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CONDITIONAL INTEGRAL TRANSFORMS OF FUNCTIONALS ON A FUNCTION SPACE OF TWO VARIABLES

Bong Jin Kim

Abstract. Let $C(Q)$ denote Yeh-Wiener space, the space of all real-valued continuous functions $x(s,t)$ on $Q \equiv [0,S]\times[0,T]$ with $x(s,0)=x(0,t)=0$ for every $(s,t)\in Q$. For each partition $\tau=\tau_{m,n}=\{(s_i,t_j)|i=1,\dots,m, j=1,\dots,n\}$ of Q with $0=s_0<s_1<\dots<s_m=S$ and $0=t_0<t_1<\dots<t_n=T$, define a random vector $X_\tau : C(Q)\to\mathbb{R}^{mn}$ by

$$X_\tau(x)=(x(s_1,t_1),\dots,x(s_m,t_n)).$$

In this paper we study the conditional integral transform and the conditional convolution product for a class of cylinder type functionals defined on $K(Q)$ with a given conditioning function X_τ above, where $K(Q)$is the space of all complex valued continuous functions of two variables on Q which satify $x(s,0)=x(0,t)=0$ for every $(s,t)\in Q$. In particular we derive a useful equation which allows to calculate the conditional integral transform of the conditional convolution product without ever actually calculating convolution product or conditional convolution product.

1. Definitions and preliminaries

Let $Q=[0,S]\times[0,T]$ and let $C(Q)$ denote Yeh-Wiener space; that is, the space of all real-valued continuous functions $x(s,t)$ on Q with $x(s,0)=x(0,t)=0$ for every $(s,t)\in Q$.

Yeh [12] defined a Gaussian measure m_y on $C(Q)$ such that a stochastic process $\{x(s,t):(s,t)\in Q\}$ has mean $E[x(s,t)]=\int_{C(Q)}x(s,t)dm_y(x)=0$ for eavery $(s,t)\in Q$ and covariance $E[x(s,t)x(u,v)]=\min\{s,u\}\min\{t,v\}$.

This process is called the standard Yeh-Wiener (or two-time parameter Wiener) process on Q.

Let $\mathcal{M}$ denote the class of all Yeh-Wiener measurable subsets of $C(Q)$ and we denote the Yeh-Wiener integral of a Yeh-Wiener integrable functional F by

$$E[F]=E_x[F(x)]=\int_{C(Q)}F(x)\,dm_y(x). \tag{1}$$

Let $K(Q)$ be the space of all complex valued continuous functions of two variables x defined on Q which satisfy $x(s,0)=x(0,t)=0$ for all $(s,t)\in Q$ and let α and β be

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nonzero complex numbers. In [5] Kim, Kim and Yoo studied the integral transforms of functionals defined on $K(Q)$. In [4] Kim, Kim and Skoug introduced the concept of a conditional integral transform and conditional convolution product for a class of functionals defined on $K[0,T]$, the space of complex-valued continuous functions on $[0,T]$ which vanish at zero. In this paper, we define the conditional integral transform $\mathcal{F}_{\alpha.\beta}(F|X)$ and the conditional convolution product $((F*G)_\alpha|X)$ for the functionals defined on $K(Q)$ for a given conditioning function X defined on $C(Q)$, and establish the existence of the conditional integral transform and conditional convolution product for a cylinder type functionals. Moreover we will show that the conditional integral transform and conditional convolution product for the cylinder type functionals are also a cylinder type functionals. Finally we drive some useful equations involving the two concepts above. We finsh this section by setting the definitions of integral transform $\mathcal{F}_{\alpha.\beta}(F)$ and the convolution product $(F*G)_\alpha$ for functionals defined on $K(Q)$ and for the nonzero complex numbers α and β.

Definition 1.1. Let F be a functional defined on $K(Q)$. Then for each $y \in K(Q)$, we define the integral transform $\mathcal{F}_{\alpha.\beta}F$ of F by

$$\mathcal{F}_{\alpha.\beta}(F)(y) \equiv \mathcal{F}_{\alpha.\beta}F(y) = E_x[F(\alpha x + \beta y)] \tag{2}$$

if it exists [1,4,6,8].

