{\partial x^2}. \end{aligned}$$

Here, $\langle \mathcal{L}_1\psi\rangle$ can be computed by

$$\langle \mathcal{L}_1\psi\rangle = -u\sqrt{2}\left\langle \Lambda\psi'\right\rangle + u\sqrt{2}\rho\left\langle f\psi'\right\rangle x\frac{\partial}{\partial x},$$

and then, the correction order term $P_1(x,v)$ satisfies

$$\begin{aligned}\mathcal{L}_{\mathrm{TBS}}(\widetilde{\sigma}_f)P_1(x,v) &= -u\sqrt{2}\left\langle \Lambda\psi'\right\rangle\left(\partial_v P_0 + \frac{x^2}{2}\partial_{xx}P_0\right)\\ &\quad + u\sqrt{2}\rho\left\langle f\psi'\right\rangle\left(x\partial_{xv}P_0 + x^2\partial_{xx}P_0 + \frac{x^3}{2}\partial_{xxx}P_0\right)\\ &:= \mathcal{H}\end{aligned}$$

with the boundary condition $P_1(x,\mathbb{V}) = 0$. This completes the proof. □

Next, we obtain the explicit form solution of the leading-order price P_0 and the correction order price P_1.

Lemma 3.3. *The leading order term $P_0(x,v)$ is a solution of PDE problem* (3.3) *in Theorem 3.1 and is expressed by*

$$P_0(x,v) = x^c e^{\left((c-1)\left(r+\frac{c\widetilde{\sigma}_f^2}{2}\right)\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\right)}\mathcal{N}(d_1(x,v)) - Ke^{-r\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}}\mathcal{N}(d_2(x,v)), \qquad (3.5)$$

where

$$d_1(x,v) = \frac{\ln\left(\frac{x^c}{K}\right) + c\left(r + (c-\frac{1}{2})\widetilde{\sigma}_f^2\right)\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}}{c\sqrt{\mathbb{V}-v}},\quad d_2(x,v) = d_1(x,v) - c\sqrt{\mathbb{V}-v},$$

$$\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{\eta^2}{2}}\,\mathrm{d}\eta.$$

Proof. If we define the state variable ω as $\omega := \frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}$, by the chain rule, the first-order derivative of P_0 with respect to ω is $\frac{\partial P_0}{\partial v} = \frac{1}{\widetilde{\sigma}_f^2}\frac{\partial P_0}{\partial \omega}$. Next, if we apply the relationship to the homogeneous PDE (3.3), we have

$$r\left(x\frac{\partial P_0}{\partial x}(x,\omega) - P_0(x,\omega)\right) + \frac{\partial P_0}{\partial \xi}(x,\omega) + \frac{\widetilde{\sigma}_f^2}{2}x^2\frac{\partial^2 P_0}{\partial x^2}(x,\omega) = 0,\quad v<\mathbb{V} \qquad (3.6)$$

with $P_0(x,\mathbb{V}) = (x^c - K)^+$. Interestingly, since the variable ω functions as a time variable, equation (3.6) is equivalent to the Black–Scholes PDE with the power index. According to Ibrahim [18], the closed–form price of the power call option, denoted by $P_{\mathrm{PO}}(x,t)$, is expressed by

$$\begin{aligned}P_{\mathrm{PO}}(x,t) &= x^c e^{(c-1)(r+\frac{c\sigma^2}{2})(T-t)}\mathcal{N}(d_1(x,t)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x,t)),\\ d_1(x,t) &= \frac{\ln\left(\frac{x^c}{K}\right) + c\left(r+(c-\frac{1}{2})\sigma^2\right)(T-t)}{c\sqrt{T-t}},\quad d_2(x,t) = d_1(x,t) - c\sqrt{T-t}.\end{aligned}$$

by replacing the time to maturity $(T-t)$ and the constant volatility σ with ω and $\widetilde{\sigma}_f^2$, respectively, P_0 is changed by

$$P_0(x,\omega) = x^c e^{(c-1)\left(r+\frac{c\widetilde{\sigma}_f^2}{2}\right)\omega}\mathcal{N}(d_1(x,\omega)) - Ke^{-r\omega}\mathcal{N}(d_2(x,\omega)),$$

and from the definition of $\omega\,(=\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2})$ mentioned above, we obtain the desired results in (3.5).

