

COEFFICIENT ESTIMATES FOR GENERALIZED LIBERA TYPE BI-CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. In a recent paper, Sakar and Güney introduced a new class of bi-close-to-convex functions and determined the estimates for the general Taylor-Maclaurin coefficients of functions therein. The main purpose of this note is to give a generalization of this class. Also we point out the proof by Sakar and Güney is incorrect and present a correct proof.

1. Introduction

Assume that \mathcal{H} is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let \mathcal{A} denote the subclass of \mathcal{H} consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}).$$

We also denote by \mathcal{S} the subclass of \mathcal{A} whose members are univalent in \mathbb{U} .

A function $f \in \mathcal{A}$ is said to be *starlike of order* β ($0 \leq \beta < 1$) if it satisfies the inequality

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}$ that are starlike of order β by $\mathcal{S}^*(\beta)$. It is well-known that $\mathcal{S}^*(\beta) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}$.

A function $f \in \mathcal{A}$ is said to be *close-to-convex of order* α ($0 \leq \alpha < 1$) if there exists a function $g \in \mathcal{S}^*$ such that the inequality

$$\Re \left(\frac{zf'(z)}{g(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

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holds. We denote the class which consists of all functions $f \in \mathcal{A}$ that are close-to-convex of order α by $\mathcal{C}(\alpha)$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$ (see [10]).

Let $0 \leq \alpha, \beta < 1$. A function $f \in \mathcal{A}$ is said to be *close-to-convex of order α and type β* if there exists a function $g \in \mathcal{S}^*(\beta)$ such that the inequality

$$\Re \left(\frac{zf'(z)}{g(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

holds. We denote the class which consists of all functions $f \in \mathcal{A}$ that are close-to-convex of order α and type β by $\mathcal{C}(\alpha, \beta)$. This class is introduced by Libera [18].

In particular, when $\beta = 0$ we have $\mathcal{C}(\alpha, 0) = \mathcal{C}(\alpha)$ of close-to-convex functions of order α , and also we get $\mathcal{C}(0, 0) = \mathcal{C}$ of close-to-convex functions introduced by Kaplan [17].

Let $0 \leq \alpha < 1$, $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \leq \lambda \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SC}(\gamma, \lambda, \alpha)$ if it satisfies the condition

$$\Re \left(1 + \frac{1}{\gamma} \left(\frac{z[(1-\lambda)f(z) + \lambda zf'(z)]'}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right) \right) > \alpha \quad (z \in \mathbb{U}).$$

This class is introduced by Altıntaş et al. [1]. Clearly, we have the following relationships: $\mathcal{SC}(1, 0, \alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{SC}(1, 0, 0) = \mathcal{S}^*$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk \mathbb{U} . Indeed, the Koebe one-quarter theorem [10] ensures that the image of \mathbb{U} under every univalent function f contains a disk with radius $1/4$. Thus, every function $f \in \mathcal{A}$ has an inverse f^{-1} , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

The inverse function $F = f^{-1}$ is given by

$$\begin{aligned} F(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \\ (2) \quad &= w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in \mathbb{U} if both f and f^{-1} are univalent in \mathbb{U} . Let Σ denote the class of bi-univalent functions in \mathbb{U} given by (1). For a brief history and interesting examples of functions in the class, see [4, 24].

The Faber polynomials introduced by Faber [11] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications like [5, 6, 14–16, 27] applying the Faber polynomial expansions to analytic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

Making use of the Faber polynomial expansion of function $f \in \mathcal{A}$ with the form (1), the coefficients of its inverse map $F = f^{-1}$ may be expressed as follows (see [2, 3]):

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots, a_n) w^n.$$

In general, for any $p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, an expansion of K_{n-1}^p is given by (see [2])

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + \dots + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1},$$

where $D_{n-1}^p = D_{n-1}^p(a_2, a_3, \dots, a_n)$. In view of [25], we see that

$$D_{n-1}^m(a_2, \dots, a_n) = \sum \frac{m!}{j_1! \dots j_{n-1}!} a_2^{j_1} \dots a_n^{j_{n-1}}$$

and the sum is taken over all non-negative integers j_1, \dots, j_{n-1} satisfying

$$\begin{cases} j_1 + j_2 + \dots + j_{n-1} = m, \\ j_1 + 2j_2 + \dots + (n-1)j_{n-1} = n-1. \end{cases}$$

It is clear that $D_{n-1}^{n-1}(a_2, \dots, a_n) = a_2^{n-1}$.