Definition 1.2. Let F and G be functionals defined on $K(Q)$. Then for each $y \in K(Q)$, we define their convolution product $(F*G)_\alpha$ by

$$(F*G)_\alpha(y) = E_x\Big[F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)\Big] \tag{3}$$

if it exists [1,4,6,8].

2. Conditional Yeh-Wiener integrals and conditional integral transforms

Let $X : C(Q) \to \mathbb{R}^k$ be a Yeh-Wiener measurable functional and let $F : C(Q) \to \mathbb{C}$ be a Yeh-Wiener integrable functional. Then we have the conditional Yeh-Wiener integral $E[F|X](\vec{\xi})$of F given X from a well-known probability theory [7].

Let $\tau = \tau_{m,n} = \{(s_i, t_j)|i = 1, \ldots, m, j = 1, \ldots, n\}$ be a partittion of Q with $0 = s_0 < s_1 < \ldots < s_m = S$ and $0 = t_0 < t_1 < \ldots < t_n = T$, and let $X_\tau : C(Q) \to \mathbb{R}^{mn}$ be a random vector defined by

$$X_\tau(x) = (x(s_1,t_1), \ldots, x(s_m,t_n)). \tag{1}$$

Define the quasi-polyhedric function $[x]$ by

$$\begin{aligned}[x](s,t) =& x(s_{i-1},t_{j-1}) + [(s-s_{i-1})(t-t_{j-1})/((s_i-s_{i-1})(t_j-t_{j-1}))]\Delta_{ij}x(s,t)\\ &+[(s-s_{i-1})/(s_i-s_{i-1})](x(s_i,t_{j-1})-x(s_{i-1},t_{j-1}))\\ &+[(t-t_{j-1})/(t_j-t_{j-1})](x(s_{i-1},t_j)-x(s_{i-1},t_{j-1}))\end{aligned} \tag{2}$$

on each $(s,t) \in Q_{ij} = (s_{i-1}, s_i] \times (t_{j-1}, t_j]$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ where $\Delta_{ij}x(s,t) = x(s_i,t_j) - x(s_{i-1},t_j) - x(s_i,t_{j-1}) + x(s_{i-1},t_{j-1})$ and $[x](s,t) = 0$ if $st = 0$.

Similarly, for $\vec{\xi} = (\xi_{1,1}, \ldots, \xi_{m,n}) \in \mathbb{R}^{mn}$, define the quasi-polyhedric function $[\vec{\xi}]$ by (2.2), $x(s_i,t_j)$ replaced by ξ_{ij} and $[\vec{\xi}](s,t) = 0$ if $st = 0$.

We note that both $[x]$ and $[\vec{\xi}]$ belong to $C(Q)$ for each $x \in C(Q)$ and each $\vec{\xi}$ in $\mathbb{R}^{mn}$.

THEOREM 2.1. (Theorem 1 in [9]) *If $\{x(s,t),(s,t) \in Q\}$ is the standard Yeh-Wiener process then the processes $\{x(s,t) - [x](s,t)|(s,t) \in Q\}$ and $X_\tau(x) = (x(s_1,t_1),\dots, x(s_m,t_n))$ are stochastically independent.*

REMARK 2.2. As a consequence of the Theorem 2.1 above, Park and Skoug [9] showed that the two processes $\{x(s,t) - [x](s,t)|(s,t) \in Q\}$ and $\{[x](s,t)|(s,t) \in Q\}$ are also independent.

In [9], Park and Skoug gave a simple formula for expressing conditional Yeh-Wiener integrals in terms of ordinary Yeh-Wiener integrals; namely that for the conditioning function X_τ given (2.1) above,

$$E[F(x)|X_\tau(x)](\vec{\xi}) = E_x[F(x - [x] + [\vec{\xi}])]. \tag{3}$$

The equality (2.3) above means that both sides are Borel measurable function of $\vec{\xi} \in \mathbb{R}^{mn}$ and they are equal except for Boerl null sets.

In this paper, we shall be concerned exclusively with $X_\tau(x)$ given by (2.1) for the conditioning function.