□

Lemma 3.4. *The correction term* $P_1(x,v)$*, which is a solution of PDE problem* (3.4)*, is expressed by*

$$P_1(x,v) = -\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\mathcal{H} \tag{3.7}$$

Proof. Let us define $A_1 = -u\sqrt{2}\,\langle\Lambda\psi'\rangle$ and $A_2 = u\rho\sqrt{2}\,\langle f\psi'\rangle$. The the non-homogeneous source term $\mathcal{H}$ presented in Theorem 3.2 is rewritten by

$$\mathcal{H} = A_1\left(\partial_v P_0 + \frac{x^2}{2}\partial_{xx}P_0\right) + A_2\left(x\partial_{xv}P_0 + x^2\partial_{xx}P_0 + \frac{x^3}{2}\partial_{xxx}P_0\right)$$

Now, to obtain the solution of PDE in (3.4), we first compute $\mathcal{L}_{\mathrm{TBS}}\mathcal{H}$ as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{TBS}}\mathcal{H} =& A_1\left[\mathcal{L}_{\mathrm{TBS}}\left(\frac{\partial P_0}{\partial v}\right) + \mathcal{L}_{\mathrm{TBS}}\left(\frac{x^2}{2}\frac{\partial^2 P_0}{\partial x^2}\right)\right] \\ &+ A_2\left[\mathcal{L}_{\mathrm{TBS}}\left(x\frac{\partial^2 P_0}{\partial x\partial v}\right) + \mathcal{L}_{\mathrm{TBS}}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right) + \mathcal{L}_{\mathrm{TBS}}\left(\frac{x^3}{2}\frac{\partial^3 P_0}{\partial x^3}\right)\right].\end{aligned}$$

Then, by using the commuting property mentioned in Fouque et al. [4], in (3.5), $\mathcal{L}_{\mathrm{TBS}}\mathcal{H} = 0$ holds. Next, the relation

$$\begin{aligned}\mathcal{L}_{\mathrm{TBS}}\left(-\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\mathcal{H}\right) =& rx\left(-\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\right)\frac{\partial\mathcal{H}}{\partial x} + \mathcal{H} - (\mathbb{V}-v)\frac{\partial\mathcal{H}}{\partial v} \\ &- (\mathbb{V}-v)\frac{x^2}{2}\frac{\partial^2\mathcal{H}}{\partial x^2} - r\left(-\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\mathcal{H}\right) \\ =& \mathcal{H} - \frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\mathcal{L}_{\mathrm{TBS}}\mathcal{H} \\ =& \mathcal{H}\ \ (\text{by using } \mathcal{L}_{\mathrm{TBS}}\mathcal{H} = 0)\end{aligned}$$

is satisfied. Therefore, $-\frac{\mathbb{V}-v}{\widetilde{\sigma}^2}\mathcal{H}$ is a solution of the PDE (3.4), and then the correction term $P_1(x,v)$ is expressed by

$$P_1(x,v) = -\frac{\mathbb{V}-v}{\widetilde{\sigma}_f^2}\mathcal{H}.$$

□

Now, if we unite the findings of leading-order term in (3.5) and the correction order term in (3.7), then we derive the first–order approximation of TPO P, denoted by $\widetilde{P}^{\epsilon}$, represented by

$$\widetilde{P}^{\epsilon} := P_0 + \sqrt{\epsilon}P_1. \tag{3.8}$$

The proof of the accuracy approximation price given in (3.10) is given by Fouque et al. [4]. If the payoff function h is continuously differentiable, then the pricing accuracy of the first-order approximation is described by

$$\left|P - (P_0 + \sqrt{\epsilon}P_1)\right| \leq \mathcal{O}(\epsilon).$$

Furthermore, if the payoff function h is not continuously differentiable at the strike price, i.e., h is not a smooth function, we can verify the error estimation by the regularization procedure, which has to take account of the pointwise accuracy of the approximated option price due to fact that the payoff function has an angle at the strike price (cf. Fouque et al. [4]). By using the payoff regularization, we can extend the accuracy of the first-order approximated price given in (3.8) by replacing the maturity T by $T - z$. Then, the error estimate of the first–order approximation in (3.8) is expressed by

$$\left|P - (P_0 + \sqrt{\epsilon}P_1)\right| \leq \mathcal{O}(\epsilon \ln \epsilon).$$

In next section, we provide a numerical experiments of the price approximation through the Monte–Carlo simulation instead of providing the detailed proof of it theoretically (Refer to Table 1 in Section 4).

