GERAGHTY TYPE CONTRACTIONS IN b−METRIC-LIKE SPACES

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Abstract. The main intent of this paper is to prove an existence and unique-
ness fixed point result under Geraghty contractions in b−metric-like spaces, which
remains an extended version of corresponding results in b−metric spaces and metric-
like spaces. Using two types of Geraghty contractions, an approach is adopted to
verify some fixed point results in b−metric-like spaces. Our main result is an exten-
sion of an earlier result given by Geraghty in b−metric-like-space. An example is
also provided to demonstrate the validity of our main result. Moreover, as an appli-
cation of our main result, the existence of solution of a Fredholm integral equation
is established which may further be utilized to study the seismic response of dams
during earthquakes.

1. Introduction and Preliminaries

The immense utility and natural wide range of applications of Banach contraction
principle always inspire researchers to prove enriched and improved versions of this
principle under different type of settings in varied conditions. Various researchers have
used this principle to prove their results in various classes of metric spaces(see [3, 4, 6–
9, 11, 13–16, 19–22, 25, 26, 28, 29]). By now, there already exist several classes of metric
spaces such as: b−metric spaces [12], partial metric spaces [27], partial b−metric
spaces [30] and metric like spaces [5]. The concept of b−metric-like spaces was given
by Alghamdi et al. in [2] while Hussain et al. [23] discussed the basic topological
arrangement of b−metric-like spaces besides proving some fixed point results.

A new set of auxiliary functions is defined to replace the Cauchy condition for
convergence in a complete metric besides adding the sub-additive properties to these
functions which are utilized to prove new results in metric spaces under Geraghty
contractions (e.g., [18]). Thus far, many researchers have generalized such results
under Geraghty contractions to several classes of metric spaces (see [1, 10, 17, 31]).
In the same continuation, we further prove new results in b−metric-like spaces using
Geraghty contractions.

The following definitions are relevant to our present work.

DEFINITION 1.1. [5] A metric-like ψ on a set Q, having atleast one element, is a mapping $\varphi : Q \times Q \to [0, \infty)$ such that for all $\psi, \sigma, \tau \in Q$, the following conditions are fulfilled:

(i) $\varphi(\psi, \sigma) = 0$ implies $\psi = \sigma$,
(ii) $\varphi(\psi, \sigma) = \varphi(\sigma, \psi)$,
(iii) $\varphi(\psi, \sigma) \leq \varphi(\psi, \tau) + \varphi(\tau, \sigma)$.

As usual, the pair (Q, φ) is called a metric-like space.

DEFINITION 1.2. [2] Let Q be a set having atleast one element and $l \geq 1$ be a real. A function $\varphi_b : Q \times Q \to [0, \infty)$ is a $b-$metric-like if, for all $\psi, \sigma, \tau \in Q$, following conditions are fulfilled:

(i) $\varphi_b(\psi, \sigma) = 0$,
(ii) $\varphi_b(\psi, \sigma) = \varphi_b(\sigma, \psi)$,
(iii) $\varphi_b(\psi, \sigma) \leq l[\varphi_b(\psi, \tau) + \varphi_b(\tau, \sigma)]$.

As earlier, $b-$metric-like space is the pair (Q, φ_b).

DEFINITION 1.3. [2] Assume (Q, φ_b) to be a $b-$metric-like space with a constant $l \geq 1$,. Let $\{\psi_n\}$ be a sequence in Q with $\psi \in Q$. Then

1. $\{\psi_n\}$ is called convergent to q with respect to ω_{φ_b}, if $\lim\limits_{n \to \infty} \varphi_b(\psi_n, \psi) = \varphi_b(\psi, \psi)$,

2. $\{\psi_n\}$ is said to be Cauchy if $\lim\limits_{n,m \to \infty} \varphi_b(\psi_n, \psi)$ exists and is finite,

3. (Q, φ_b) is called complete if for every Cauchy sequence $\{\psi_n\}$ in Q there exists $\psi \in Q$ such that $\lim\limits_{n,m \to \infty} \varphi_b(\psi_n, \psi_m) = \lim\limits_{n \to \infty} \varphi_b(\psi_n, \psi) = \varphi_b(\psi, \psi)$.

