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LARGE TIME BEHAVIOR TO THE 2D MICROPOLAR BOUSSINESQ FLUIDS

XUEWEN WANG[1,2], KEKE LEI[1,2,†], AND PIGONG HAN[1,2]

[1] ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, CHINA

[2] SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, CHINA
Email address: {xwwang,†leikeke,pghan}@amss.ac.cn

ABSTRACT. In this paper, we prove the global existence of classical solutions to the 2D incompressible Boussinesq equations for the micropolar fluid. Furthermore, applying the Fourier splitting methods, we obtain the lager time decay properties.

1. INTRODUCTION

The standard incompressible Boussinesq equations for the micropolar fluid in $\mathbb{R}^n$ $(n = 2, 3)$ can be written as

$$\begin{cases} u_t + (u \cdot \nabla)u - (\mu + \chi)\Delta u + \nabla \pi = 2\chi \text{curl } w + \theta e_n, \\ w_t + (u \cdot \nabla)w - \gamma \Delta w + (n-1)\kappa \nabla \text{div } w + 4\chi w = 2\chi \nabla^{\perp} u, \\ \theta_t + (u \cdot \nabla)\theta - \eta \Delta \theta = 0, \\ \text{div } u = 0, \\ (u, w, \theta)(x, 0) = (u_0, w_0, \theta_0)(x), \end{cases} \tag{1.1}$$

where u, w, θ denote flowed velocity, micro-rotational velocity, temperature, respectively. π is the pressure, and $e_n = (0, \cdots, 0, 1)$. μ, χ, η denote kinematic viscosity, vortex viscosity, thermal diffusivity, respectively. γ and κ are spin viscosities.

When the temperature field is considered, problem (1.1) describes a motion of the micropolar fluid under the framework of the Boussinesq approximation, which is closely related to many classical equations. When the temperature is neglected ($\theta = 0$), the system (1.1) reduces to the incompressible micropolar equations. The global well-posedness for the two-dimensional case and for the 3D regularity results are considered in [1, 2, 3]. If $w = 0$, (1.1) becomes the incompressible Boussinesq system, and there are many important results on the Boussinesq system with the full or partial dissipation, please see [4, 5, 6] for details.

The optimal decay rates of weak solutions to the incompressible Navier-Stokes equations were first obtained by M. E. Schonbek [7] via the Fourier splitting method. For the incompressible micropolar equations, R. H. Guterres, J. R. Nunes, C. F. Perusato [8] established the

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† Corresponding author.

large time decay for Leray solutions, that is,

$$\lim_{t\to\infty} \|(u,w)(t)\|_{L^2(\mathbb{R}^n)} = 0, \qquad \lim_{t\to\infty} t^{\frac{1}{2}} \|w(t)\|_{L^2(\mathbb{R}^n)} = 0, \qquad n = 2, 3.$$

When $\theta = 0$ and $\gamma = 0$ in (1.1), the decay rates of solutions for the two and three dimensional cases were given in [9, 10], respectively. For the incompressible Boussinesq equations, the optimal large time decay rates of weak solutions in $\mathbb{R}^3$ were obtained by L. Brandolese and M. E. Schonbek [11]. More decay properties of related models are referred to [12, 13, 14, 15, 16] and the references therein.

In this paper, we consider the following variant of (1.1) with the damping term in $\mathbb{R}^2 \times \mathbb{R}_+$,

$$\begin{cases} u_t + (u\cdot\nabla)u - (\mu+\chi)\Delta u + \nabla\pi = 2\chi \text{curl } w + \theta e_2, \\ w_t + (u\cdot\nabla)w - \gamma\Delta w + 4\chi w = 2\chi\nabla^{\perp}u, \\ \theta_t + (u\cdot\nabla)\theta + \sigma\theta = 0, \\ \text{div } u = 0, \\ (u,w,\theta)(x,0) = (u_0,w_0,\theta_0)(x), \end{cases} \tag{1.2}$$

where $\sigma > 0$ is the damping coefficient, and $\nabla^{\perp}u = \partial_1 u_2 - \partial_2 u_1$, $\text{div } u = \partial_1 u_1 + \partial_2 u_2$, $\text{curl } w = (\partial_2 w, -\partial_1 w)$.

Firstly, we define a weak solution of the problem (1.2).

Definition 1.1. *Let $u_0, w_0, \theta_0 \in L^2(\mathbb{R}^2)$. We say that (u,w,θ) is a weak solution for problem* (1.2)*, if for every $T > 0$,*

$$u, w \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;H^1(\mathbb{R}^2)), \;\; \theta \in L^\infty(0,T;L^2(\mathbb{R}^2)),$$

satisfies (1.2) *in the sense of distribution. Moreover, the corresponding energy inequalities hold.*

Our first result is on the global existence of the strong solution for (1.2), which is stated as follows.

Theorem 1.2. *Let $u_0, w_0, \theta_0 \in H^s(\mathbb{R}^2)$, $s > 2$,* $\text{div } u_0 = 0$*, $T > 0$. Then there exists a unique global solution (u,w,θ) to* (1.2)*, which satisfies*

$$u, w \in L^\infty(0,T;H^s(\mathbb{R}^2)) \cap L^2(0,T;H^{s+1}(\mathbb{R}^2)), \;\; \theta \in L^\infty(0,T;H^s(\mathbb{R}^2)).$$

Next, we give the second main result of the present paper as follows.

Theorem 1.3. *Let $u_0, w_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ satisfy* $\text{div } u_0 = 0$*. Then, for any $t > 0$,*

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}},$$
$$\|\nabla u(t)\|_{L^2} + \|w(t)\|_{H^1} \le C(1+t)^{-1},$$

where $C = C(\mu,\chi,\gamma,\kappa,\sigma,\|u_0\|_{L^1\cap H^1},\|w_0\|_{L^1\cap H^1},\|\theta_0\|_{L^1\cap L^2})$.

Remark 1.4. *Due to lack of a dissipation term $\Delta\theta$ for* (1.2)*, we are not able to derive the H^1 decay estimates for θ.*

The structure of this paper is organized as follows. In Section 2, we list some preliminary lemmas, which are necessary in our argument. Section 3 devotes to the establishment of the global existence of strong solutions for (1.2) (i.e., Theorem 1.2). In Section 4, the H^1 decay rates of (u,w) are obtained.

2. Some preliminary lemmas

In this section, we give some important lemmas, which are necessary to our analysis in the next sections.

Lemma 2.1. *Let* $0 < T \le \infty$, $y \in W^{1,1}(0,T)$, $y \ge 0$, *which satisfies*

$$y'(t) + v(t) \le g(t)y(t) + h(t), \tag{2.1}$$

where $v, g, h \in L^1(0,T)$, $v, g, h \ge 0$. *Then*

$$y(t) + \int_0^t v(s)ds \le \left[y(0) + \int_0^t h(s)ds\right] e^{\int_0^t g(s)ds}.$$

Proof. Multiplying (2.1) both sides by $e^{-\int_0^t g(\tau)d\tau}$, we obtain

$$\frac{d}{dt}\left[y(t)e^{-\int_0^t g(\tau)d\tau}\right] + v(t)e^{-\int_0^t g(\tau)d\tau} \le h(t)e^{-\int_0^t g(\tau)d\tau}.$$

Integrating on $[0,t]$ gives

$$y(t)e^{-\int_0^t g(\tau)d\tau} + \int_0^t v(s)e^{-\int_0^s g(\tau)d\tau}ds \le y(0) + \int_0^t h(s)e^{-\int_0^s g(\tau)d\tau}ds,$$

from which, we derive

$$\begin{aligned} y(t) + \int_0^t v(s)e^{\int_s^t g(\tau)d\tau}ds &\le y(0)e^{\int_0^t g(\tau)d\tau} + \int_0^t h(s)e^{\int_s^t g(\tau)d\tau}ds \\ &\le \left[y(0) + \int_0^t h(s)ds\right]e^{\int_0^t g(\tau)d\tau}. \end{aligned}$$

Then,

$$y(t) + \int_0^t v(s)ds \le y(t) + \int_0^t v(s)e^{\int_s^t g(\tau)d\tau}ds \le \left[y(0) + \int_0^t h(s)ds\right]e^{\int_0^t g(s)ds}.$$

□

The second result is on the commutator estimates, which is given in [17].

Lemma 2.2. *Let* $f, g \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $s > 0$, $\operatorname{div} g = 0$, *and* $\nabla g \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Define* $\Lambda^s = (-\Delta)^{s/2}$ *and the commutator*

$$[\Lambda^s, g\cdot\nabla]f = \Lambda^s[(g\cdot\nabla)f] - (g\cdot\nabla)\Lambda^s f.$$

Then,

$$\|[\Lambda^s, g\cdot\nabla]f\|_{L^2} \le C(\|\nabla g\|_{L^\infty}\|f\|_{H^s} + \|\nabla g\|_{H^s}\|f\|_{L^\infty}).$$

Furthermore, if $\nabla f \in L^\infty(\mathbb{R}^2)$, *there holds*

$$\|[\Lambda^s, g\cdot\nabla]f\|_{L^2} \le C(\|\nabla g\|_{L^\infty}\|f\|_{H^s} + \|g\|_{H^s}\|\nabla f\|_{L^\infty}),$$

where C *is independent of* f *and* g.