In particular, the first three terms of K_{n-1}^{-n} are

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3), \quad K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

Hamidi and Jahangiri [13] introduced the class of bi-close-to-convex functions of order α as follows: For α ($0 \leq \alpha < 1$), a function $f \in \mathcal{A}$ is said to be *bi-close-to-convex of order α* if both f and its inverse map $F = f^{-1}$ are close-to-convex of order α in \mathbb{U} . We denote the class which consists of all functions $f \in \Sigma$ that are bi-close-to-convex of order α by $\mathcal{C}_\Sigma(\alpha)$. In particular, we set $\mathcal{C}_\Sigma(0) = \mathcal{C}_\Sigma$ for the class of bi-close-to-convex functions. For recent works on bi-close-to-convex functions, please see [7–9, 12, 13, 21–23, 26].

In a very recent paper, the author introduced Libera type bi-close-to-convex functions as follows.

DEFINITION 1.1. [8] Let $0 \leq \alpha, \beta < 1$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{C}_\Sigma(\alpha, \beta)$ of *bi-close-to-convex functions of order α and type β* (or *Libera type bi-close-to-convex functions*) if there exists the functions $g, G \in \mathcal{S}^*(\beta)$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \quad \text{and} \quad \Re\left(\frac{wF'(w)}{G(w)}\right) > \alpha \quad (z, w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (2).

In particular, we get the class $\mathcal{C}_\Sigma(\alpha, 0) = \mathcal{C}_\Sigma(\alpha)$ of bi-close-to-convex functions of order α .

REMARK 1.2. We note that when $\beta = \alpha, g = f$ and $G = F$, the class $\mathcal{C}_\Sigma(\alpha, \beta)$ reduces to the class $\mathcal{S}_\Sigma^*(\alpha)$ of bi-starlike functions of order α ($0 \leq \alpha < 1$) which consists of functions $f \in \Sigma$ satisfying

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{and} \quad \Re\left(\frac{wF'(w)}{F(w)}\right) > \alpha \quad (z, w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (2).

Now, we introduce a new generalization of Libera type bi-close-to-convex functions of complex order as follows.

DEFINITION 1.3. Let $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{SC}_{\Sigma}^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$ if there exists the functions $g, G \in \mathcal{SC}(\tau, \delta, \beta)$ such that

$$(3) \quad \Re \left(1 + \frac{1}{\gamma} \left(\frac{z [(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} - 1 \right) \right) > \alpha \quad (z \in \mathbb{U})$$

and

$$(4) \quad \Re \left(1 + \frac{1}{\gamma} \left(\frac{w [(1-\lambda)F(w) + \lambda w F'(w)]'}{(1-\lambda)G(w) + \lambda w G'(w)} - 1 \right) \right) > \alpha \quad (w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (2).

REMARK 1.4. If we set $\beta = 0$, $\delta = 0$ and $\gamma = \tau = 1$ in Definition 1.3, then the class $\mathcal{SC}_{\Sigma}^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$ reduces to the class $\mathcal{T}_{\Sigma}(\lambda, \alpha)$ which consists of functions $f \in \Sigma$ satisfying

$$\Re \left(\frac{z [(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

and

$$\Re \left(\frac{w [(1-\lambda)F(w) + \lambda w F'(w)]'}{(1-\lambda)G(w) + \lambda w G'(w)} \right) > \alpha \quad (w \in \mathbb{U}),$$

where $g, G \in \mathcal{S}^*$ and the function $F = f^{-1}$ is defined by (2). This class is introduced by Sakar and Güney [21]. The authors investigated the coefficient bounds for a_n of functions belong to the class $\mathcal{T}_{\Sigma}(\lambda, \alpha)$. They proved their main result by making use of the assertion: if an analytic function f of the form (1) is in the class $\mathcal{T}(\lambda, \alpha)$, that is, if it satisfies the condition

$$\Re \left(\frac{z [(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} \right) > \alpha, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \quad (z \in \mathbb{U}),$$

and if $a_k = 0$ ($2 \leq k \leq n-1$), then the coefficients $b_k = 0$ ($2 \leq k \leq n-1$). But we can provide a counterexample to illuminate the above assertion is wrong. For example, by choosing the functions f and g as

$$f(z) = z \quad \text{and} \quad g(z) = z - \frac{z^2}{2},$$

clearly, we see that $g \in \mathcal{S}^*$ and $f \in \mathcal{T}(1/2, 1/2)$. It is worthy to note that for these functions $a_2 = 0$ but $b_2 = -1/2 \neq 0$ (see Figure 1).

REMARK 1.5. If we set $\lambda = \delta = 0$ and $\gamma = \tau = 1$, then the class $\mathcal{SC}_{\Sigma}^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$ reduces to the class $\mathcal{C}_{\Sigma}(\alpha, \beta)$ of Libera type bi-close-to-convex functions defined in Definition 1.1.