REMARK 2.3. Using simple formula (2.3), when $m = n = 1$, that is, $X_\tau(x) = x(S,T)$, we can get the following conditional Yeh-Wiener integral

$$\begin{aligned} E[F(x)|x(S,T)](\xi) =& E_x[F(x(\cdot,*) - \frac{\cdot *}{ST}x(S,T) + \frac{\cdot *}{ST}\xi)] \\ =& \int_{C(Q)} F\big(x(\cdot,*) - \frac{\cdot *}{ST}x(S,T) + \frac{\cdot *}{ST}\xi\big) dm_y(x) \end{aligned} \tag{4}$$

DEFINITION 2.4. Let F and G be functionals defined on $K(Q)$. Then we define conditional integral transform $\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\vec{\xi})$ of F and the conditional convolution product $((F*G)_\alpha|X_\tau)(y,\vec{\xi})$ of $(F*G)_\alpha$ given X_τ, respectively, by formulas

$$\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\vec{\xi}) = E_x[F(\alpha x + \beta y)|X_\tau(x)](\vec{\xi}) \tag{5}$$

and

$$((F*G)_\alpha|X_\tau)(y,\vec{\xi}) = E_x\Big[F\Big(\frac{y+\alpha x}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha x}{\sqrt{2}}\Big)|X_\tau(x)\Big](\vec{\xi}) \tag{6}$$

if they exist [4,11].

REMARK 2.5. Let F and G be defined on $K(Q)$. Then using the simple formrla (2.3), we can get the expressions of the conditional integral transform $\mathcal{F}_{\alpha.\beta}(F|X_\tau)$ of F and the conditional convolution product $((F*G)_\alpha|X_\tau)$ of $(F*G)_\alpha$ by (2.7) and (2.8), respectively, namely for the conditioning function $X_\tau(x)$ given by (2.1).

$$\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\vec{\xi}) = E_x[F(\alpha(x - [x] + [\vec{\xi}]) + \beta y)] \tag{7}$$

and

$$((F*G)_\alpha|X_\tau)(y,\vec{\xi}) = E_x\Big[F\Big(\frac{y+\alpha(x-[x]+[\vec{\xi}])}{\sqrt{2}}\Big)G\Big(\frac{y-\alpha(x-[x]+[\vec{\xi}])}{\sqrt{2}}\Big)\Big]. \tag{8}$$

In particular, if $m = n = 1$, then we have the followings

$$\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\xi) = \int_{C(Q)} F\Big(\alpha(x(\cdot,*) - \frac{\cdot *}{ST}x(S,T) + \frac{\cdot *}{ST}\xi) + \beta y\Big) dm_y(x)$$

and

$$((F*G)_\alpha|X_\tau)(y,\xi) = \int_{C(Q)} F\Big(\frac{y + \alpha(x(\cdot,*) - \frac{\cdot *}{ST}x(S,T) + \frac{\cdot *}{ST}\xi)}{\sqrt{2}}\Big) G\Big(\frac{y - \alpha(x(\cdot,*) - \frac{\cdot *}{ST}x(S,T) + \frac{\cdot *}{ST}\xi)}{\sqrt{2}}\Big) dm_y(x).$$

Under the mild conditions on F and G, the following theorem shows that the conditional integral transform of conditional convolution product is the product of conditional integral transforms.

Theorem 2.6. *Assume that for F and G on $K(Q)$, $\mathcal{F}_{\alpha.\beta}(((F*G)_\alpha|X_\tau)(\cdot,\vec{\xi_1})|X_\tau)$, $\mathcal{F}_{\alpha.\beta}(F|X_\tau)$ and $\mathcal{F}_{\alpha.\beta}(G|X_\tau)$ all exist for $\vec{\xi_1}$, $\vec{\xi_2} \in \mathbb{R}^{mn}$. Then*

$$\begin{aligned} &\mathcal{F}_{\alpha.\beta}(((F*G)_\alpha|X_\tau)(\cdot,\vec{\xi_1})|X_\tau)(y,\vec{\xi_2}) \\ &= \mathcal{F}_{\alpha.\beta}(F|X_\tau)\big(\frac{y}{\sqrt{2}}, \frac{\vec{\xi_2}+\vec{\xi_1}}{\sqrt{2}}\big)\mathcal{F}_{\alpha.\beta}(G|X_\tau)\big(\frac{y}{\sqrt{2}}, \frac{\vec{\xi_2}-\vec{\xi_1}}{\sqrt{2}}\big) \end{aligned} \tag{9}$$

for all $y \in K(Q)$ and $\vec{\xi_1}, \vec{\xi_2} \in \mathbb{R}^{mn}$.