Next, a well known heat semigroup estimate is presented, which can be found in [18].

Lemma 2.3. *Let $n \geq 1$, $u \in L^p(\mathbb{R}^n)$ and $(e^{\nu t\Delta})_{t\geq 0}$ is the heat semigroup. Given any $1 \leq p \leq q \leq \infty$ and any multi-index $\alpha \in \mathbb{N}^n$, we have*

$$\|D^\alpha e^{\nu t\Delta} u\|_{L^q} \leq K\|u\|_{L^p} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}}, \quad \forall\, t > 0,$$

where $K \in \mathbb{R}^+$ only depends on n, ν, p, q, and $|\alpha|$.

Finally, we state an elementary inequality.

Lemma 2.4. *Let $\beta \geq 0$ and $\lambda > 0$, then for $\forall\, t > 0$, we have*

$$\int_0^t e^{-\lambda(t-s)}(1+s)^{-\beta} ds \leq C(1+t)^{-\beta}.$$

Proof. Since $(1+t-s)^a e^{-\lambda(t-s)} \leq C$, $a > 1$, we have

$$\begin{aligned}
\int_0^t e^{-\lambda(t-s)}(1+s)^{-\beta} ds &\leq C\int_0^t (1+t-s)^{-a}(1+s)^{-\beta} ds \\
&= C\int_0^t (1+s)^{-a}(1+t-s)^{-\beta} ds \\
&\leq C\left(1+\frac{t}{2}\right)^{-\beta}\int_0^{\frac{t}{2}} (1+s)^{-a} ds + C\left(1+\frac{t}{2}\right)^{-a}\int_{\frac{t}{2}}^t (1+t-s)^{-\beta} ds \\
&\leq C(1+t)^{-\beta} + C(1+t)^{-a}\int_{\frac{t}{2}}^t (1+t-s)^{-\beta} ds \\
&:= C(1+t)^{-\beta} + I.
\end{aligned}$$

Next, we estimate the second term I. If $\beta > 1$,

$$I \leq C(1+t)^{-a}\int_{\frac{t}{2}}^t (1+t-s)^{-\beta} ds \leq C(1+t)^{-a}\left[1-\left(1+\frac{t}{2}\right)^{1-\beta}\right] \leq C(1+t)^{-a} \leq C(1+t)^{-\beta},$$

where $\beta \leq a$. If $\beta < 1$,

$$I \leq C(1+t)^{-a}\int_{\frac{t}{2}}^t (1+t-s)^{-\beta} ds \leq C(1+t)^{-a}\left[\left(1+\frac{t}{2}\right)^{1-\beta}-1\right] \leq C(1+t)^{1-a-\beta} \leq C(1+t)^{-\beta}.$$

If $\beta = 1$,

$$I \leq C(1+t)^{-a}\int_{\frac{t}{2}}^t (1+t-s)^{-1} ds \leq C(1+t)^{-a}\ln\left(1+\frac{t}{2}\right) \leq C(1+t)^{-\beta},$$

where $\beta < a$. Hence,

$$\int_0^t e^{-\lambda(t-s)}(1+s)^{-\beta} ds \leq C(1+t)^{-\beta}, \quad \forall\, \beta \geq 0.$$

□

3. EXISTENCE OF THE STRONG SOLUTION FOR (1.2)

To prove Theorem 1.2, we first establish the following $a\ priori$ estimates for system (1.2).

Lemma 3.1. *Let* $u_0, w_0 \in H^1(\mathbb{R}^2)$, $\theta_0 \in L^p(\mathbb{R}^2)$ $(1 \le p \le \infty)$, $\operatorname{div} u_0 = 0$. *Assume* (u, w, θ) *satisfies* (1.2). *Then,*

$$\|\theta(t)\|_{L^p} = e^{-\sigma t}\|\theta_0\|_{L^p}, \quad \forall\, t > 0.$$

Proof. Multiplying both sides of $(1.2)_3$ by $|\theta|^{p-2}\theta$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta\|_{L^p}^p + \sigma\|\theta\|_{L^p}^p = -\int_{\mathbb{R}^2}[(u\cdot\nabla)\theta]|\theta|^{p-2}\theta dx = -\frac{1}{p}\int_{\mathbb{R}^2} u\cdot\nabla|\theta|^p dx = 0.$$

Then,

$$\frac{d}{dt}\|\theta\|_{L^p}^p + \sigma p\|\theta\|_{L^p}^p = 0.$$

Multiplying by $e^{\sigma p t}$ and integrating from 0 to t, we deduce

$$\|\theta(t)\|_{L^p} = e^{-\sigma t}\|\theta_0\|_{L^p}. \tag{3.1}$$

□

Lemma 3.2. *Suppose* $u_0, w_0 \in H^1(\mathbb{R}^2)$, $\theta_0 \in L^2(\mathbb{R}^2)$ *satisfy* $\operatorname{div} u_0 = 0$. *Let* (u, w, θ) *be a smooth solution of system* (1.2). *Then for every* $T > 0$

$$u, w \in L^\infty(0,T; H^1(\mathbb{R}^2)) \cap L^2(0,T; H^2(\mathbb{R}^2)), \quad \theta \in L^\infty(0,T; L^2(\mathbb{R}^2)),$$

and

$$\sup_{t\in[0,T]}(\|u(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 + \|\theta(t)\|_{L^2}^2) + \int_0^T(\|\nabla u\|_{H^1}^2 + \|\nabla w\|_{H^1}^2 + \|\theta\|_{L^2}^2)dt \le C,$$

where $C = C(\mu, \chi, \gamma, \kappa, \sigma, \|u_0\|_{H^1}, \|w_0\|_{H^1}, \|\theta_0\|_{L^2})$.

Proof. Taking the L^2-inner product of $(1.2)_1$ with u and $(1.2)_2$ with w, and adding these results together, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + (\mu+\chi)\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 + 4\chi\|w\|_{L^2}^2 \\
&= \int_{\mathbb{R}^2} 2\chi \operatorname{curl} w\cdot u dx + \int_{\mathbb{R}^2} 2\chi\nabla^\perp u\cdot w dx + \int_{\mathbb{R}^2}\theta e_2\cdot u dx \\
&= 4\chi\int_{\mathbb{R}^2}\nabla^\perp u\cdot w dx + \int_{\mathbb{R}^2}\theta e_2\cdot u dx \\
&\le \frac{\mu+2\chi}{2}\|\nabla u\|_{L^2}^2 + \frac{8\chi^2}{\mu+2\chi}\|w\|_{L^2}^2 + \|\theta\|_{L^2}\|u\|_{L^2},
\end{aligned}$$

where we used

$$\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot u dx = \int_{\mathbb{R}^2}(u\cdot\nabla)w\cdot w dx = 0,$$

and

$$\int_{\mathbb{R}^2}\operatorname{curl} w\cdot u dx = \int_{\mathbb{R}^2}\nabla^\perp u\cdot w dx.$$

Then,

$$\frac{d}{dt}(\|u\|_{L^2}^2+\|w\|_{L^2}^2)+\mu\|\nabla u\|_{L^2}^2+2\gamma\|\nabla w\|_{L^2}^2+\frac{8\chi\mu}{\mu+2\chi}\|w\|_{L^2}^2 \\ \leq 2\|\theta\|_{L^2}\|u\|_{L^2}\leq \|\theta\|_{L^2}(\|u\|_{L^2}^2+1)\leq e^{-\sigma t}\|\theta_0\|_{L^2}(\|u\|_{L^2}^2+1).$$

Setting $c_1=\min\{\mu,2\gamma,\frac{8\chi\mu}{\mu+2\chi}\}$, we deduce

$$\frac{d}{dt}(\|u\|_{L^2}^2+\|w\|_{L^2}^2)+c_1(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|w\|_{L^2}^2)\leq e^{-\sigma t}\|\theta_0\|_{L^2}(\|u\|_{L^2}^2+1). \quad (3.2)$$

Then, using Lemma 2.1, we derive

$$\sup_{t\in[0,T]}(\|u\|_{L^2}^2+\|w\|_{L^2}^2)+c_1\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)dt\leq (\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2+1)e^{\frac{1}{\sigma}\|\theta_0\|_{L^2}}. \quad (3.3)$$