2. Preliminary Lemmas

Let the class \mathcal{P} be defined by

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{U})\}.$$

Assume that

$$(5) \quad p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U}).$$

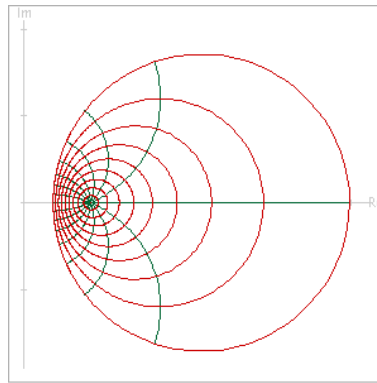


FIGURE 1.

LEMMA 2.1. (Carathéodory Lemma [19]) Let $p \in \mathcal{P}$ given by (5). Then

$$|c_n| \leq 2 \quad (n \in \mathbb{N}).$$

LEMMA 2.2. [10] If $p \in \mathcal{P}$ given by (5) and $\mu \in \mathbb{C}$, then

$$|c_2 - \mu c_1^2| \leq 2 \max \{1, |2\mu - 1|\}.$$

LEMMA 2.3. [1] If $g \in \mathcal{SC}(\tau, \delta, \beta)$ ($0 \leq \beta < 1$, $0 \leq \delta \leq 1$, $\tau \in \mathbb{C}^*$) with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then

$$|b_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1 - \beta)]}{(n - 1)! [1 + \delta(n - 1)]} \quad (n \in \mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, \dots\}).$$

LEMMA 2.4. If $g \in \mathcal{SC}(\tau, \delta, \beta)$ ($0 \leq \beta < 1$, $0 \leq \delta \leq 1$, $\tau \in \mathbb{C}^*$) with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then for $\mu \in \mathbb{C}$

$$|b_3 - \mu b_2^2| \leq \frac{|\tau|(1 - \beta)}{1 + 2\delta} \max \left\{ 1, \left| 1 + 2\tau(1 - \beta) \left(1 - \frac{2(1 + 2\delta)}{(1 + \delta)^2} \mu \right) \right| \right\}.$$

Proof. Let $0 \leq \beta < 1$, $0 \leq \delta \leq 1$ and $\tau \in \mathbb{C}^*$. If $g \in \mathcal{SC}(\tau, \delta, \beta)$, then we have

$$\Re \left(1 + \frac{1}{\tau} \left(\frac{zG'_\delta(z)}{G_\delta(z)} - 1 \right) \right) > \beta \quad (z \in \mathbb{U}),$$

where

$$G_\delta(z) = (1 - \delta)g(z) + \delta z g'(z).$$

Then there exist a positive real part function $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \in \mathcal{P}$ in \mathbb{U} such that

$$(6) \quad 1 + \frac{1}{\tau} \left(\frac{zG'_\delta(z)}{G_\delta(z)} - 1 \right) = \beta + (1 - \beta)h(z) = 1 + (1 - \beta) \sum_{n=1}^{\infty} h_n z^n.$$

From (6), we have

$$(7) \quad b_2 = \frac{\tau(1 - \beta)}{1 + \delta} h_1,$$

$$(8) \quad b_3 = \frac{\tau(1 - \beta)}{2(1 + 2\delta)} [h_2 + \tau(1 - \beta)h_1^2].$$

Taking into account (7) and (8), we obtain

$$(9) \quad b_3 - \mu b_2^2 = \frac{\tau(1-\beta)}{2(1+2\delta)}(h_2 - \nu h_1^2),$$

where

$$\nu = -\tau(1-\beta) \left(1 - \frac{2(1+2\delta)}{(1+\delta)^2} \mu \right).$$

Our result now follows by an application of Lemma 2.2. This completes the proof of Lemma 2.4. □

3. Main Results

THEOREM 3.1. For $0 \leq \alpha, \beta < 1, 0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$, let $f \in \mathcal{SC}_{\Sigma}^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$. If $a_k = 0$ ($2 \leq k \leq n-1$), then for $n \geq 3$,

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1-\beta)]}{n! [1 + \delta(n-1)]} + \frac{2|\gamma|(1-\alpha)}{n [1 + (n-1)\lambda]} + \frac{1}{n [1 + (n-1)\lambda]} \sum_{l=1}^{n-2} \frac{[1 + (n-l-1)\lambda] \prod_{j=0}^{n-l-2} [j + 2|\tau|(1-\beta)]}{(n-l-1)! [1 + \delta(n-l-1)]} \Omega_l^\lambda.$$

where

$$(10) \quad \Omega_l^\lambda = \min \left\{ \left| K_l^{-1}((1+\lambda)b_2, \dots, (1+l\lambda)b_{l+1}) \right|, \left| K_l^{-1}((1+\lambda)B_2, \dots, (1+l\lambda)B_{l+1}) \right| \right\}.$$