EXAMPLE 1.4. [2] Let $Q = [0, \infty)$ and $\varphi_b : Q \times Q \to [0, \infty)$ is defined by

$$\varphi_b(\psi, \sigma) = (\psi + \sigma)^2 \text{ for all } \psi, \sigma \in Q.$$

Then the pair (Q, φ_b) is a $b-$metric like space with $l = 2$.

2. Main Results

Let k be the set of all functions $\rho : [0, \infty) \to [0, \frac{1}{l})$ (for any $l \geq 1$) which satisfies the condition: $\limsup\limits_{n \to \infty} \rho(q_n) = \frac{1}{l}$ implies that $q_n \to 0$ as $n \to \infty$ (see [18]).

DEFINITION 2.1. Let (Q, φ_b) be a $b-$metric-like space with a constant $l \geq 1$. A self mapping $G : Q \to Q$ is said to be a Geraghty type contraction if the following condition holds:

(1) $\varphi_b(G\psi, G\sigma) \leq \rho(M(\psi, \sigma)) M(\psi, \sigma)$ for all $\psi, \sigma \in Q$

where

$$M(\psi, \sigma) = \max\left\{ \varphi_b(\psi, \sigma), \varphi_b(\psi, G\psi), \varphi_b(\sigma, G\sigma), \frac{1}{2l}(\varphi_b(\psi, G\sigma) + \varphi_b(\sigma, G\psi)) \right\},$$

and $\rho \in$ k.

Now, we state and proof one of our main results as follows:

THEOREM 2.2. *Assume that the pair (Q, φ_b) forms a complete $b-$metric-like space with a constant $l \geq 1$. If the mapping $G : Q \to Q$ is a Geraghty type contraction. Then there is a fixed point in G which is unique.*

Proof. Let $q_0 \in Q$ be arbitrary. Consider the sequence $\{\psi_n\}$ where

$$\psi_n = G\psi_{n-1} = G^n\psi_0, \ \forall \ n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N}$ for which $\psi_{n+1} = \psi_n$, implies ψ_n is already a fixed point and we are through.

Now, let $\psi_{n+1} \neq \psi_n$. Then by using condition (1) for every $n \in \mathbb{N}$, we have

$$(2) \qquad \varphi_b\left(\psi_{n+1}, \psi_n\right) = \varphi_b\left(G\psi_{n-1}, G\psi_n\right) \leq \rho\left(M\left(\psi_{n-1}, \psi_n\right)\right) M\left(\psi_{n-1}, \psi_n\right),$$

where

$$
\begin{aligned}
M\left(\psi_{n-1}, \psi_n\right) \ &= \ \max\Bigg\{ \varphi_b(\psi_{n-1}, \psi_n), \varphi_b(\psi_{n-1}, G\psi_{n-1})\varphi_b(\psi_n, G\psi_n) \\
&\qquad \frac{\varphi_b\left(\psi_{n-1}, G\psi_n + \varphi_b\left(\psi_n, G\psi_{n-1}\right)\right)}{2l} \Bigg\} \\
&\leq \ \max\Bigg\{ \varphi_b\left(\psi_{n-1}, \psi_n\right), \varphi_b\left(\psi_n, \psi_{n+1}\right), \\
&\qquad \frac{\varphi_b\left(\psi_{n-1}, \psi_n\right), \varphi_b\left(\psi_n, \psi_{n+1}\right)}{2l} \Bigg\} \\
&= \ \max\{\varphi_b(\psi_{n-1}, \psi_n), \varphi_b(\psi_n, \psi_{n+1})\}.
\end{aligned}
$$