Multiplying $(1.2)_1$, $(1.2)_2$ by $-\Delta u$, $-\Delta w$, respectively, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+(\mu+\chi)\|\Delta u\|_{L^2}^2+\gamma\|\Delta w\|_{L^2}^2+4\chi\|\nabla w\|_{L^2}^2\\
=&-4\chi\int_{\mathbb{R}^2}\text{curl } w\cdot\Delta u dx+\int_{\mathbb{R}^2}(u\cdot\nabla)u\cdot\Delta u dx+\int_{\mathbb{R}^2}(u\cdot\nabla)w\cdot\Delta w dx-\int_{\mathbb{R}^2}\theta e_2\cdot\Delta u dx\\
\leq&4\chi\|\nabla w\|_{L^2}\|\Delta u\|_{L^2}+\|(u\cdot\nabla)u\|_{L^2}\|\Delta u\|_{L^2}+\|(u\cdot\nabla)w\|_{L^2}\|\Delta w\|_{L^2}+\|\theta\|_{L^2}\|\Delta u\|_{L^2}\\
\leq&\frac{\mu+2\chi}{2}\|\Delta u\|_{L^2}^2+\frac{8\chi^2}{\mu+2\chi}\|\nabla w\|_{L^2}^2+\frac{\mu}{8}\|\Delta u\|_{L^2}^2+\frac{\gamma}{4}\|\Delta w\|_{L^2}^2\\
&\quad+C\|(u\cdot\nabla)u\|_{L^2}^2+C\|(u\cdot\nabla)w\|_{L^2}^2+C\|\theta\|_{L^2}^2.
\end{aligned}$$

Using Gagliardo-Nirenberg inequality yields

$$C\|(u\cdot\nabla)u\|_{L^2}^2\leq C\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2\leq C\|u\|_{L^2}\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}\leq\frac{\mu}{8}\|\Delta u\|_{L^2}^2+C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4,$$

and

$$\begin{aligned}
C\|(u\cdot\nabla)w\|_{L^2}^2\leq C\|u\|_{L^4}^2\|\nabla w\|_{L^4}^2&\leq C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}\|\Delta w\|_{L^2}\\
&\leq\frac{\gamma}{4}\|\Delta w\|_{L^2}^2+C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+\frac{\mu}{4}\|\Delta u\|_{L^2}^2+\frac{\gamma}{2}\|\Delta w\|_{L^2}^2+\frac{4\mu\chi}{\mu+2\chi}\|\nabla w\|_{L^2}^2\\
\leq&C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+C\|\theta\|_{L^2}^2.
\end{aligned}$$

Letting $c_2=\min\{\frac{\mu}{2},\gamma,\frac{8\mu\chi}{\mu+2\chi}\}$, we have

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+c_2(\|\Delta u\|_{L^2}^2+\|\Delta w\|_{L^2}^2+\|\nabla w\|_{L^2}^2) \\ \leq C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+Ce^{-2\sigma t}\|\theta_0\|_{L^2}^2. \quad (3.4)$$

Then,

$$\sup_{t\in[0,T]}(\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta w\|_{L^2}^2+\|\nabla w\|_{L^2}^2)dt\le C, \tag{3.5}$$

where we used (3.3) and Lemma 2.1. Combining (3.1), (3.3) and (3.5), we have

$$\sup_{t\in[0,T]}(\|u(t)\|_{H^1}^2+\|w(t)\|_{H^1}^2+\|\theta(t)\|_{L^2}^2)+\int_0^T(\|\nabla u\|_{H^1}^2+\|\nabla w\|_{H^1}^2+\|\theta\|_{L^2}^2)dt\le C.$$

□

Next, we establish the global L^r and $W^{1,r}$-estimates for $\Omega=\nabla^\perp u$ and ∇w. Taking $\nabla^\perp$ on $(1.2)_1$ and ∇ on $(1.2)_2$, we have

$$\begin{cases}\partial_t\Omega+(u\cdot\nabla)\Omega-(\mu+\chi)\Delta\Omega=-2\chi\Delta w+\partial_1\theta,\\ \partial_t\nabla w+\nabla(u\cdot\nabla w)-\gamma\nabla\Delta w+4\chi\nabla w=2\chi\nabla\Omega.\end{cases} \tag{3.6}$$

Lemma 3.3. *Under the assumptions of Lemma 3.2, if $u_0,w_0\in W^{1,r}(\mathbb{R}^2)$, $\theta_0\in L^r(\mathbb{R}^2)$ for $2<r<\infty$, then*

$$\sup_{t\in[0,T]}(\|\Omega\|_{L^r}+\|\nabla w\|_{L^r})\le C,$$

and

$$\sup_{t\in[0,T]}(\|u(t)\|_{L^\infty}+\|w(t)\|_{L^\infty})\le C,$$

where $C=C(T,r,\mu,\chi,\gamma,\kappa,\sigma,\|u_0\|_{W^{1,r}},\|w_0\|_{W^{1,r}},\|\theta_0\|_{L^r})$.

Proof. Multiplying $(3.6)_1$ by $|\Omega|^{r-2}\Omega$, integrating over $\mathbb{R}^2$ by parts, using Young's inequality and Hölder inequality, we obtain

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^2}|\Omega|^r dx+(r-1)(\mu+\chi)\int_{\mathbb{R}^2}|\nabla\Omega|^2|\Omega|^{r-2}dx\\
=&-2\chi\int_{\mathbb{R}^2}(\Delta w)|\Omega|^{r-2}\Omega dx+\int_{\mathbb{R}^2}(\partial_1\theta)|\Omega|^{r-2}\Omega dx\\
\le&2(r-1)\chi\int_{\mathbb{R}^2}|\nabla w||\nabla\Omega||\Omega|^{r-2}dx+(r-1)\int_{\mathbb{R}^2}|\theta||\nabla\Omega||\Omega|^{r-2}dx\\
\le&\frac{(r-1)(\mu+\chi)}{2}\int_{\mathbb{R}^2}|\nabla\Omega|^2|\Omega|^{r-2}dx+C\int_{\mathbb{R}^2}|\nabla w|^2|\Omega|^{r-2}dx+C\int_{\mathbb{R}^2}|\theta|^2|\Omega|^{r-2}dx\\
\le&\frac{(r-1)(\mu+\chi)}{2}\int_{\mathbb{R}^2}|\nabla\Omega|^2|\Omega|^{r-2}dx+C\|\nabla w\|_{L^r}^2\|\Omega\|_{L^r}^{r-2}+C\|\theta\|_{L^r}^2\|\Omega\|_{L^r}^{r-2}.\end{aligned}$$

By (3.1), we have

$$\frac{d}{dt}\|\Omega\|_{L^r}^r\le C\|\nabla w\|_{L^r}^2\|\Omega\|_{L^r}^{r-2}+Ce^{-2\sigma t}\|\theta_0\|_{L^r}^2\|\Omega\|_{L^r}^{r-2}.$$

Dividing both sides by $\|\Omega\|_{L^r}^{r-2}$, we get

$$\frac{d}{dt}\|\Omega\|_{L^r}^2\le C\|\nabla w\|_{L^r}^2+Ce^{-2\sigma t}\|\theta_0\|_{L^r}^2. \tag{3.7}$$

Multiplying $(3.6)_2$ by $|\nabla w|^{r-2}(\nabla w)$, we derive

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla w|^r dx+(r-1)\gamma\int_{\mathbb{R}^2}|\Delta w|^2|\nabla w|^{r-2}dx+4\chi\int_{\mathbb{R}^2}|\nabla w|^r dx\\
=&-\int_{\mathbb{R}^2}\nabla(u\cdot\nabla w)|\nabla w|^{r-2}(\nabla w)dx+2\chi\int_{\mathbb{R}^2}(\nabla\Omega)|\nabla w|^{r-2}(\nabla w)dx\\
\leq&(r-1)\int_{\mathbb{R}^2}|u||\nabla w||\Delta w||\nabla w|^{r-2}dx+2(r-1)\chi\int_{\mathbb{R}^2}|\Omega||\Delta w||\nabla w|^{r-2}dx\\
\leq&\frac{(r-1)\gamma}{2}\int_{\mathbb{R}^2}|\Delta w|^2|\nabla w|^{r-2}dx+C\int_{\mathbb{R}^2}|u|^2|\nabla w|^2|\nabla w|^{r-2}dx+C\int_{\mathbb{R}^2}|\Omega|^2|\nabla w|^{r-2}dx\\
\leq&\frac{(r-1)\gamma}{2}\int_{\mathbb{R}^2}|\Delta w|^2|\nabla w|^{r-2}dx+C\|u\|_{L^\infty}^2\|\nabla w\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}+C\|\Omega\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}\\
\leq&\frac{(r-1)\gamma}{2}\int_{\mathbb{R}^2}|\Delta w|^2|\nabla w|^{r-2}dx+C\|u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla w\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}+C\|\Omega\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{dt}\|\nabla w\|_{L^r}^r&\leq C\|u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla w\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}+C\|\Omega\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}\\
&\leq C(\|u\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\|\nabla w\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}+C\|\Omega\|_{L^r}^2\|\nabla w\|_{L^r}^{r-2}.
\end{aligned}$$