Proof. For $0 \leq \alpha, \beta < 1, 0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$, let the function f given by (1) satisfies the hypothesis of the theorem, that is, let f belongs to the class $\mathcal{SC}_{\Sigma}^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$. Then there exist the functions

$$(11) \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{SC}(\tau, \delta, \beta) \quad \text{and} \quad G(w) = w + \sum_{n=2}^{\infty} B_n w^n \in \mathcal{SC}(\tau, \delta, \beta),$$

such that (3) and (4) hold. The Faber polynomial expansion for

$$1 + \frac{1}{\gamma} \left(\frac{z [(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} - 1 \right)$$

and

$$1 + \frac{1}{\gamma} \left(\frac{w [(1-\lambda)F(w) + \lambda w F'(w)]'}{(1-\lambda)G(w) + \lambda w G'(w)} - 1 \right)$$

are obtained by

$$(12) \quad 1 + \frac{1}{\gamma} \left(\frac{z [(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)g(z) + \lambda z g'(z)} - 1 \right) = 1 + \sum_{n=2}^{\infty} \left\{ \frac{1 + (n-1)\lambda}{\gamma} (na_n - b_n) + \sum_{l=1}^{n-2} \frac{1 + (n-l-1)\lambda}{\gamma} K_l^{-1}((1+\lambda)b_2, \dots, (1+l\lambda)b_{l+1}) [(n-l)a_{n-l} - b_{n-l}] \right\} z^{n-1}$$

and

$$\begin{aligned}
 & 1 + \frac{1}{\gamma} \left(\frac{w [(1 - \lambda) F(w) + \lambda w F'(w)]'}{(1 - \lambda) G(w) + \lambda w G'(w)} - 1 \right) = 1 + \sum_{n=2}^{\infty} \left\{ \frac{1 + (n - 1) \lambda}{\gamma} (nA_n - B_n) \right. \\
 (13) \quad & \left. + \sum_{l=1}^{n-2} \frac{1 + (n - l - 1) \lambda}{\gamma} K_l^{-1} ((1 + \lambda) B_2, \dots, (1 + l\lambda) B_{l+1}) [(n - l) A_{n-l} - B_{n-l}] \right\} w^{n-1},
 \end{aligned}$$

respectively. On the other hand by (3) and (4), we see that there exist two positive real part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P} \quad \text{and} \quad q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{P}$$

in \mathbb{U} such that

$$\begin{aligned}
 & 1 + \frac{1}{\gamma} \left(\frac{z [(1 - \lambda) f(z) + \lambda z f'(z)]'}{(1 - \lambda) g(z) + \lambda z g'(z)} - 1 \right) = \alpha + (1 - \alpha) p(z) \\
 (14) \quad & = 1 + (1 - \alpha) \sum_{n=1}^{\infty} c_n z^n,
 \end{aligned}$$

and

$$\begin{aligned}
 & 1 + \frac{1}{\gamma} \left(\frac{w [(1 - \lambda) F(w) + \lambda w F'(w)]'}{(1 - \lambda) G(w) + \lambda w G'(w)} - 1 \right) = \alpha + (1 - \alpha) q(w) \\
 (15) \quad & = 1 + (1 - \alpha) \sum_{n=1}^{\infty} d_n w^n.
 \end{aligned}$$

We note that

$$(16) \quad |c_n| \leq 2 \quad \text{and} \quad |d_n| \leq 2 \quad (n \in \mathbb{N})$$

by Lemma 2.1. Comparing the corresponding coefficients of (12) and (14), for any $n \geq 2$, yields

$$\begin{aligned}
 (17) \quad & \frac{1 + (n - 1) \lambda}{\gamma} (na_n - b_n) \\
 & + \sum_{l=1}^{n-2} \frac{1 + (n - l - 1) \lambda}{\gamma} K_l^{-1} ((1 + \lambda) b_2, \dots, (1 + l\lambda) b_{l+1}) [(n - l) a_{n-l} - b_{n-l}] \\
 & = (1 - \alpha) c_{n-1}.
 \end{aligned}$$

Similarly, it follows from (13) and (15) that

$$\begin{aligned}
 (18) \quad & \frac{1 + (n - 1) \lambda}{\gamma} (nA_n - B_n) \\
 & + \sum_{l=1}^{n-2} \frac{1 + (n - l - 1) \lambda}{\gamma} K_l^{-1} ((1 + \lambda) B_2, \dots, (1 + l\lambda) B_{l+1}) [(n - l) A_{n-l} - B_{n-l}] \\
 & = (1 - \alpha) d_{n-1}.
 \end{aligned}$$