4. Implications

In this section, we numerically demonstrate the accuracy of the pricing formula of TPO by comparing the analytic–closed form solution from the asymptotic analysis with the solution obtained from the Monte–Carlo simulation. In addition, we analyze the price behaviors of the TPO with regard to several parameters.

In Table 1, we present the values of the Monte-Carlo price P_{MC} and the TPO $\widetilde{P}^{\epsilon}$ to investigate the pricing accuracy of our approximation solution $\widetilde{P}^{\epsilon}$. Referring to Saunders [19], Bernard and Cui [7] and Kim et al. [17], the selected parameters for the Monte–Carlo simulation are expressed by

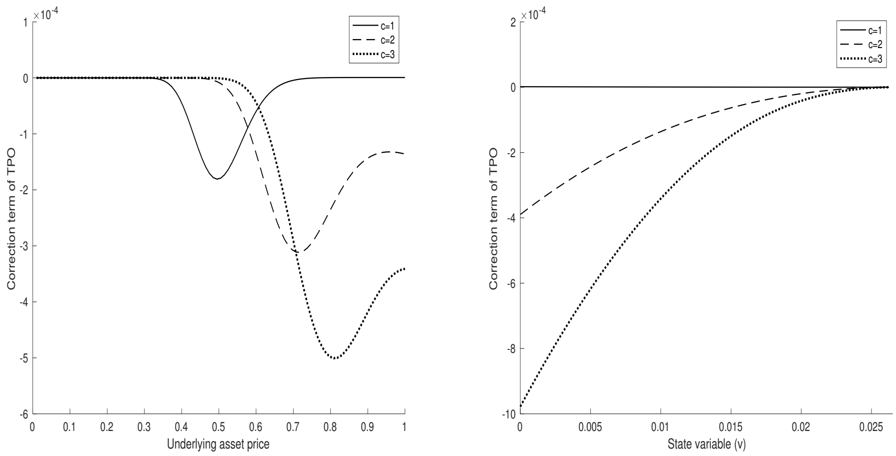
$$X_0 = 1, V_0 = 1, K = 0.7, \mathbb{V} = 0.02, v = 0.01, \widehat{\sigma} = 0.1, u = \sqrt{2}, c = 2, r = 0.01,$$
$$\rho = 0.2, \langle \Lambda\psi' \rangle = -0.006, \langle f\psi' \rangle = 0.003, m = \log(0.1).$$

Monte–Carlo simulation is conducted by 50,000 paths and all the computations are performed in an Apple M1 and 8 GB memory. In the table, we can notice that the difference between the Monte-Carlo price P_{MC} and the TPO price $\widetilde{P}^{\epsilon}$, expressed by $|P_{\mathrm{MC}} - \widetilde{P}^{\epsilon}|$, goes to zero as the parameter ϵ decreases. Next, $\mathrm{RE}\,[\%] = \frac{|P_{\mathrm{MC}} - \widetilde{P}^{\epsilon}|}{P_{\mathrm{MC}}} \times 100$ refers to a percentage of a relative error between the Monte–Carlo price P_{MC} and the TPO price $\widetilde{P}^{\epsilon}$. Then, the values of RE approaches to zero rapidly if the parameter ϵ gets smaller. It implies that the approximated TPO prices

TABLE 1. The error comparison between the Monte–Carlo price (P_{MC}) and the VTO price ($\widetilde{P}^{\epsilon}$) with respect to ϵ. Referring to Saunders [19], Bernard and Cui [7]. and Kim et al [17], we used parameters as follows: $X_0 = 1, V_0 = 1, K = 0.7, \mathbb{V} = 0.02, v = 0.01, \widehat{\sigma} = 0.1, u = \sqrt{2}, c = 2, r = 0.01, \rho = 0.2, \langle \Lambda\psi' \rangle = -0.006, \langle f\psi' \rangle = 0.003$, and $m = \log(0.1)$.