Proof. Equation (2.9) follows from the following calculations;

$$\begin{aligned} &\mathcal{F}_{\alpha.\beta}(((F*G)_\alpha|X_\tau)(\cdot,\vec{\xi_1})|X_\tau)(y,\vec{\xi_2}) \\ &= E_x[((F*G)_\alpha|X_\tau)(\alpha(x-[x]+[\vec{\xi_2}]) + \beta y, \vec{\xi_1})] \\ &= E_x\Big[E_w\Big(F\big(\frac{\beta y}{\sqrt{2}} + \alpha(\frac{x-[x]+[\vec{\xi_2}]}{\sqrt{2}} + \frac{w-[w]+[\vec{\xi_1}]}{\sqrt{2}})\big) \\ &\quad G\big(\frac{\beta y}{\sqrt{2}} + \alpha(\frac{x-[x]+[\vec{\xi_2}]}{\sqrt{2}} - \frac{w-[w]+[\vec{\xi_1}]}{\sqrt{2}})\big)\Big)\Big] \\ &= E_x\Big[E_w\Big[F\big(\frac{\beta y}{\sqrt{2}} + \alpha(\frac{x+w}{\sqrt{2}} - \frac{[x]+[w]}{\sqrt{2}} + \frac{[\vec{\xi_2}]+[\vec{\xi_1}]}{\sqrt{2}})\big)\Big]\Big] \\ &\quad E_x\Big[E_w\Big[G\big(\frac{\beta y}{\sqrt{2}} + \alpha(\frac{x-w}{\sqrt{2}} - \frac{[x]-[w]}{\sqrt{2}} + \frac{[\vec{\xi_2}]-[\vec{\xi_1}]}{\sqrt{2}})\big)\Big]\Big] \\ &= E_x\Big[F\big(\frac{\beta y}{\sqrt{2}} + \alpha(x-[x] + \frac{[\vec{\xi_2}]+[\vec{\xi_1}]}{\sqrt{2}})\big)\Big] E_x\Big[G\big(\frac{\beta y}{\sqrt{2}} + \alpha(x-[x] + \frac{[\vec{\xi_2}]-[\vec{\xi_1}]}{\sqrt{2}})\big)\Big] \end{aligned} \tag{10}$$

The first and second equalities in (2.10) follow from (2.7) and (2.8) respectively. The third equality follows since $x-[x]+w-[w]$ and $x-[x]-w+[w]$ are independent processes as can be seen by checking their covariance. Finally, the last equality follows because the processes $\frac{x+w}{\sqrt{2}}$ and $\frac{x-w}{\sqrt{2}}$ are each equivalent to the process x. Then from (2.7), we see that the last equation can be expressed as (2.9), which completes the proof.

□

3. Conditional integral transforms for the cylinder type functionals

Now we decribe the class of functionals that we work with in this paper. For some positive integer n, let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be an orthonormal set of real valued functions in $L_2(Q)$ and assume that each θ_j is of bounded variation in the sence of Hardy and Krause on Q [10]. Then for each $y \in K(Q)$ and $j = 1, 2, \ldots, n$, the Riemann-Stieltjes integral $\langle \theta_j, y\rangle \equiv \int_Q \theta_j(s,t)dy(s,t)$ exists. Furthermore

$$(1)\qquad \begin{aligned} |\langle \theta_j, y\rangle| =& \Big|\theta_j(S,T)y(S,T) - \int_0^T y(S,t)d\theta_j(S,t) \\ &- \int_0^S y(s,T)d\theta_j(s,T) + \int_Q y(s,t)d\theta_j(s,t)\Big| \le C_j\|y\|_\infty \end{aligned}$$

with

$$C_j = |\theta_j(S,T)| + \mathrm{var}(\theta_j(S,\cdot),[0,T]) + \mathrm{var}(\theta_j(\cdot,T),[0,S]) + \mathrm{var}(\theta_j, Q).$$

For $0 \le \sigma < 1$, let $E_\sigma(Q)$ be the space of all cylinder type functionals $F : K(Q) \to \mathbb{C}$ of the form

$$(2)\qquad F(y) = f(\langle \theta_1, y\rangle, \ldots, \langle \theta_n, y\rangle) = f(\langle \vec{\theta}, y\rangle)$$

where $f(\lambda_1, \ldots, \lambda_n) = f(\vec{\lambda})$ is an entire function of n complex variables $\lambda_1, \ldots, \lambda_n$ of expoential type; that is to say

$$|f(\vec{\lambda})| \le A_f \exp\{B_f \sum_{j=1}^n |\lambda_j|^{1+\sigma}\}$$

for some positive constants A_f and B_f.