Similar to the proof of (3.7), using Lemma 3.2, we have

$$\frac{d}{dt}\|\nabla w\|_{L^r}^2\leq C\|\Delta u\|_{L^2}^2\|\nabla w\|_{L^r}^2+C\|\nabla w\|_{L^r}^2+C\|\Omega\|_{L^r}^2. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\frac{d}{dt}(\|\Omega\|_{L^r}^2+\|\nabla w\|_{L^r}^2)\leq C(1+\|\Delta u\|_{L^2}^2)(\|\Omega\|_{L^r}^2+\|\nabla w\|_{L^r}^2)+Ce^{-2\sigma t}\|\theta_0\|_{L^r}^2.$$

By Lemma 3.2, we get

$$\sup_{t\in[0,T]}(\|\Omega\|_{L^r}^2+\|\nabla w\|_{L^r}^2)\leq C. \tag{3.9}$$

Applying Calderón-Zygmund inequality, we have

$$\|\nabla u\|_{L^r}\leq C_r\|\Omega\|_{L^r},\quad 1<r<\infty. \tag{3.10}$$

By Gagliardo-Nirenberg inequality, we get

$$\|u\|_{L^\infty}\leq C\|u\|_{L^2}^{\frac{r-2}{2r-2}}\|\nabla u\|_{L^r}^{\frac{r}{2r-2}},\quad \|w\|_{L^\infty}\leq C\|w\|_{L^2}^{\frac{r-2}{2r-2}}\|\nabla w\|_{L^r}^{\frac{r}{2r-2}}. \tag{3.11}$$

From (3.9)–(3.11), and using Lemma 3.2, we obtain

$$\sup_{t\in[0,T]}(\|u\|_{L^\infty}+\|w\|_{L^\infty})\leq C.$$

□

Lemma 3.4. *Under the assumptions of Lemma 3.2, if* $u_0, w_0 \in W^{2,r}(\mathbb{R}^2)$*,* $\theta_0 \in W^{1,r}(\mathbb{R}^2)$ *for* $2 < r < \infty$*, then*

$$\sup_{t\in[0,T]} (\|\nabla\Omega\|_{L^r} + \|\Delta w\|_{L^r} + \|\nabla\theta\|_{L^r}) \leq C,$$

and

$$\sup_{t\in[0,T]} (\|\nabla u(t)\|_{L^\infty} + \|\nabla w(t)\|_{L^\infty}) \leq C,$$

where $C = C(T, r, \mu, \chi, \gamma, \kappa, \sigma, \|u_0\|_{W^{2,r}}, \|w_0\|_{W^{2,r}}, \|\theta_0\|_{W^{1,r}})$*.*

Proof. Taking ∇ of $(3.6)_1$, then multiplying by $|\nabla\Omega|^{r-2}(\nabla\Omega)$, integrating over $\mathbb{R}^2$ by parts, using Young's inequality and Hölder inequality, we obtain

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\Omega|^r dx + (r-1)(\mu+\chi)\int_{\mathbb{R}^2} |\Delta\Omega|^2|\nabla\Omega|^{r-2}dx \\
=& -\int_{\mathbb{R}^2} \nabla(u\cdot\nabla\Omega)|\nabla\Omega|^{r-2}(\nabla\Omega)dx - 2\chi\int_{\mathbb{R}^2} (\nabla\Delta w)|\nabla\Omega|^{r-2}(\nabla\Omega)dx + \int_{\mathbb{R}^2} \nabla(\partial_1\theta)|\nabla\Omega|^{r-2}(\nabla\Omega)dx \\
\leq&(r-1)\int_{\mathbb{R}^2} |u||\nabla\Omega||\Delta\Omega||\nabla\Omega|^{r-2}dx + 2(r-1)\chi\int_{\mathbb{R}^2} |\Delta w||\Delta\Omega||\nabla\Omega|^{r-2}dx \\
&\quad + (r-1)\int_{\mathbb{R}^2} |\partial_1\theta||\Delta\Omega||\nabla\Omega|^{r-2}dx \\
\leq&\frac{(r-1)(\mu+\chi)}{2}\int_{\mathbb{R}^2} |\Delta\Omega|^2|\nabla\Omega|^{r-2}dx + C\int_{\mathbb{R}^2} |u|^2|\nabla\Omega|^2|\nabla\Omega|^{r-2}dx \\
&\quad + C\int_{\mathbb{R}^2} |\Delta w|^2|\nabla\Omega|^{r-2}dx + C\int_{\mathbb{R}^2} |\nabla\theta|^2|\nabla\Omega|^{r-2}dx \\
\leq&\frac{(r-1)(\mu+\chi)}{2}\int_{\mathbb{R}^2} |\Delta\Omega|^2|\Omega|^{r-2}dx + C\|u\|_{L^\infty}^2\|\nabla\Omega\|_{L^r}^r + C\|\Delta w\|_{L^r}^2\|\nabla\Omega\|_{L^r}^{r-2} + C\|\nabla\theta\|_{L^r}^2\|\nabla\Omega\|_{L^r}^{r-2} \\
\leq&\frac{(r-1)(\mu+\chi)}{2}\int_{\mathbb{R}^2} |\Delta\Omega|^2|\Omega|^{r-2}dx + C(1+\|u\|_{L^\infty}^2)\|\nabla\Omega\|_{L^r}^r + C\|\Delta w\|_{L^r}^r + C\|\nabla\theta\|_{L^r}^r.
\end{aligned}$$

Then, by Lemma 3.3, we have

$$\frac{d}{dt}\|\nabla\Omega\|_{L^r}^r \leq C(\|\nabla\Omega\|_{L^r}^r + \|\Delta w\|_{L^r}^r + \|\nabla\theta\|_{L^r}^r). \tag{3.12}$$

Taking ∇ of $(3.6)_2$, multiplying by $|\Delta w|^{r-2}(\Delta w)$, we obtain

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^2}|\Delta w|^r dx+(r-1)\gamma\int_{\mathbb{R}^2}|\nabla\Delta w|^2|\Delta w|^{r-2}dx+4\chi\int_{\mathbb{R}^2}|\Delta w|^r dx\\
=&-\int_{\mathbb{R}^2}\Delta(u\cdot\nabla w)|\Delta w|^{r-2}(\Delta w)dx+2\chi\int_{\mathbb{R}^2}(\Delta\Omega)|\Delta w|^{r-2}(\Delta w)dx\\
\le&(r-1)\int_{\mathbb{R}^2}\nabla(u\cdot\nabla w)|\nabla\Delta w||\Delta w|^{r-2}dx+2(r-1)\chi\int_{\mathbb{R}^2}(\nabla\Omega)|\nabla\Delta w||\Delta w|^{r-2}dx\\
\le&(r-1)\int_{\mathbb{R}^2}|\nabla u||\nabla w||\nabla\Delta w||\Delta w|^{r-2}dx+(r-1)\int_{\mathbb{R}^2}|u||\Delta w||\nabla\Delta w||\Delta w|^{r-2}dx\\
&+2(r-1)\chi\int_{\mathbb{R}^2}(\nabla\Omega)|\nabla\Delta w||\Delta w|^{r-2}dx\\
\le&\frac{(r-1)\gamma}{2}\int_{\mathbb{R}^2}|\nabla\Delta w|^2|\Delta w|^{r-2}dx+C\int_{\mathbb{R}^2}|u|^2|\Delta w|^2|\Delta w|^{r-2}dx\\
&+C\int_{\mathbb{R}^2}|\nabla\Omega|^2|\Delta w|^{r-2}dx+C\int_{\mathbb{R}^2}|\nabla u|^2|\nabla w|^2|\Delta w|^{r-2}dx\\
\le&\frac{(r-1)\gamma}{2}\int_{\mathbb{R}^2}|\nabla\Delta w|^2|\Delta w|^{r-2}dx+C\|u\|_{L^\infty}^2\|\Delta w\|_{L^r}^r\\
&+C\|\nabla\Omega\|_{L^r}^2\|\Delta w\|_{L^r}^{r-2}+C\|\nabla u\|_{L^{2r}}^2\|\nabla w\|_{L^{2r}}^2\|\Delta w\|_{L^r}^{r-2}\\
\le&\frac{(r-1)\gamma}{2}\int_{\mathbb{R}^2}|\nabla\Delta w|^2|\Delta w|^{r-2}dx+C(1+\|u\|_{L^\infty}^2)\|\Delta w\|_{L^r}^r+C\|\nabla\Omega\|_{L^r}^r+C\|\nabla u\|_{L^{2r}}^r\|\nabla w\|_{L^{2r}}^r.
\end{aligned}$$

Then, using (3.10) and Lemma 3.3, we derive

$$\frac{d}{dt}\|\Delta w\|_{L^r}^r\le C(\|\Delta w\|_{L^r}^r+\|\nabla\Omega\|_{L^r}^r+1). \tag{3.13}$$

Taking ∇ of $(1.2)_3$, multiplying by $|\nabla\theta|^{r-2}(\nabla\theta)$, we have