ϵ	P_{MC}	$\widetilde{P}^{\epsilon}$	$\lvert P_{\mathrm{MC}} - \widetilde{P}^{\epsilon} \rvert$	RE [%]
0.1	0.358419	0.335476	0.022943	6.401164
0.07	0.341780	0.334396	0.007384	2.160512
0.03	0.336031	0.332485	0.003545	1.055088
0.007	0.332248	0.330613	0.001634	0.491943
0.003	**0.331218**	**0.330009**	**0.001209**	**0.364966**

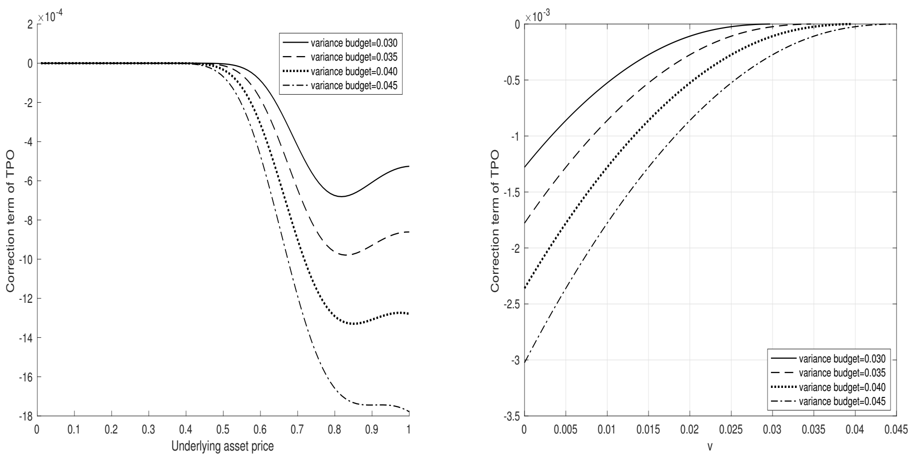
comes close to the Monte–Carlo price, which may be considered as the good approximation of real solution, and then our solution is derived accurately.



(a) Correction term of TPO with $v = 0.01$ (b) Correction term of TPO with $x = 1$

FIGURE 1. The impact of index c on the correction term of timer power option (TPO). The used parameters are as follows: $r = 0.01$, $u = 0.1$, $\langle \Lambda\psi' \rangle = 0.006$, $\langle f\psi' \rangle = 0.003$, $\rho = -0.3$, $K = 0.5$, $\mathbb{V} = 0.0265$, $\widehat{\sigma} = 0.1$, $\epsilon = 0.003$

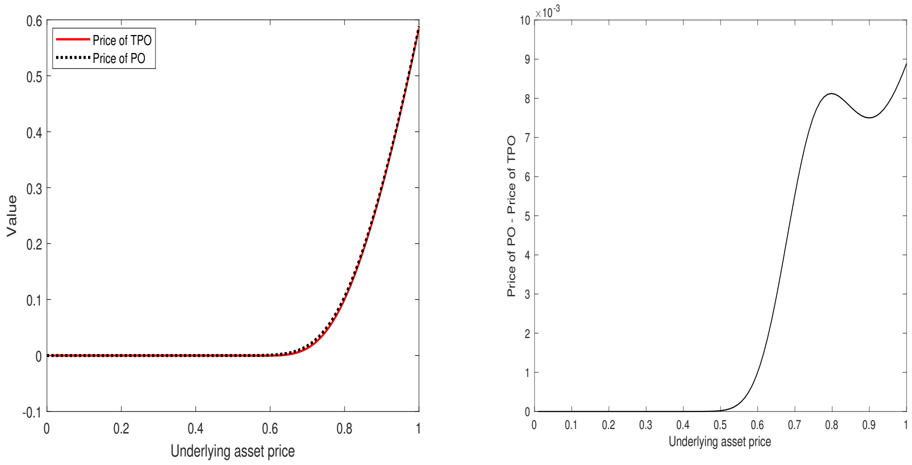
Figure 1 describes the effect of the index c for the underlying asset price or the realized variance on the correction term of TPO. Figure 1(a) displays the influence of the power c on the correction term with respect to the underlying asset price. It can be seen that the sensitivity of the correction term becomes significant as the power value c increases, displaying a bigger hump phenomenon. Next, Figure 1(b) presents the influence of the power c on the correction term with respect to the state variable v. In the figure, if c gets larger, the curve of the correction term tends to be an increasing function and be more sensitive. Also, as the state variable v approaches the predetermined variance budget $\mathbb{V}$, the price of the corrections term converges to zero regardless of the choice of c. Therefore, from the both of figures, it can be observed that the index c in the payoff (2.3) is regarded as a very significant parameter and the impact of the parameter on the TPO is very crucial.