Now we introduce a well known Yeh-Wiener integration formula for the functionals $f(\langle \vec{\theta}, x\rangle)$;

$$(3)\qquad \int_{C(Q)} f(\langle \vec{\theta}, x\rangle)dm_y(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u})\exp\{-\frac{1}{2}\|\vec{u}\|^2\}d\vec{u}$$

where $\|\vec{u}\|^2 = \sum_{j=1}^n u_j^2$ and $d\vec{u} = du_1 \cdots du_n$.

We encounter the Rimann-Stieltjes integrals $\langle \theta_j, x - [x] + [\vec{\xi}]\rangle$, when we evaluate the conditional integral transform of F given X_τ, as we will see below in Theoerms 3.4 through 3.8.

It turns out that $\tilde{\theta}$, the sectional average of θ, simplifies both the statement and the proof of the conditional integral transforms.

Definition 3.1. Let $\tau = \tau_{m,n} = \{(s_i, t_j)| i = 1, \ldots, m, j = 1, \ldots, n\}$ be a partition of Q. Then for each function $\theta \in L_2(Q)$, define the sectional average of θ by

$$(4)\qquad \tilde{\theta}(s,t) = \frac{1}{(s_k - s_{k-1})(t_l - t_{l-1})} \int_{Q_{kl}} \theta(u,v)dudv$$

on each $Q_{kl} = (s_{k-1}, s_k] \times (t_{l-1}, t_l]$ and $\tilde{\theta}(s,t) = 0$ if $st = 0$.

For each θ_j in (3.2), using (2.2) and (3.4), we can see that

$$(5)\qquad \langle \theta_j, x - [x] + [\vec{\xi}]\rangle = \langle \theta_j - \tilde{\theta}_j, x\rangle + \langle \theta_j, [\vec{\xi}]\rangle$$

for each $x \in C(Q)$.

Example 3.2. If we define $\theta(s,t)=\frac{1}{\sqrt{ST}}\in L_2(Q)$ then we can get the followings;

(1) For the partition $\tau=\tau_{1,1}=\{(S,T)\}$, $\theta-\tilde{\theta}=0$ and $\langle\theta,[\xi]\rangle=\frac{\xi}{\sqrt{ST}}$. where $\xi_{1,1}=\xi$. Thus we have

$$\langle\theta,x-[x]+[\xi]\rangle=\frac{\xi}{\sqrt{ST}}.$$

(2) For the partition $\tau=\tau_{2,2}=\{(s_i,t_j)|i,j=1,2\}$, $\theta-\tilde{\theta}=0$ and $\langle\theta,[\vec{\xi}]\rangle=\frac{\xi_{2,2}}{\sqrt{ST}}$. Thus we have

$$\langle\theta,x-[x]+[\vec{\xi}]\rangle=\frac{\xi_{2,2}}{\sqrt{ST}}.$$

Now even though the set of functions $\{\theta_1,\dots,\theta_n\}$ are orthonormal, the set of functions $\{\theta_1-\tilde{\theta_1},\dots,\theta_n-\tilde{\theta_n}\}$ need not even be orthogonal. However, using the Gram-Schmidt orthonormalization process, we can find a set of orthomormal functions $\{\phi_1,\dots,\phi_m\}$, each of bounded variation on Q, with $\operatorname{span}\{\theta_1-\tilde{\theta_1},\dots,\theta_n-\tilde{\theta_n}\}=\operatorname{span}\{\phi_1,\dots,\phi_m\}$. Then we can find $m\times n$ matrix $A_{m_n}=(a_{ij})$ with

$$\vec{\theta}-\vec{\tilde{\theta}}=(\sum_{j=1}^{m}a_{j,1}\phi_j,\dots,\sum_{j=1}^{m}a_{j,n}\phi_j)=\vec{\phi}A_{m_n} \tag{6}$$

where $\vec{\theta}-\vec{\tilde{\theta}}=(\theta_1-\tilde{\theta_1},\dots,\theta_n-\tilde{\theta_n})$. Of course the functions $\{\phi_1,\dots,\phi_m\}$ are not unique, but our results are independent of which orthonormal sets are chosen. For related and a detailed work, see [4].