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\theta|^r dx+\sigma\int_{\mathbb{R}^2}|\nabla\theta|^r dx&=-\int_{\mathbb{R}^2}\nabla(u\cdot\nabla\theta)|\nabla\theta|^{r-2}(\nabla\theta)dx\\
&=-\int_{\mathbb{R}^2}(\nabla u\cdot\nabla\theta)|\nabla\theta|^{r-2}(\nabla\theta)dx\\
&\le\int_{\mathbb{R}^2}|\nabla u||\nabla\theta|^r dx\\
&\le\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^r}^r.
\end{aligned} \tag{3.14}$$

Combining (3.12)–(3.14), we get

$$\frac{d}{dt}(\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r)\le C(1+\|\nabla u\|_{L^\infty})(\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r+1).$$

Notice that the following estimate holds (see [4]):

$$\|\nabla u\|_{L^\infty}\le C(1+\|\Delta u\|_{L^2})\big[1+\ln^{\frac{1}{2}}(1+\|\Delta u\|_{L^r})\big]+C\|\nabla u\|_{L^2},\quad 2<r<\infty.$$

Then, using Lemma 3.3, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r)\\ \le& C(1+\|\Delta u\|_{L^2})\big[1+\ln^{\frac{1}{2}}(1+\|\Delta u\|_{L^r})\big](\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r+1)\\ \le& C(1+\|\Delta u\|_{L^2}^2)\big[1+\ln(1+\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r)\big](\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r+1).\end{aligned}$$

Setting $J(t)=\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r+1$, we have

$$\frac{d}{dt}[\ln J]\le C(1+\|\Delta u\|_{L^2}^2)(1+\ln J).$$

By Lemmas 2.1, 3.2, we find

$$\sup_{t\in[0,T]}(\|\nabla\Omega\|_{L^r}^r+\|\Delta w\|_{L^r}^r+\|\nabla\theta\|_{L^r}^r)\le C. \tag{3.15}$$

By Gagliardo-Nirenberg inequality, we get

$$\|\nabla u\|_{L^\infty}\le C\|\nabla u\|_{L^2}^{\frac{r-2}{2r-2}}\|\Delta u\|_{L^r}^{\frac{r}{2r-2}},\quad \|\nabla w\|_{L^\infty}\le C\|\nabla w\|_{L^2}^{\frac{r-2}{2r-2}}\|\Delta w\|_{L^r}^{\frac{r}{2r-2}}. \tag{3.16}$$

From (3.10), (3.15) and (3.16), we obtain

$$\sup_{t\in[0,T]}(\|\nabla u\|_{L^\infty}+\|\nabla w\|_{L^\infty})\le C.$$

□

Lemma 3.5. *Assume* $u_0,w_0,\theta_0\in H^s(\mathbb{R}^2)$, $s>2$, *and* $\operatorname{div}u_0=0$. *Let* (u,w,θ) *be a smooth solution of* (1.2). *Then, for each* $T>0$,

$$\sup_{t\in[0,T]}(\|u\|_{H^s}+\|w\|_{H^s}+\|\theta\|_{H^s})\le C,$$

and

$$\int_0^T(\|\nabla u\|_{H^s}^2+\|\nabla w\|_{H^s}^2)dt\le C,$$

where $C=C(T,\mu,\chi,\gamma,\kappa,\sigma,\|u_0\|_{H^s},\|w_0\|_{H^s},\|\theta_0\|_{H^s})$.

Proof. Applying the operator $\Lambda^s=(-\Delta)^{\frac{s}{2}}$ to both sides of (1.2), multiplying $(1.2)_1$, $(1.2)_2$, $(1.2)_3$ by $\Lambda^s u$, $\Lambda^s w$ and $\Lambda^s\theta$, respectively, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{H^s}^2+\|w\|_{H^s}^2+\|\theta\|_{H^s}^2)+(\mu+\chi)\|\nabla u\|_{H^s}^2+\gamma\|\nabla w\|_{H^s}^2+4\chi\|w\|_{H^s}^2+\sigma\|\theta\|_{H^s}^2\\ =&-\int_{\mathbb{R}^2}[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s udx-\int_{\mathbb{R}^2}[\Lambda^s,u\cdot\nabla]w\cdot\Lambda^s wdx-\int_{\mathbb{R}^2}[\Lambda^s,u\cdot\nabla]\theta\cdot\Lambda^s\theta dx\\ &+4\chi\int_{\mathbb{R}^2}\Lambda^s[\nabla^\perp u]\cdot\Lambda^s wdx+\int_{\mathbb{R}^2}\Lambda^s[\theta e_2]\cdot\Lambda^s udx\\ =&I_1+I_2+I_3+I_4+I_5,\end{aligned} \tag{3.17}$$

where we used the facts:

$$\int_{\mathbb{R}^2}u\cdot\nabla\Lambda^s u\cdot\Lambda^s udx=0,\quad \int_{\mathbb{R}^2}u\cdot\nabla\Lambda^s w\cdot\Lambda^s wdx=0,\quad \int_{\mathbb{R}^2}u\cdot\nabla\Lambda^s\theta\cdot\Lambda^s\theta dx=0,$$

and

$$\int_{\mathbb{R}^2} \Lambda^s[\text{curl } w] \cdot \Lambda^s u dx = \int_{\mathbb{R}^2} \Lambda^s[\nabla^\perp u] \cdot \Lambda^s w dx.$$

Now we begin to estimate I_1–I_5. For I_1, by Lemma 2.2 and Hölder inequality, we get

$$\begin{aligned} I_1 &\leq \|[\Lambda^s, u \cdot \nabla]u\|_{L^2} \|\Lambda^s u\|_{L^2} \\ &\leq C(\|\nabla u\|_{L^\infty} \|u\|_{H^s} + \|u\|_{H^s} \|\nabla u\|_{L^\infty}) \|u\|_{H^s} \\ &\leq C\|\nabla u\|_{L^\infty} \|u\|_{H^s}^2. \end{aligned}$$

By Lemma 2.2, Hölder inequality and Young's inequality, we obtain

$$\begin{aligned} I_2 &\leq \|[\Lambda^s, u \cdot \nabla]w\|_{L^2} \|\Lambda^s w\|_{L^2} \\ &\leq C(\|\nabla u\|_{L^\infty} \|w\|_{H^s} + \|u\|_{H^s} \|\nabla w\|_{L^\infty}) \|w\|_{H^s} \\ &\leq C\|\nabla u\|_{L^\infty} \|w\|_{H^s}^2 + C\|\nabla w\|_{L^\infty} \|u\|_{H^s} \|w\|_{H^s} \\ &\leq C(\|\nabla u\|_{L^\infty} + \|\nabla w\|_{L^\infty})(\|u\|_{H^s}^2 + \|w\|_{H^s}^2), \end{aligned}$$

$$\begin{aligned} I_3 &\leq \|[\Lambda^s, u \cdot \nabla]\theta\|_{L^2} \|\Lambda^s \theta\|_{L^2} \\ &\leq C(\|\nabla u\|_{L^\infty} \|\theta\|_{H^s} + \|\nabla u\|_{H^s} \|\theta\|_{L^\infty}) \|\theta\|_{H^s} \\ &\leq C\|\nabla u\|_{L^\infty} \|\theta\|_{H^s}^2 + C\|\theta\|_{L^\infty} \|\nabla u\|_{H^s} \|\theta\|_{H^s} \\ &\leq \frac{\mu}{4} \|\nabla u\|_{H^s}^2 + C(\|\nabla u\|_{L^\infty} + \|\theta\|_{L^\infty}^2) \|\theta\|_{H^s}^2, \end{aligned}$$

$$I_4 \leq 4\chi \|\Lambda^s[\nabla^\perp u]\|_{L^2} \|\Lambda^s w\|_{L^2} \leq \frac{\mu + 2\chi}{2} \|\nabla u\|_{H^s}^2 + \frac{8\chi^2}{\mu + 2\chi} \|w\|_{H^s}^2,$$

and

$$I_5 \leq \|\Lambda^s[\theta e_2]\|_{L^2} \|\Lambda^s u\|_{L^2} \leq \frac{\sigma}{2} \|\theta\|_{H^s}^2 + C\|u\|_{H^s}^2.$$

Substituting the estimates of $I_1 - I_5$ into (3.17), using (3.1), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|\theta\|_{H^s}^2) + \frac{\mu}{4} \|\nabla u\|_{H^s}^2 + \gamma \|\nabla w\|_{H^s}^2 + \frac{4\mu\chi}{\mu + 2\chi} \|w\|_{H^s}^2 + \frac{\sigma}{2} \|\theta\|_{H^s}^2 \\ \leq &C(\|\nabla u\|_{L^\infty} + \|\nabla w\|_{L^\infty} + \|\theta\|_{L^\infty}^2 + 1)(\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|\theta\|_{H^s}^2) \\ = &C(\|\nabla u\|_{L^\infty} + \|\nabla w\|_{L^\infty} + e^{-2\sigma t} \|\theta_0\|_{L^\infty}^2 + 1)(\|u\|_{H^s}^2 + \|w\|_{H^s}^2 + \|\theta\|_{H^s}^2). \end{aligned}$$