If $\varphi_b\left(\psi_{n-1}, \psi_n\right) \leq \varphi_b\left(\psi_n, \psi_{n+1}\right)$, then $M(\psi_{n-1}, \psi_n) = \varphi_b(\psi_n, \psi_{n+1})$. Now, on using (2), we have

$$
\begin{aligned}
\varphi_b(\psi_n, \psi_{n+1}) \ &\leq \ \rho(M(\psi_{n-1}, \psi_n))M(\psi_{n-1}, \psi_n) \\
&\leq \ \frac{1}{l}\varphi_b(\psi_n, \psi_{n+1})
\end{aligned}
$$

for $n \in \mathbb{N}$ which contradicts. Thus, we have $M\left(\psi_{n-1}, \psi_n\right) = \varphi_b\left(\psi_n, \psi_{n-1}\right).$ Therefore, using (2), we get

$$
\begin{aligned}
\varphi_b(\psi_n, \psi_{n+1}) \ &\leq \ \rho(M(\psi_{n-1}, \psi_n))\varphi_b(\psi_{n-1}, \psi_n) \\
(3) \qquad &\leq \ \frac{1}{l}\varphi_b\left(\psi_{n-1}, \psi_n\right) < \varphi_b\left(\psi_{n-1}, \psi_n\right),
\end{aligned}
$$

which shows that $\{\varphi_b\left(\psi_{n-1}, \psi_n\right)\}$ is a decreasing sequence. Hence there exists $q \geq 0$ for which $\varphi_b\left(\psi_{n-1}, \psi_n\right) \to q$ as $n \to \infty$. We assert that, $q = 0$. Let on contrary, $q > 0$, then on employing (3), we have

$$q \leq \limsup_{n\to\infty} \rho\left(M\left(\psi_{n-1}, \psi_n\right)\right) q,$$

so that

$$\frac{1}{l} \leq 1 \leq \limsup_{n\to\infty} \rho\left(M\left(\psi_{n-1}, \psi_n\right)\right) \leq \frac{1}{l}.$$

Since, $\rho \in \mathrm{k}$, therefore $\lim_{n\to\infty} M(\psi_{n-1}, \psi_n) = 0$ implies $\lim_{n\to\infty} \varphi_b\left(\psi_{n-1}, \psi_n\right) = 0$, which is contradictory yielding thereby $q = 0$. Next, to show that $\{\psi_n\}$ is a Cauchy sequence. Assume on contrary that the sequence $\{\psi_n\}$ is not Cauchy. Then, we have $\varepsilon > 0$ for

the sub sequences $\{\psi_{n(a)}\}$ and $\{\psi_{m(a)}\}$ of $\{\psi_n\}$ for which $n(a)$ is the smallest index with $n(a) > m(a) > a$ such that

$$(4) \qquad\qquad \varphi_b\left(\psi_{m(a)}, \psi_{n(a)}\right) \geq \varepsilon,$$

while

$$(5) \qquad\qquad \varphi_b\left(\psi_{m(a)}, \psi_{n(a)-1}\right) < \varepsilon.$$

Using triangular inequality and condition (5), we have

$$\varepsilon \leq l(\varphi_b\left(\psi_{m(a)}, \psi_{n(a)+1}\right) + \varphi_b\left(\psi_{m(a)+1}, \psi_{n(a)}\right)),$$

so that

$$(6) \qquad\qquad \frac{\varepsilon}{l} \leq \limsup_{a\to\infty} \varphi_b(\psi_{m(a)+1}, \psi_{n(a)}).$$

Therefore,

$$\limsup_{a\to\infty} M(\psi_{m(a)}, \psi_{n(a)-1}) \leq \varepsilon.$$

Using (1) and (6), we have

$$
\begin{aligned}
\frac{\varepsilon}{l} &\leq \limsup_{a\to\infty} \varphi_b(\psi_{m(a)+1}, \psi_{n(a)}) \\
&\leq \limsup_{a\to\infty} \rho(M(\psi_{m(a)}, \psi_{n(a)-1}))M(\psi_{m(a)}, \psi_{n(a)-1}) \\
&\leq \varepsilon \limsup_{a\to\infty} \rho(M(\psi_{m(a)}, \psi_{n(a)-1})),
\end{aligned}
$$

implying thereby,

$$\frac{1}{l} \leq \limsup_{a\to\infty} \rho(M\left(\psi_{m(a)}, \psi_{n(a)-1}\right)) \leq \frac{1}{l}.$$