Example 3.3. If we let $\{\theta_1,\theta_2\}$ be an orthonormal set of $L_2(Q)$, where $\theta_1(s,t)=\frac{\chi_{(0,s_1]\times(0,T]}(s,t)}{\sqrt{s_1T}}$ and $\theta_2(s,t)=\frac{\chi_{(s_1,S]\times(0,T]}(s,t)}{\sqrt{(S-s_1)T}}$, with $\theta_j(s,t)=0$, $j=1,2$ if $st=0$, where $\chi_{(0,s_1]\times(0,T]}$ is the characteristic function of $(0,s_1]\times(0,T]$.

Then for the partition $\tau=\tau_{1,1}=\{(S,T)\}$, we have the orthogonal set $\{\theta_1-\tilde{\theta}_1,\theta_2-\tilde{\theta}_2\}$ where $\theta_1(s,t)-\tilde{\theta}_1(s,t)=\frac{S-s_1}{S}\theta_1(s,t)-\frac{\sqrt{s_1(S-s_1)}}{S}\theta_2(s,t)$ and $\theta_2(s,t)-\tilde{\theta}_2(s,t)=-\frac{\sqrt{s_1(S-s_1)}}{S}\theta_1(s,t)+\frac{s_1}{S}\theta_2(s,t)$. In this case, $\operatorname{span}\{\theta_1-\tilde{\theta}_1,\theta_2-\tilde{\theta}_2\}=\operatorname{span}\{\phi_1\}$ where $\phi_1=\theta_1-\tilde{\theta}_1/\|\theta_1-\tilde{\theta}_1\|$.

Theorem 3.4. *Let $F\in E_\sigma(Q)$ be given by (3.2). Then the conditional integral transform $\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\vec{\xi})$ exists, belongs to $E_\sigma(Q)$ and is given by the formula $\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\vec{\xi})=h_f(\vec{\xi}:\langle\vec{\theta},y\rangle)$ for all $y\in K(Q)$ and $\vec{\xi}\in\mathbb{R}^{mn}$, where*

$$h_f(\vec{\xi}:\vec{\lambda})=(2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m}f(\alpha\vec{u}A_{m_n}+\alpha\langle\vec{\theta},[\vec{\xi}]\rangle+\beta\vec{\lambda})\exp\{-\frac{1}{2}\|\vec{u}\|^2\}d\vec{u} \tag{7}$$

with $\|\vec{u}\|^2=\sum_{j=1}^{m}u_j^2$ and $d\vec{u}=du_1\dots du_m$.

Proof. Using the definition of the conditional integral transform together with equations (3.4) through (3.6) we have,

$$\begin{aligned}\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y,\vec{\xi})&=E_x[f(\alpha(\langle\vec{\theta},x-[x]+[\vec{\xi}]\rangle)+\beta\langle\vec{\theta},y\rangle)]\\&=E_x[f(\alpha(\langle\vec{\theta}-\vec{\tilde{\theta}},x\rangle)+\alpha\langle\vec{\theta},[\vec{\xi}]\rangle+\beta\langle\vec{\theta},y\rangle)]\\&=E_x[f(\alpha(\langle\vec{\phi}A_{m_n},x\rangle)+\alpha\langle\vec{\theta},[\vec{\xi}]\rangle+\beta\langle\vec{\theta},y\rangle)]\end{aligned} \tag{8}$$

for each $y \in K(Q)$ and a.e. $\vec{\xi} \in \mathbb{R}^{mn}$.