By the hypothesis $a_k = 0$ ($2 \leq k \leq n-1$), we find from (17) and (18) that

$$(19) \quad \begin{aligned} & \frac{1 + (n-1)\lambda}{\gamma} (na_n - b_n) \\ & - \sum_{l=1}^{n-2} \frac{1 + (n-l-1)\lambda}{\gamma} b_{n-l} K_l^{-1} ((1+\lambda)b_2, \dots, (1+l\lambda)b_{l+1}) \\ & = (1-\alpha)c_{n-1}, \end{aligned}$$

and

$$(20) \quad \begin{aligned} & \frac{1 + (n-1)\lambda}{\gamma} (nA_n - B_n) \\ & - \sum_{l=1}^{n-2} \frac{1 + (n-l-1)\lambda}{\gamma} B_{n-l} K_l^{-1} ((1+\lambda)B_2, \dots, (1+l\lambda)B_{l+1}) \\ & = (1-\alpha)d_{n-1}, \end{aligned}$$

respectively. Also the equality $a_k = 0$ ($2 \leq k \leq n-1$) implies that $A_n = -a_n$. Thus (19) and (20) gives

$$(21) \quad \begin{aligned} & \frac{n[1 + (n-1)\lambda]}{\gamma} a_n = \frac{1 + (n-1)\lambda}{\gamma} b_n + (1-\alpha)c_{n-1} \\ & + \sum_{l=1}^{n-2} \frac{1 + (n-l-1)\lambda}{\gamma} b_{n-l} K_l^{-1} ((1+\lambda)b_2, \dots, (1+l\lambda)b_{l+1}) \end{aligned}$$

and

$$(22) \quad \begin{aligned} & -\frac{n[1 + (n-1)\lambda]}{\gamma} a_n = \frac{1 + (n-1)\lambda}{\gamma} B_n + (1-\alpha)d_{n-1} \\ & + \sum_{l=1}^{n-2} \frac{1 + (n-l-1)\lambda}{\gamma} B_{n-l} K_l^{-1} ((1+\lambda)B_2, \dots, (1+l\lambda)B_{l+1}), \end{aligned}$$

respectively.

On the other hand, by the hypothesis (11), since $g, G \in \mathcal{SC}(\tau, \delta, \beta)$ we obtain the coefficient inequalities

$$|b_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1-\beta)]}{(n-1)! [1 + \delta(n-1)]} \quad \text{and} \quad |B_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1-\beta)]}{(n-1)! [1 + \delta(n-1)]}$$

from Lemma 2.3. Considering the above coefficient bounds and the inequalities in (16), from (21) and (22) we get

$$\begin{aligned} & \frac{n[1 + (n-1)\lambda]}{|\gamma|} |a_n| \\ & \leq \frac{1 + (n-1)\lambda}{|\gamma|} |b_n| + (1-\alpha) |c_{n-1}| \\ & + \sum_{l=1}^{n-2} \frac{1 + (n-l-1)\lambda}{|\gamma|} |b_{n-l}| |K_l^{-1} ((1+\lambda)b_2, \dots, (1+l\lambda)b_{l+1})| \end{aligned}$$

$$\begin{aligned}
 & \leq \frac{[1 + (n - 1) \lambda] \prod_{j=0}^{n-2} [j + 2 |\tau| (1 - \beta)]}{|\gamma| (n - 1)! [1 + \delta (n - 1)]} + 2(1 - \alpha) \\
 (23) \quad & + \sum_{l=1}^{n-2} \frac{[1 + (n - l - 1) \lambda] \prod_{j=0}^{n-l-2} [j + 2 |\tau| (1 - \beta)]}{|\gamma| (n - l - 1)! [1 + \delta (n - l - 1)]} |K_l^{-1} ((1 + \lambda) b_2, \dots, (1 + l\lambda) b_{l+1})|
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{n [1 + (n - 1) \lambda]}{|\gamma|} |a_n| \\
 & \leq \frac{1 + (n - 1) \lambda}{|\gamma|} |B_n| + (1 - \alpha) |d_{n-1}| \\
 & + \sum_{l=1}^{n-2} \frac{1 + (n - l - 1) \lambda}{|\gamma|} |B_{n-l}| |K_l^{-1} ((1 + \lambda) B_2, \dots, (1 + l\lambda) B_{l+1})| \\
 & \leq \frac{[1 + (n - 1) \lambda] \prod_{j=0}^{n-2} [j + 2 |\tau| (1 - \beta)]}{|\gamma| (n - 1)! [1 + \delta (n - 1)]} + 2(1 - \alpha)
 \end{aligned}$$