Since, $\rho \in k$, we have

$$\limsup_{a\to\infty} M\left(\psi_{m(a)}, \psi_{n(a)-1})\right) = \frac{1}{l} \implies \limsup_{a\to\infty} \varphi_b\left(\psi_{m(a)}, \psi_{n(a)-1})\right) = 0.$$

Using (4) and triangular inequality, we have

$$\varepsilon \leq \varphi_b(\psi_{m(a)}, \psi_{n(a)}) \leq l(\varphi_b\left(\psi_{m(a)}, \psi_{n(a)-1}\right) + \varphi_b(\psi_{n(a)-1}, \psi_{n(a)})),$$

yielding thereby $\lim_{a\to\infty} \varphi_b(\psi_{m(a)}, \psi_{n(a)}) = 0$ which contradicts (4). Hence, $\{\psi_n\}$ is a Cauchy sequence. Using (1), we have

$$\leq l(\varphi_b\left(\omega, G\psi_n\right) + s\rho(M\left(\psi_n, \omega\right)) M\left(\psi_n, \omega\right).$$

Letting $n \to \infty$ in the preceding inequality, we get

$$(7) \quad \varphi_b(\omega, G\omega) \leq s\limsup_{n\to\infty} \varphi_b(\omega, \psi_{n+1}) + s\limsup_{n\to\infty} \rho(M(\psi_n, \omega)) \limsup_{n\to\infty} M(\psi_n, \omega),$$

where

$$\begin{aligned} \limsup_{n\to\infty} M(\psi_n,\omega) &= \limsup_{n\to\infty} \max\Big\{ \varphi_b\left(\psi_n,\omega\right), \varphi_b\left(\psi_n,G\psi_n\right), \varphi_b\left(\omega,G\omega\right) \\ &\qquad\qquad \frac{\varphi_b(\psi_n,G\omega)+\varphi_b(\omega,G\psi_n)}{2l} \Big\} \\ &\le \limsup_{n\to\infty} \max\Big\{ \varphi_b\left(\psi_n,\omega\right), \varphi_b\left(\psi_n,\psi_{n+1}\right), \varphi_b\left(\omega,G\omega\right), \\ &\qquad\qquad \frac{l\varphi_b(\psi_n,\omega)+l\varphi_b(\omega,G\omega)+\varphi_b(\omega,\psi_{n+1})}{2l} \Big\} \\ &\le \varphi_b(\omega,G\omega). \end{aligned}$$

Hence, using (7) we have;

$$\varphi_b\left(\omega,G\omega\right) \le s\limsup_{n\to\infty}\rho(M(\psi_n,\omega))\varphi_b(\omega,G\omega).$$

Consequently, $\frac{1}{l} \le \varepsilon\limsup\limits_{n\to\infty}\rho(M(\psi_n,\omega)) \le \frac{1}{l}$. Since, $\rho \in$ k, we concluded that $\lim\limits_{n\to\infty} M\left(\psi_n,\omega\right)=0$, which implies that $G\omega=\omega$.

To see that $\omega \in Q$ is unique, suppose there exists $\omega \ne \psi$ in Q such that $G\omega=\omega$ and $G\psi=\psi$. From (1), we get

$$\varphi_b(\omega,\psi)=\varphi_b(G\omega,G\psi) \le \rho(M(\omega,\psi))M(\omega,\psi).$$

Recall that

$$\begin{aligned} M(\omega,\psi) &= \max\Big\{ \varphi_b(\omega,\psi), \varphi_b(\omega,G\omega), \varphi_b(\psi,G\psi), \frac{\varphi_b(\omega,G\psi)+(\varphi_b(\psi,G\omega)}{2l} \Big\} \\ &\le \varphi_b(\omega,\psi). \end{aligned}$$

Therefore, we have $\varphi_b(\omega,\psi) < \frac{1}{l}\varphi_b(\omega,\psi)$ a contradiction. Hence, $\omega=\psi$, so that a unique fixed point ω is available in G. $\qquad\square$

EXAMPLE 2.3. Consider $Q=\left[0,\frac{1}{2}\right]$ and $\varphi_b:G\times G\to[0,\infty)$ is defined as

$$\varphi_b(\psi,\sigma)=\mid\psi+\sigma\mid^2$$

for all $\psi,\sigma\in[0,\frac{1}{2}]$. It is easy to check that (Q,φ_b) is a complete $b-$metric-like space with $l=2$.