Finally by the formula (3.3), that is to say, a Yeh-Wiener integration formula for the functional of form $f(\langle\vec{\theta}, x\rangle)$, we see the last expression of (3.8) equals $h_f(\vec{\xi} : \langle\vec{\theta}, y\rangle)$, where $h_f(\vec{\xi} : \cdot)$ is given by (3.7). Then by Theorem 3.15 in [2], $h_f(\vec{\xi} : \vec{\lambda})$ is an entire function. Furthermore using the integration $\int_{\mathbb{R}} e^{-av^2+bv}dv < \infty$ for $a > 0$ and $b \in \mathbb{R}$, we can see, $h_f(\vec{\xi} : \vec{\lambda})$ is an element of $E_\sigma(Q)$ as a function of $\vec{\lambda}$ [see [3, 4]]. Hence $\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y, \vec{\xi}) \in E_\sigma(Q)$ as a function of y. □

Example 3.5. Let $F(y) = f(\langle\theta, y\rangle)$, where $f(\lambda) = e^\lambda$ for $\lambda \in \mathbb{C}$ for $\theta(s, t) = \frac{1}{\sqrt{ST}} \in L_2(Q)$.

Then $F \in E_\sigma(Q)$ and $\langle\frac{1}{\sqrt{ST}}, x - [x] + [\xi]\rangle = \frac{\xi}{\sqrt{ST}}$. Using (2.7) and (3.3) we have the following conditional integral transform given conditioning function $X_\tau(x) = x(S, T)$,

$$\begin{aligned}\mathcal{F}_{\alpha.\beta}(F|X_\tau)(y, \xi) &= \int_{C(Q)} e^{\alpha\langle\frac{1}{\sqrt{ST}}, x-[x]+[\xi]\rangle+\beta y} dm_Y(x) \\ &= \int_{C(Q)} e^{\frac{\alpha\xi}{\sqrt{ST}}+\beta y} dm_Y(x) = e^{\frac{\alpha\xi}{\sqrt{ST}}} e^{\beta y}.\end{aligned} \tag{9}$$

In our next theorem we show that the conditional convolution product of functionals from $E_\sigma(Q)$ for the conditioning function X_τ given by (2.1) is an element of $E_\sigma(Q)$.

Theorem 3.6. *Let $F, G \in E_\sigma(Q)$ be given by (3.2) with corresponding entire functions f and g, respectively. Then the conditional convolution product $((F * G)_\alpha|X_\tau)(y, \vec{\xi})$ exists for all $y \in K(Q)$ and a.e. $\vec{\xi} \in \mathbb{R}^{mn}$, belongs to $E_\sigma(Q)$ and is given by*

$$((F * G)_\alpha|X_\tau)(y, \vec{\xi}) = k_{f*g}(\vec{\xi} : \langle\vec{\theta}, y\rangle) \tag{10}$$

where

(11)

$$\begin{aligned}&k_{f*g}(\vec{\xi} : \vec{\lambda}) \\ =&(2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} f\Big(\frac{\vec{\lambda} + \alpha\vec{u}A_{m_n} + \alpha\langle\theta_j, [\vec{\xi}]\rangle}{\sqrt{2}}\Big) g\Big(\frac{\vec{\lambda} - \alpha\vec{u}A_{m_n} - \alpha\langle\theta_j, [\vec{\xi}]\rangle}{\sqrt{2}}\Big) \exp\{-\frac{1}{2}\|\vec{u}\|^2\} d\vec{u}\end{aligned}$$

with $\|\vec{u}\|^2 = \sum_{j=1}^{m} u_j^2$ and $d\vec{u} = du_1 \dots du_m$.

Proof. Using the definition of the conditional convolution product we have the following

$$\begin{aligned}&((F * G)_\alpha|X_\tau)(y, \vec{\xi}) \\ =&E_x\Big[f\Big(\frac{\langle\vec{\theta}, y\rangle + \alpha\langle\vec{\phi}A_{m_n}, x\rangle + \alpha\langle\vec{\theta}, [\vec{\xi}]\rangle}{\sqrt{2}}\Big) g\Big(\frac{\langle\vec{\theta}, y\rangle - \alpha\langle\vec{\phi}A_{m_n}, x\rangle - \alpha\langle\vec{\theta}, [\vec{\xi}]\rangle}{\sqrt{2}}\Big)\Big].\end{aligned} \tag{12}$$

Then by the equations (3.4), (3.6), (2.8) and a well-known Yeh-Wiener integration formula (3.3), we see that the last expression above equals $k_{f*g}(\vec{\xi} : \langle\vec{\theta}, y\rangle)$. Then by Theorem 3.15 in [2], $k_{f*g}(\vec{\xi} : \vec{\lambda})$ is an entire function and an element of $E_\sigma(Q)$ as a function of $\vec{\lambda}$ [see [3, 4]].