(24)

$$+ \sum_{l=1}^{n-2} \frac{[1 + (n - l - 1) \lambda] \prod_{j=0}^{n-l-2} [j + 2 |\tau| (1 - \beta)]}{|\gamma| (n - l - 1)! [1 + \delta (n - l - 1)]} |K_l^{-1} ((1 + \lambda) B_2, \dots, (1 + l\lambda) B_{l+1})|,$$

respectively. Consequently, by comparing (23) and (24), we get the coefficient bounds for $|a_n|$ as asserted in Theorem 3.1. \square

By setting $\delta = 0$, $\beta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we get the following result.

COROLLARY 3.2. *For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let $f \in \mathcal{T}_\Sigma(\lambda, \alpha)$. If $a_k = 0$ ($2 \leq k \leq n - 1$), then for $n \geq 3$,*

$$\begin{aligned}
 |a_n| \leq & 1 + \frac{2(1 - \alpha)}{n [1 + (n - 1) \lambda]} \\
 & + \frac{1}{n [1 + (n - 1) \lambda]} \sum_{l=1}^{n-2} [1 + (n - l - 1) \lambda] (n - l) \Omega_l^\lambda.
 \end{aligned}$$

where Ω_l^λ is defined by (10).

By setting $\lambda = \delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we have the following consequence.

COROLLARY 3.3. [8] For $0 \leq \alpha, \beta < 1$, let $f \in \mathcal{C}_\Sigma(\alpha, \beta)$. If $a_k = 0$ ($2 \leq k \leq n-1$), then for $n \geq 3$,

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2(1 - \beta)]}{n!} + \frac{2(1 - \alpha)}{n} + \frac{1}{n} \sum_{l=1}^{n-2} \frac{\prod_{j=0}^{n-l-2} [j + 2(1 - \beta)]}{(n-l-1)!} \min \{ |K_l^{-1}(b_2, \dots, b_{l+1})|, |K_l^{-1}(B_2, \dots, B_{l+1})| \}.$$

By setting $\beta = 0$, $\lambda = \delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we get the following result.

COROLLARY 3.4. [26] For $0 \leq \alpha < 1$, let $f \in \mathcal{C}_\Sigma(\alpha)$. If $a_k = 0$ ($2 \leq k \leq n-1$), then for $n \geq 3$,

$$|a_n| \leq 1 + \frac{2(1 - \alpha)}{n} + \frac{1}{n} \sum_{l=1}^{n-2} (n-l) \min \{ |K_l^{-1}(b_2, \dots, b_{l+1})|, |K_l^{-1}(B_2, \dots, B_{l+1})| \}.$$

By setting $b_k = B_k = 0$ ($2 \leq k \leq n-1$) in Theorem 3.1, we get the following result.

COROLLARY 3.5. For $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$, let $f \in \mathcal{SC}_\Sigma^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$. If $a_k = b_k = B_k = 0$ ($2 \leq k \leq n-1$), then for $n \geq 3$,

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1 - \beta)]}{n! [1 + \delta(n-1)]} + \frac{2|\gamma|(1 - \alpha)}{n[1 + (n-1)\lambda]}.$$

By setting $b_k = B_k = 0$ ($2 \leq k \leq n-1$), $\beta = 0$, $\delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.1, we get the following result. It corrects the errors of [21, Theorem 2.1]. More precisely, Theorem 2.1 in [21] holds only with the additional condition $b_k = B_k = 0$ ($2 \leq k \leq n-1$).

COROLLARY 3.6. (Correction of [21, Theorem 2.1]) For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let $f \in \mathcal{T}_\Sigma(\lambda, \alpha)$. If $a_k = b_k = B_k = 0$ ($2 \leq k \leq n-1$), then for $n \geq 3$,

$$|a_n| \leq 1 + \frac{2(1 - \alpha)}{n[1 + (n-1)\lambda]}.$$

COROLLARY 3.7. For $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$, let $f \in \mathcal{SC}_\Sigma^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$. Also suppose that

$$(25) \quad G(w) = g^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3) w^3 - (5b_2^3 - 5b_2 b_3 + b_4) w^4 + \dots$$

If $a_k = b_k = 0$ ($2 \leq k \leq n-1$), then for $n \geq 3$,

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\tau|(1 - \beta)]}{n! [1 + \delta(n-1)]} + \frac{2|\gamma|(1 - \alpha)}{n[1 + (n-1)\lambda]}.$$

As a special case to Theorem 3.1, we determine the initial coefficient bounds of functions belonging to the class $\mathcal{SC}_\Sigma^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$ of bi-close-to-convex functions of order α and type β .