By Lemmas 2.1, 3.4, we have

$$\sup_{t \in [0,T]} (\|u\|_{H^s} + \|w\|_{H^s} + \|\theta\|_{H^s}) \leq C,$$

and

$$\int_0^T (\|\nabla u\|_{H^s}^2 + \|\nabla w\|_{H^s}^2) dt \leq C.$$

□

Proof of Theorem 1.2. We consider the following regularized Boussinesq equations for the micropolar fluid

$$\begin{cases} \partial_t u + (J_\varepsilon u \cdot \nabla)u - (\mu+\chi)\Delta u + \nabla\pi = 2\chi \text{curl } w + \theta e_2, \\ \partial_t w + (J_\varepsilon u \cdot \nabla)w - \gamma\Delta w + 4\chi w = 2\chi\nabla^\perp u, \\ \partial_t \theta + (J_\varepsilon u \cdot \nabla)\theta + \sigma\theta = 0, \\ \text{div } u = 0, \\ (u, w, \theta)(x, 0) = (u_0, w_0, \theta_0), \end{cases} \tag{3.18}$$

where $J_\varepsilon u := (I + \varepsilon(-\Delta)^{\frac{1}{2}})^{-1}$. Problem (3.18) admits a strong solution $(u^\epsilon, w^\epsilon, \theta^\epsilon)$, the arguments are similar to the proof of Theorem 2.5.1 in Chapter V in [19], so we omit the details in here. By Lemma 3.5, we find that all estimates are independent of ϵ. Letting $\epsilon \longrightarrow 0^+$, we have

$$(u^\epsilon, w^\epsilon, \theta^\epsilon) \longrightarrow (u, w, \theta) \quad \text{weakly}-* \text{ in } L^\infty(0, T; H^s(\mathbb{R}^2)),$$
$$(u^\epsilon, w^\epsilon) \longrightarrow (u, w) \quad \text{weakly in } L^2(0, T; H^{s+1}(\mathbb{R}^2)).$$

Furthermore, by Hölder inequality and Sobolev embedding inequality, we can get

$$u^\epsilon \otimes u^\epsilon, \;\; u^\epsilon w^\epsilon, \;\; u^\epsilon\theta^\epsilon \in L^2(0, T; L^2(\mathbb{R}^2)),$$

which are bounded by constant C (independent of ϵ). Then we obtain from (3.18) that

$$\int_0^t (\|\partial_s u^\epsilon\|_{H^{-1}}^2 + \|\partial_s w^\epsilon\|_{H^{-1}}^2 + \|\partial_s \theta^\epsilon\|_{H^{-1}}^2) ds \le C,$$

where C is independent of ϵ. Notice that the embedding $H^1(K) \hookrightarrow L^2(K) \hookrightarrow H^{-1}(K)$ is compact for any compact set K. By the well-known Aubin-Lions compactness lemma (see [20]), we derive

$$(u^\epsilon, w^\epsilon, \theta^\epsilon) \to (u, w, \theta) \quad \text{strongly in } L^2(0, T; L^2_{\text{loc}}).$$

Then passing to a limit in the sense of distributions for (3.18), we find that (u, w, θ) is a weak solution of (1.2), which is also a strong solution by Lemma 3.5.

Next, we prove the uniqueness of solutions for (1.2). Suppose $(u^1, w^1, \theta^1, \pi^1)$ and $(u^2, w^2, \theta^2, \pi^2)$ are two solutions of (1.2). Set

$$\widetilde{u} = u^1 - u^2, \quad \widetilde{w} = w^1 - w^2, \quad \widetilde{\theta} = \theta^1 - \theta^2, \quad \widetilde{\pi} = \pi^1 - \pi^2.$$

Then $(\widetilde{u}, \widetilde{w}, \widetilde{\theta})$ satisfies

$$\begin{cases} \widetilde{u}_t + (u^1 \cdot \nabla)\widetilde{u} + (\widetilde{u} \cdot \nabla)u^2 - (\mu+\chi)\Delta\widetilde{u} + \nabla\widetilde{\pi} = 2\chi \text{curl } \widetilde{w} + \widetilde{\theta} e_2, \\ \widetilde{w}_t + (u^1 \cdot \nabla)\widetilde{w} + (\widetilde{u} \cdot \nabla)w^2 - \gamma\Delta\widetilde{w} + 4\chi\widetilde{w} = 2\chi\nabla^\perp\widetilde{u}, \\ \widetilde{\theta}_t + (u^1 \cdot \nabla)\widetilde{\theta} + (\widetilde{u} \cdot \nabla)\theta^2 + \sigma\widetilde{\theta} = 0, \\ \text{div } \widetilde{u} = 0, \\ (\widetilde{u}, \widetilde{w}, \widetilde{\theta})(x, 0) = 0. \end{cases} \tag{3.19}$$

Multiplying $(3.19)_1$, $(3.19)_2$, $(3.19)_3$ by $\widetilde{u}$, $\widetilde{w}$ and $\widetilde{\theta}$, respectively, integrating over $\mathbb{R}^2$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)+(\mu+\chi)\|\nabla\widetilde{u}\|_{L^2}^2+\gamma\|\nabla\widetilde{w}\|_{L^2}^2+4\chi\|\widetilde{w}\|_{L^2}^2+\sigma\|\widetilde{\theta}\|_{L^2}^2\\
=&-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)u^2\cdot\widetilde{u}dx-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)w^2\cdot\widetilde{w}dx-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)\theta^2\cdot\widetilde{\theta}dx+2\chi\int_{\mathbb{R}^2}\mathrm{curl}\,\widetilde{w}\cdot\widetilde{u}dx\\
&+2\chi\int_{\mathbb{R}^2}\nabla^{\perp}\widetilde{u}\cdot\widetilde{w}dx+\int_{\mathbb{R}^2}\widetilde{\theta}e_2\cdot\widetilde{u}dx\\
=&-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)u^2\cdot\widetilde{u}dx-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)w^2\cdot\widetilde{w}dx-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)\theta^2\cdot\widetilde{\theta}dx\\
&+4\chi\int_{\mathbb{R}^2}\nabla^{\perp}\widetilde{u}\cdot\widetilde{w}dx+\int_{\mathbb{R}^2}\widetilde{\theta}e_2\cdot\widetilde{u}dx.
\end{aligned}
$$

Then by Hölder inequality, Young's inequality and Gagliardo-Nirenberg inequality, we have for $s>2$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)+\frac{\mu}{4}\|\nabla u\|_{L^2}^2+\gamma\|\nabla w\|_{L^2}^2+\frac{4\mu\chi}{\mu+2\chi}\|w\|_{L^2}^2+\frac{\sigma}{2}\|\theta\|_{L^2}^2\\
\le&-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)u^2\cdot\widetilde{u}dx-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)w^2\cdot\widetilde{w}dx-\int_{\mathbb{R}^2}(\widetilde{u}\cdot\nabla)\theta^2\cdot\widetilde{\theta}dx+\frac{1}{2\sigma}\|\widetilde{u}\|_{L^2}^2\\
\le&\|\nabla u^2\|_{L^\infty}\|\widetilde{u}\|_{L^2}^2+\|\nabla w^2\|_{L^\infty}\|\widetilde{u}\|_{L^2}\|\widetilde{w}\|_{L^2}+\|\nabla\theta^2\|_{L^\infty}\|\widetilde{u}\|_{L^2}\|\widetilde{\theta}\|_{L^2}+\frac{1}{2\sigma}\|\widetilde{u}\|_{L^2}^2\\
\le&C(\|u^2\|_{L^2}^{1-\frac{2}{s}}\|u^2\|_{H^s}^{\frac{2}{s}}+\|w^2\|_{L^2}^{1-\frac{2}{s}}\|w^2\|_{H^s}^{\frac{2}{s}}+\|\theta^2\|_{L^2}^{1-\frac{2}{s}}\|\theta^2\|_{H^s}^{\frac{2}{s}}+1)(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2).
\end{aligned}
$$

Because of Lemmas 2.1, 3.2, 3.5, we obtain

$$\|\widetilde{u}\|_{L^2}^2+\|\widetilde{w}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\le C(\|\widetilde{u}_0\|_{L^2}^2+\|\widetilde{w}_0\|_{L^2}^2+\|\widetilde{\theta}_0\|_{L^2}^2)=0.$$

Thus,

$$u^1=u^2,\quad w^1=w^2,\quad \theta^1=\theta^2.$$

□

4. LARGE TIME DECAY OF RESULTS

In this section, we prove Theorem 1.3 by applying the Fourier splitting methods. The proof is divided into the following lemmas.

Lemma 4.1. *Under the assumptions of Theorem 1.3, it holds for all $t>0$,*

$$\|u(t)\|_{L^2}+\|w(t)\|_{L^2}\le C(1+t)^{-\frac{1}{2}},$$

where $C=C(\mu,\chi,\gamma,\kappa,\sigma,\|u_0\|_{L^1\cap L^2},\|w_0\|_{L^1\cap L^2},\|\theta_0\|_{L^1\cap L^2})$.