Hence $((F * G)_\alpha|X_\tau)(y, \vec{\xi}) \in E_\sigma(Q)$ as a function of y.

□

In view of Theorems 3.4 and 3.6 above, conditional integral transforms and convolution products of functionals from $E_\sigma(Q)$ for the given conditioning function $X_\tau(x)$ are also belong to $E_\sigma(Q)$. Then by Theorem 2.6 we have the following result.

Our next result (3.13) below is useful because it allows us to calculate $\mathcal{F}_{\alpha.\beta}(((F*G)_\alpha|X_\tau)(\cdot,\vec{\xi_1})|X_\tau)(y,\vec{\xi_2})$ without ever actually calculating $(F*G)_\alpha$ or $((F*G)_\alpha|X_\tau)$.

THEOREM 3.7. *Let F and G be as in Theorem 3.3. Then we have*

$$\begin{aligned}&\mathcal{F}_{\alpha.\beta}(((F*G)_\alpha|X_\tau)(\cdot,\vec{\xi_1})|X_\tau)(y,\vec{\xi_2})\\ =&\mathcal{F}_{\alpha.\beta}(F|X_\tau)(\frac{y}{\sqrt{2}},\frac{\vec{\xi_2}+\vec{\xi_1}}{\sqrt{2}})\mathcal{F}_{\alpha.\beta}(G|X_\tau)(\frac{y}{\sqrt{2}},\frac{\vec{\xi_2}-\vec{\xi_1}}{\sqrt{2}})\end{aligned} \tag{13}$$

for each $y\in K(Q)$ and a.e. $\vec{\xi_1},\vec{\xi_2}\in\mathbb{R}^{mn}$.

Proof. The left hand side of (3.13) exists by Theorems 3.4 and 3.6 while the right hand side of (3.13) exists by Theorem 3.4. The equality in equation (3.13) then follows from (2.9). □

Our next formula (3.14), giving the conditional convolution product of conditional integral transforms follows from Theorems 3.4, 3.6 and a well-known Yeh-Wiener integration formula.

THEOREM 3.8. *Let F and G be as in Theorem 3.4. Then for all $y\in K(Q)$ and a.e. $\vec{\xi_i}\in\mathbb{R}^{mn}$, $i=1,2,3$*

$$\begin{aligned}&((\mathcal{F}_{\alpha.\beta}(F|X_\tau)(\cdot,\vec{\xi_1})*\mathcal{F}_{\alpha.\beta}(G|X_\tau)(\cdot,\vec{\xi_2}))_\alpha|X_\tau)(y,\vec{\xi_3})\\ =&(2\pi)^{-\frac{3m}{2}}\int_{\mathbb{R}^{3m}} f(\alpha\vec{u}_2A_{m_n}+\alpha\langle\vec{\theta},[\vec{\xi_1}]\rangle+\frac{\beta}{\sqrt{2}}(\langle\vec{\theta},y\rangle+\alpha\vec{u}_1A_{m_n}+\alpha\langle\vec{\theta},[\vec{\xi_3}]\rangle))\\ &g(\alpha\vec{u}_3A_{m_n}+\alpha\langle\vec{\theta},[\vec{\xi_2}]\rangle+\frac{\beta}{\sqrt{2}}(\langle\vec{\theta},y\rangle-\alpha\vec{u}_1A_{m_n}-\alpha\langle\vec{\theta},[\vec{\xi_3}]\rangle))\\ &exp\{-\frac{1}{2}\sum_{j=1}^{3}\|\vec{u}_j\|^2\}d\vec{u}_1d\vec{u}_2d\vec{u}_3\end{aligned} \tag{14}$$

where $\|\vec{u}_i\|^2=\sum_{j=1}^{m}u_{ij}^2$and $d\vec{u}_i=du_{i1}\dots du_{im}$, $i=1,2,3$.

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Bong Jin Kim
Department of Data Science, Daejin University, Pocheon 11159, Korea
E-mail: `bjkim@@daejin.ac.kr`