THEOREM 3.8. For $0 \leq \alpha, \beta < 1$, $0 \leq \lambda, \delta \leq 1$ and $\gamma, \tau \in \mathbb{C}^*$, let the function f given by (1) be in the function class $\mathcal{SC}_{\Sigma}^{\gamma, \tau}(\lambda, \alpha; \delta, \beta)$ and suppose that the function G be defined by (25). Then one has the following

$$(26) \quad |a_2| \leq \min \left\{ \frac{|\gamma|(1-\alpha)}{1+\lambda} + \frac{|\tau|(1-\beta)}{1+\delta}, \sqrt{\frac{4|\tau|(1-\beta)}{3(1+\delta)} \left(\frac{|\gamma|(1+\lambda)(1-\alpha)}{1+2\lambda} + \frac{|\tau|(1-\beta)}{1+\delta} \right) + \frac{2|\gamma|(1-\alpha)}{3(1+2\lambda)}} \right\}$$

and

$$(27) \quad |a_3| \leq \min \left\{ \frac{|\gamma|(1-\alpha)}{3(1+2\lambda)} [1 + \max\{1, |\mu|\}] + \frac{|\tau|(1-\beta)}{3(1+2\delta)} \max\{1, |\rho|\} + \frac{2|\gamma|(1-\alpha)|\tau|(1-\beta)}{(1+\lambda)(1+\delta)}, \frac{2|\gamma|(1-\alpha)}{3(1+2\lambda)} + \frac{4|\gamma|(1+\lambda)(1-\alpha)|\tau|(1-\beta)}{3(1+2\lambda)(1+\delta)} + \frac{|\tau|(1-\beta)[1+2|\tau|(1-\beta)]}{3(1+2\delta)} \right\},$$

where

$$\mu = 1 + \frac{3(1+2\lambda)}{(1+\lambda)^2} \gamma(1-\alpha), \quad \rho = 1 + \frac{1+2\delta+2\delta^2}{(1+\delta)^2} \tau(1-\beta).$$

Proof. If we set $n = 2$ and $n = 3$ in (17) and (18), respectively, we get

$$(28) \quad 2a_2 = \frac{\gamma(1-\alpha)}{1+\lambda} c_1 + b_2$$

$$(29) \quad 3a_3 = \frac{\gamma(1-\alpha)}{1+2\lambda} c_2 + \frac{\gamma(1+\lambda)(1-\alpha)}{1+2\lambda} c_1 b_2 + b_3$$

$$(30) \quad -2a_2 = \frac{\gamma(1-\alpha)}{1+\lambda} d_1 - b_2$$

$$(31) \quad 6a_2^2 - 3a_3 = \frac{\gamma(1-\alpha)}{1+2\lambda} d_2 - \frac{\gamma(1+\lambda)(1-\alpha)}{1+2\lambda} d_1 b_2 + 2b_2^2 - b_3.$$

From (28) and (30), we find

$$(32) \quad c_1 = -d_1$$

and

$$(33) \quad a_2 = \frac{\gamma(1-\alpha)}{2(1+\lambda)} c_1 + \frac{1}{2} b_2.$$

On the other hand, from (29) and (31), we obtain

$$(34) \quad a_2^2 = \frac{\gamma(1-\alpha)}{6(1+2\lambda)} (c_2 + d_2) + \frac{\gamma(1+\lambda)(1-\alpha)}{3(1+2\lambda)} c_1 b_2 + \frac{1}{3} b_2^2.$$

Therefore by applying triangle inequality to (33) and (34), using (16) and the fact that

$$(35) \quad |b_2| \leq \frac{2|\tau|(1-\beta)}{1+\delta}$$

obtained from Lemma 2.3 for $n = 2$, we get the desired estimate on the coefficient bound for $|a_2|$ as asserted in (26).

Next, in order to find the bound for $|a_3|$, we subtract (31) from (29). We thus get

$$6a_3 - 6a_2^2 = \frac{\gamma(1-\alpha)}{1+2\lambda}(c_2 - d_2) + \frac{\gamma(1+\lambda)(1-\alpha)}{1+2\lambda}b_2(c_1 + d_1) - 2b_2^2 + 2b_3.$$

By (32), we obtain

$$(36) \quad a_3 = a_2^2 + \frac{\gamma(1-\alpha)}{6(1+2\lambda)}(c_2 - d_2) + \frac{b_3 - b_2^2}{3}.$$