Proof. Applying the Fourier transform to $(1.2)_1$ and $(1.2)_2$ in both sides, we get

$$
\begin{cases}
\partial_t\widehat{u}+(\mu+\chi)|\xi|^2\widehat{u}=\mathscr{F}[-\nabla\pi-(u\cdot\nabla)u+2\chi\mathrm{curl}\,w+\theta e_2],\\
\partial_t\widehat{w}+\gamma|\xi|^2\widehat{w}+4\chi\widehat{w}=\mathscr{F}[-(u\cdot\nabla)w+2\chi\nabla^{\perp}u].
\end{cases}\tag{4.1}
$$

Multiplying $(4.1)_1$ and $(4.1)_2$ by $\overline{\widehat{u}}$ and $\overline{\widehat{w}}$, respectively, adding these results together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|\widehat{u}|^2+|\widehat{w}|^2)+(\mu+\chi)|\xi|^2|\widehat{u}|^2+\gamma|\xi|^2|\widehat{w}|^2+4\chi|\widehat{w}|^2\\&=\mathscr{F}[-\nabla\pi]\cdot\overline{\widehat{u}}+\mathscr{F}[-(u\cdot\nabla)u]\cdot\overline{\widehat{u}}+\mathscr{F}[2\chi\text{curl } w]\cdot\overline{\widehat{u}}+\mathscr{F}[\theta e_2]\cdot\overline{\widehat{u}}\\&\quad+\mathscr{F}[-(u\cdot\nabla)w]\cdot\overline{\widehat{w}}+\mathscr{F}[2\chi\nabla^{\perp}u]\cdot\overline{\widehat{w}},\end{aligned}\tag{4.2}$$

where $|\widehat{u}|^2=\widehat{u}\cdot\overline{\widehat{u}}$ and $|\widehat{w}|^2=\widehat{w}\overline{\widehat{w}}$.

For the first term on the right-hand-side, taking the divergence of $(1.2)_1$, we have

$$\Delta\pi=-\text{div }[(u\cdot\nabla)u]+\text{div }(\theta e_2).$$

Applying the Fourier transform to the above equality in both sides, we obtain

$$\widehat{\Delta\pi}=-i\xi\cdot\mathscr{F}[\text{div}(u\otimes u)-\theta e_2].$$

Then,

$$\widehat{\nabla\pi}=-\frac{i\xi}{|\xi|^2}\widehat{\Delta\pi}=-\frac{\xi\otimes\xi}{|\xi|^2}\mathscr{F}[\text{div}(u\otimes u)-\theta e_2].$$

This equality leads to

$$\begin{aligned}|\mathscr{F}[-\nabla\pi]\cdot\overline{\widehat{u}}|&\le|\xi||\widehat{u\otimes u}||\widehat{u}|+|\widehat{\theta}e_2||\widehat{u}|\\&\le|\xi|\|u\otimes u\|_{L^1}|\widehat{u}|+\|\theta\|_{L^1}|\widehat{u}|\le|\xi|\|u\|_{L^2}^2|\widehat{u}|+\|\theta\|_{L^1}|\widehat{u}|.\end{aligned}\tag{4.3}$$

Similarly,

$$|\mathscr{F}[\theta e_2]\cdot\overline{\widehat{u}}|\le|\widehat{\theta}||\widehat{u}|\le\|\theta\|_{L^1}|\widehat{u}|.\tag{4.4}$$

By the divergence free of u, we obtain

$$\begin{aligned}|\mathscr{F}[-(u\cdot\nabla)u]\cdot\overline{\widehat{u}}|+|\mathscr{F}[-(u\cdot\nabla)w]\cdot\overline{\widehat{w}}|&\le|\xi|(|\widehat{u\otimes u}||\widehat{u}|+|\widehat{u\otimes w}||\widehat{w}|)\\&\le|\xi|[\|u\|_{L^2}^2|\widehat{u}|+\|u\|_{L^2}\|w\|_{L^2}|\widehat{w}|]\\&\le|\xi|(\|u\|_{L^2}^2+\|u\|_{L^2}\|w\|_{L^2})(|\widehat{u}|+|\widehat{w}|).\end{aligned}\tag{4.5}$$

Using Young's inequality, we have

$$\begin{aligned}|\mathscr{F}[2\chi\text{curl } w]\cdot\overline{\widehat{u}}+\mathscr{F}[2\chi\nabla^{\perp}u]\cdot\overline{\widehat{w}}|&\le|2\chi\xi\times\widehat{w}\cdot\widehat{u}+2\chi\xi\times\widehat{u}\cdot\widehat{w}|\\&\le4\chi|\xi||\widehat{w}||\widehat{u}|\le\frac{\mu+2\chi}{2}|\xi|^2|\widehat{u}|^2+\frac{8\chi^2}{\mu+2\chi}|\widehat{w}|^2.\end{aligned}\tag{4.6}$$

Substituting (4.3)–(4.6) into (4.2), we get

$$\frac{1}{2}\frac{d}{dt}(|\widehat{u}|^2+|\widehat{w}|^2)+\frac{c_1}{2}|\xi|^2(|\widehat{u}|^2+|\widehat{w}|^2)\le C|\xi|(\|u\|_{L^2}^2+\|u\|_{L^2}\|w\|_{L^2})(|\widehat{u}|+|\widehat{w}|)+2\|\theta\|_{L^1}|\widehat{u}|.$$

Set $G(t)=\sqrt{|\widehat{u}|^2+|\widehat{w}|^2}$. Then,

$$\frac{1}{2}\frac{d}{dt}[G(t)]^2+\frac{c_1}{2}|\xi|^2[G(t)]^2\le C|\xi|(\|u\|_{L^2}^2+\|u\|_{L^2}\|w\|_{L^2})G(t)+2\|\theta\|_{L^1}G(t),$$

which implies

$$\frac{d}{dt}G(t)+\frac{c_1}{2}|\xi|^2G(t)\le C|\xi|(\|u\|_{L^2}^2+\|u\|_{L^2}\|w\|_{L^2})+2\|\theta\|_{L^1}.$$

Multiplying by $e^{\frac{c_1}{2}|\xi|^2 t}$ and integrating from 0 to t, we deduce

$$G(t) \le e^{-\frac{c_1}{2}|\xi|^2 t}G(0)+C\int_0^t e^{-\frac{c_1}{2}|\xi|^2(t-s)}|\xi|(\|u\|_{L^2}^2+\|u\|_{L^2}\|w\|_{L^2})ds+2\int_0^t e^{-\frac{c_1}{2}|\xi|^2(t-s)}\|\theta\|_{L^1}ds.$$

Then,

$$\begin{aligned} G(t) &\le C + C|\xi|\int_0^t (\|u\|_{L^2}^2 + \|u\|_{L^2}\|w\|_{L^2})ds + 2\int_0^t e^{-\sigma s}\|\theta_0\|_{L^1}ds \\ &\le C + C|\xi|\int_0^t (\|u\|_{L^2}^2 + \|u\|_{L^2}\|w\|_{L^2})ds. \end{aligned}$$

Therefore, by Lemma 3.2, we have

$$|\widehat{u}| + |\widehat{w}| \le C + C|\xi|\int_0^t (\|u\|_{L^2}^2 + \|u\|_{L^2}\|w\|_{L^2})ds \le C + Ct|\xi|. \tag{4.7}$$

Now we set

$$S(t) = \{\xi \in \mathbb{R}^2 |\ |\xi| \le r^{\frac{1}{2}}[g(t)]^{-\frac{1}{2}}\},$$

where r is a fixed number. Substituting (3.3) into (3.2), we obtain

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + c_1(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \le Ce^{-\sigma t}. \tag{4.8}$$

Applying the Fourier transform to (4.8), by Plancherel's theorem, we get

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + c_1\int_{\mathbb{R}^2}|\xi|^2(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \le Ce^{-\sigma t}.$$

Observe that

$$\begin{aligned} \int_{\mathbb{R}^2}|\xi|^2(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi &= \int_{S(t)}|\xi|^2(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi + \int_{S(t)^c}|\xi|^2(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \\ &\ge \int_{S(t)^c}|\xi|^2(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \ge \frac{r}{g(t)}\int_{S(t)^c}(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi. \end{aligned}$$

Hence,

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + \frac{c_1 r}{g(t)}\int_{\mathbb{R}^2}(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \le \frac{c_1 r}{g(t)}\int_{S(t)}(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi + Ce^{-\sigma t}. \tag{4.9}$$

Let $g(t) = (1+t)\ln(1+t)$. By (4.7), we obtain

$$\int_{S(t)}(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \le C\int_{S(t)}(1 + t^2|\xi|^2)d\xi \le C(1+t)^{-1}\ln^{-1}(1+t) + C\ln^{-2}(1+t).$$