(a) Correction term of TPO with $v = 0.01$ (b) Correction term of TPO with $x = 1$

FIGURE 2. Correction terms for TPO in terms of the underlying asset price and state variable for different variance budgets. The selected parameters are given by $r = 0.01$, $u = 0.1$, $\langle \Lambda\psi' \rangle = 0.006$, $\langle f\psi' \rangle = 0.003$, $\rho = -0.3$, $K = 0.5$, $\widehat{\sigma} = 0.1$, $c = 3$, and $\epsilon = 0.003$

Figure 2 exhibits the behavior of the correction terms for TPO in terms of the underlying asset price or state variable for given different variance budgets. In Figure 2(a), the sensitivity of the correction price becomes larger as the variance budget gets bigger in the region of ITM (In-The-Money) region. Here, the variance budget in the timer-options functions like the maturity in European vanllia option. The larger variance budget, the greater the effect of SV in the domain of ITM, and therefore it implied that, for the area of ITM, the influence of SV for TPO is more significant as the variance budget is on the rise. In addition, in Figure 2(a), one can see that the hump phenomenon has a tendency to happen in the region of ITM, showing the larger the variance budget $\mathbb{V}$, the bigger the width of the hump. Next, Figure 2(b) shows that the graph of the correction term price is more sensitive to the state variable v for greater variance budget $\mathbb{V}$, and the price change is almost zero as the state variable gets close to the variance budget.



(a) (b)

FIGURE 3. (a) Pricing impacts of PO and TPO; (b) Pricing differences of PO and TPO. The selected parameters are as follows: $r = 0.01$, $u = 0.1$, $\langle \Lambda\psi' \rangle = 0.006$, $\langle f\psi' \rangle = 0.003$, $\rho = -0.3$, $K = 0.5$, $\mathbb{V} = 0.0625$, $\widehat{\sigma} = 0.1$, $c = 3$, and $\epsilon = 0.003$.

Figure 3 exhibits the price change for the TPO with respect to the underlying asset price, comparing it with that of the standard European power option (PO) given by Ibrahim [18]. In Figure 3(a), we investigate the price difference between the PO and the TPO against the underlying asset price. In the figure, the PO and the TPO prices increase as the underlying asset price grows, and the value of the TPO tends to be less than that of PO. It implies that the price of TPO may be underpriced compared with the price of PO, especially, in the ITM region, verifying the results of Sawyer [6]. Moreover, in Figure 3(b), we provide the price gap between the PO and the TPO graphically. The figure suggests that the difference of two value has a tendency to get largers rapidly as the underlying asset price increases.

5. CONCLUSION

In this study, we obtain the fair price of timer power options (TPO) under a generalized stochastic volatility (SV). The TPO is a kind of derivatives considering the index of the power to the underlying asset on the standard timer option, and it allows the investors to obtain the stable profit under the fluctuations of the sudden and unexpected market, such as the global financial crisis and the COVID–19 pandemic. We derive the first-order approximation for the TPO by using the asymptotic analysis, and verify that our derived solution is accurately found by comparing it with Monte–Carlo price. Finally, we investigate the impact of the SV against various model parameters, including the value of the power to the underlying asset, on the TPO.

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