If we set the value of a_2^2 from (33) in (36), then we have

$$\begin{aligned} a_3 &= \frac{\gamma^2(1-\alpha)^2}{4(1+\lambda)^2}c_1^2 + \frac{1}{4}b_2^2 + \frac{\gamma(1-\alpha)}{2(1+\lambda)}c_1b_2 \\ &\quad + \frac{\gamma(1-\alpha)}{6(1+2\lambda)}c_2 - \frac{\gamma(1-\alpha)}{6(1+2\lambda)}d_2 + \frac{1}{3}b_3 - \frac{1}{3}b_2^2 \\ &= \frac{\gamma(1-\alpha)}{6(1+2\lambda)}\left(c_2 + \frac{3\gamma(1+2\lambda)(1-\alpha)}{2(1+\lambda)^2}c_1^2\right) + \frac{1}{3}\left(b_3 - \frac{1}{4}b_2^2\right) \\ &\quad + \frac{\gamma(1-\alpha)}{2(1+\lambda)}c_1b_2 - \frac{\gamma(1-\alpha)}{6(1+2\lambda)}d_2. \end{aligned}$$

So using Lemma 2.2, Lemma 2.4, (16) and (35), we get

$$(37) \quad |a_3| \leq \frac{|\gamma|(1-\alpha)}{3(1+2\lambda)}[1 + \max\{1, |\mu|\}] + \frac{|\tau|(1-\beta)}{3(1+2\delta)}\max\{1, |\rho|\} + \frac{2|\gamma|(1-\alpha)|\tau|(1-\beta)}{(1+\lambda)(1+\delta)},$$

where

$$\mu = 1 + \frac{3(1+2\lambda)}{(1+\lambda)^2}\gamma(1-\alpha), \quad \rho = 1 + \frac{1+2\delta+2\delta^2}{(1+\delta)^2}\tau(1-\beta).$$

If we set the value of a_2^2 from (34) in (36), then we have

$$a_3 = \frac{\gamma(1-\alpha)}{3(1+2\lambda)}c_2 + \frac{\gamma(1+\lambda)(1-\alpha)}{3(1+2\lambda)}c_1b_2 + \frac{1}{3}b_3.$$

So using (16), (35) and the fact that

$$|b_3| \leq \frac{|\tau|(1-\beta)[1+2|\tau|(1-\beta)]}{1+2\delta}$$

obtained from Lemma 2.3 for $n = 3$, we obtain

$$(38) \quad |a_3| \leq \frac{2|\gamma|(1-\alpha)}{3(1+2\lambda)} + \frac{4|\gamma|(1+\lambda)(1-\alpha)|\tau|(1-\beta)}{3(1+2\lambda)(1+\delta)} + \frac{|\tau|(1-\beta)[1+2|\tau|(1-\beta)]}{3(1+2\delta)}.$$

Hence (37) and (38) give the desired estimate on the coefficient $|a_3|$ as asserted in (27). \square

By setting $\delta = 0$, $\beta = 0$ and $\gamma = \tau = 1$ in Theorem 3.8, we get the following result.

COROLLARY 3.9. For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let the function f given by (1) be in the function class $\mathcal{T}_\Sigma(\lambda, \alpha)$ and suppose that the function G be defined by (25). Then one has the following

$$|a_2| \leq \min \left\{ \frac{2 - \alpha + \lambda}{1 + \lambda}, \sqrt{\frac{2(1 - \alpha)(3 + 2\lambda)}{3(1 + 2\lambda)} + \frac{4}{3}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{2(2 - \alpha + 2\lambda)}{3(1 + 2\lambda)} + \frac{(1 - \alpha)(3 - \alpha + 2\lambda)}{(1 + \lambda)^2}, \frac{2(1 - \alpha)(3 + 2\lambda)}{3(1 + 2\lambda)} + 1 \right\}.$$

By setting $\lambda = \delta = 0$ and $\gamma = \tau = 1$ in Theorem 3.8, we get the following result.

COROLLARY 3.10. [8] For $0 \leq \alpha, \beta < 1$, let the function f given by (1) be in the function class $\mathcal{C}_\Sigma(\alpha, \beta)$ and suppose that the function G be defined by (25). Then one has the following

$$|a_2| \leq \min \left\{ (2 - \alpha - \beta), \sqrt{\frac{4(1 - \beta)(2 - \alpha - \beta) + 2(1 - \alpha)}{3}} \right\}$$

and

$$|a_3| \leq \frac{1}{3} \begin{cases} (3 - 2\beta)(3 - 2\alpha - \beta) & , \quad 0 \leq \alpha \leq \frac{2+\beta}{3} \\ (1 - \alpha)(5 - 3\alpha) + (1 - \beta)(2 - \beta) + 6(1 - \alpha)(1 - \beta) & , \quad \frac{2+\beta}{3} \leq \alpha < 1 \end{cases}.$$

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