Then,

$$\begin{aligned} &\frac{d}{dt}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + \frac{\frac{3}{2}}{(1+t)\ln(1+t)}\int_{\mathbb{R}^2}(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \\ &\qquad \le C(1+t)^{-2}\ln^{-2}(1+t) + C(1+t)^{-1}\ln^{-3}(1+t) + Ce^{-\sigma t}. \end{aligned}$$

Multiplying by $\ln^{\frac{3}{2}}(1+t)$, we deduce

$$\frac{d}{dt}[\ln^{\frac{3}{2}}(1+t)(\|u\|_{L^2}^2+\|w\|_{L^2}^2)] \leq C(1+t)^{-2}\ln^{-\frac{1}{2}}(1+t)+C(1+t)^{-1}\ln^{-\frac{3}{2}}(1+t)+C\ln^{\frac{3}{2}}(1+t)e^{-\sigma t}.$$

Observe that

$$\int_0^\infty (1+t)^{-2}\ln^{-\frac{1}{2}}(1+t)dt = \int_0^\infty (1+t)^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}\ln^{-\frac{1}{2}}(1+t)dt \leq C,$$

$$\int_0^\infty (1+t)^{-1}\ln^{-\frac{3}{2}}(1+t)dt = \int_0^\infty -2d[\ln^{-\frac{1}{2}}(1+t)] \leq C,$$

and

$$\int_0^\infty \ln^{\frac{3}{2}}(1+t)e^{-\sigma t}dt \leq \int_0^\infty (1+t)^{\frac{3}{2}}e^{-\sigma t}dt \leq C.$$

Then,

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \leq C\ln^{-\frac{3}{2}}(1+t). \tag{4.10}$$

Substituting (4.10) into (4.7) and using Hölder inequality, we can re-estimate (4.7) as

$$\begin{aligned}
|\widehat{u}| + |\widehat{w}| &\leq C + C|\xi|\int_0^t \ln^{-\frac{3}{4}}(1+s)(\|u\|_{L^2} + \|w\|_{L^2})ds \\
&\leq C + C|\xi|t^{\frac{1}{2}}\left[\int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds\right]^{\frac{1}{2}}.
\end{aligned} \tag{4.11}$$

Set $g(t) = e + t$. Combining (4.9) and (4.11), we deduce

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + \frac{k}{e+t}\int_{\mathbb{R}^2}(|\widehat{u}|^2 + |\widehat{w}|^2)d\xi \\
&\leq \frac{C}{e+t}\int_{S(t)}\left[1 + \int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds\right]d\xi + Ce^{-\sigma t} \\
&\leq \frac{C}{e+t}\left[\int_{S(t)}d\xi + \int_{S(t)}d\xi\int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds\right] + Ce^{-\sigma t} \\
&\leq C(1+t)^{-2} + C(1+t)^{-2}\int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds + Ce^{-\sigma t}.
\end{aligned} \tag{4.12}$$

Multiplying (4.12) by $(1+t)^k$, we get

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^k(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)] \\
&\leq C(1+t)^{k-2} + C(1+t)^{k-2}\int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds + C(1+t)^k e^{-\sigma t}.
\end{aligned}$$

Integrating over $[0,t]$, we obtain

$$\begin{aligned}
&(1+t)^k(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) \\
&\leq C(1+t)^{k-1} + C(1+t)^{k-1}\int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds + C\int_0^t (1+s)^k e^{-\sigma s}ds.
\end{aligned}$$

Then,

$$\|u\|_{L^2}^2 + \|w\|_{L^2}^2 \le C(1+t)^{-1} + C(1+t)^{-1}\int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds. \quad (4.13)$$

Let $I(t) = \int_0^t \ln^{-\frac{3}{2}}(1+s)(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)ds$. Then,

$$I'(t) \le C(1+t)^{-1}\ln^{-\frac{3}{2}}(1+s) + C(1+t)^{-1}\ln^{-\frac{3}{2}}(1+s)I(t).$$

Because $I(0) = 0$ and

$$\int_0^\infty (1+t)^{-1}\ln^{-\frac{3}{2}}(1+t)dt \le C.$$

Thus we can deduce from Lemma 2.1 that $I(t) \le C$. Combining with (4.13) yields

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \le C(1+t)^{-1}.$$

□

Lemma 4.2. *Under the assumptions of Theorem 1.3, it holds for all* $t > 0$,

$$\|\nabla u(t)\|_{L^2} + \|\nabla w(t)\|_{L^2} \le C(1+t)^{-1},$$

where $C = C(\mu, \chi, \gamma, \kappa, \sigma, \|u_0\|_{H^1\cap L^2}, \|w_0\|_{H^1\cap L^2}, \|\theta_0\|_{L^1\cap L^2})$.

Proof. Multiplying (4.8) by $(1+t)^k$, we deduce

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^k(\|u\|_{L^2}^2 + \|w\|_{L^2}^2)] + c_1(1+t)^k(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\
\le & k(1+t)^{k-1}(\|u\|_{L^2}^2 + \|w\|_{L^2}^2) + C(1+t)^k e^{-t} \\
\le & C(1+t)^{k-2} + C(1+t)^k e^{-t}.
\end{aligned}$$

Integrating on $[0,t]$ gives

$$\int_0^t (1+s)^k(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)ds \le C + C(1+t)^{k-1} \le C(1+t)^{k-1}. \quad (4.14)$$

Multiplying (3.4) by $(1+t)^{k+1}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}[(1+t)^{k+1}(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)] + c_2(1+t)^{k+1}(\|\Delta u\|_{L^2}^2 + \|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\
\le & (k+1)(1+t)^k(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\
& + C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2[(1+t)^{k+1}(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)] + C(1+t)^{k+1}e^{-2\sigma t}\|\theta_0\|_{L^2}^2.
\end{aligned} \quad (4.15)$$

Because of

$$\int_0^\infty \|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 dt \le C\int_0^\infty \|\nabla u\|_{L^2}^2 dt \le C,$$

and

$$\int_0^\infty (1+t)^{k+1}e^{-2\sigma t}\|\theta_0\|_{L^2}^2 dt \le C.$$

From Lemma 2.1, (4.14) and (4.15), we obtain

$$(1+t)^{k+1}(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2) \leq C+C\int_0^t (1+s)^k(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)ds \leq C(1+t)^{k-1}.$$

Therefore,

$$\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \leq C(1+t)^{-2}.$$

□

Lemma 4.3. *Under the assumptions of Theorem 1.3, there holds for all* $t > 0$*,*

$$\|w(t)\|_{L^2} \leq C(1+t)^{-1},$$

where $C = C(\mu, \chi, \gamma, \kappa, \sigma, \|u_0\|_{H^1\cap L^2}, \|w_0\|_{H^1\cap L^2}, \|\theta_0\|_{L^1\cap L^2})$.

Proof. Applying Duhamel's principle gives

$$w(t) = e^{-4\chi t}e^{\gamma t\Delta}w_0 + \int_0^t e^{-4\chi(t-s)}e^{\gamma(t-s)\Delta}[2\chi\nabla^\perp u - (u\cdot\nabla)w](s)ds,$$

where $(e^{\gamma t\Delta})_{t\geq 0}$ is the heat semigroup. Then, by Lemma 2.3, we have

$$\begin{aligned}\|w(t)\|_{L^2} \leq& e^{-4\chi t}\|e^{\gamma t\Delta}w_0\|_{L^2} + \int_0^t e^{-4\chi(t-s)}\|e^{\gamma(t-s)\Delta}[2\chi\nabla^\perp u - (u\cdot\nabla)w](s)\|_{L^2}ds\\ \leq& e^{-4\chi t}\|w_0\|_{L^2} + C\int_0^t e^{-4\chi(t-s)}[\|\nabla u(s)\|_{L^2} + (t-s)^{-\frac{1}{2}}\|(u\cdot\nabla)w(s)\|_{L^1}]ds\\ \leq& e^{-4\chi t}\|w_0\|_{L^2} + C\int_0^t e^{-4\chi(t-s)}[\|\nabla u(s)\|_{L^2} + (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^2}\|\nabla w(s)\|_{L^2}]ds.\end{aligned}$$

Using Lemmas 2.4, 3.2, 4.2, we obtain for all $t > 0$

$$\begin{aligned}\|w(t)\|_{L^2} \leq& e^{-4\chi t}\|w_0\|_{L^2} + C\int_0^t e^{-4\chi(t-s)}[(1+s)^{-1} + (t-s)^{-\frac{1}{2}}(1+s)^{-1}]ds\\ \leq& e^{-4\chi t}\|w_0\|_{L^2} + C\int_0^t e^{-4\chi(t-s)}(1+s)^{-1}[1 + (t-s)^{-\frac{1}{2}}]ds\\ \leq& e^{-4\chi t}\|w_0\|_{L^2} + C\int_0^t e^{-4\chi(t-s)}(1+s)^{-1}ds\\ \leq& C(1+t)^{-1}.\end{aligned}$$

□

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