Set $G\psi=\frac{\psi}{2}$ for all $\psi\in Q$ and $\rho=\frac{1}{4}$ for all $t\ge 0$. Then,

$$\begin{aligned} \varphi_b(G\psi,G\sigma) &= \left|\frac{\psi}{2}+\frac{\sigma}{2}\right|^2 \\ &\le \frac{1}{4}\max\Big\{ |\psi+\sigma|^2, \left|\psi+\frac{\psi}{2}\right|^2, \left|\sigma+\frac{\sigma}{2}\right|^2, \\ &\qquad\qquad \frac{1}{4}\left(\left|\psi+\frac{\sigma}{2}\right|^2+\left|\sigma+\frac{\psi}{2}\right|^2\right) \Big\} \\ &\le \rho(M(\psi,\sigma))M(\psi,\sigma). \end{aligned}$$

Thus all the requirements of Theorem 2.2 are met out. Observer that, 0 is a unique fixed point in G.

Now, we state and proof a common fixed point result as follows:

THEOREM 2.4. *Assume that the pair* (Q, φ_b) *is a complete* $b-$*metric-like space with constant* $l \geq 1$. *Consider a pair of mappings* $G, H : Q \to Q$ *which satisfy*

$$(8) \qquad l\varphi_b\left(G\psi, H\sigma\right) \leq \rho\left(M\left(\psi, \sigma\right)\right) M\left(\psi, \sigma\right), \ \forall \ \psi, \sigma \in Q$$

where

$$M(\psi, \sigma) = \max\left\{\varphi_b(\psi, \sigma), \varphi_b(\psi, G\psi), \varphi_b(\sigma, H\sigma)\right\},$$

and $\rho \in k$. *If* G *(or* H*) is continuous, then there is a common fixed point of* G *and* H *in* Q *which is unique.*

Proof. Choose $q_0 \in Q$ arbitrarly. Consider the sequence $\{\psi_n\}$ by $\psi_{2n+1} = G\psi_{2n}$, and $\psi_{2n+2} = H\psi_{2n+1}$ $n \in \mathbb{N}$. If $n \in \mathbb{N}$ for which $\psi_{n+1} = \psi_n$, then ψ_n is a fixed point and we are done. So, let $\psi_{n+1} \neq \psi_n$. Then by using condition (8) for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
l\varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right) &= l\varphi_b\left(G\psi_{2n}, H\psi_{2n+1}\right) \\
&\leq \rho\left(M(\psi_{2n}, \psi_{2n+1})\right) M(\psi_{2n}, \psi_{2n+1}) \\
&= \rho\left(M(\psi_{2n}, \psi_{2n+1})\right) \max\left\{\varphi_b(\psi_{2n}, \psi_{2n+1}), \varphi_b(\psi_{2n}, G\psi_{2n}),\right. \\
&\quad \left. \varphi_b(\psi_{2n+1}, H\psi_{2n+1})\right\} \\
&= \rho\left(M(\psi_{2n}, \psi_{2n+1})\right) \max\left\{\varphi_b(\psi_{2n}, \psi_{2n+1}), \varphi_b(\psi_{2n}, \psi_{2n+1}),\right. \\
&\quad \left. \varphi_b(\psi_{2n+1}, \psi_{2n+2})\right\}.
\end{aligned}
$$
$$(9)$$

If $M\left(\psi_{2n}, \psi_{2n+1}\right) = \varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right)$, then

$$
\begin{aligned}
l\varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right) &\leq \rho\left(M\left(\psi_{2n+1}, \psi_{2n+2}\right)\right) M\left(\psi_{2n+1}, \psi_{2n+2}\right) \\
&< \frac{1}{l}\varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right),
\end{aligned}
$$

which is a contradiction. Hence, we have

$$M(\psi_{2n}, \psi_{2n+1}) = \varphi_b(\psi_{2n}, \psi_{2n+1}).$$

Using (9), we have

$$
\begin{aligned}
\varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right) &\leq \rho\left(M\left(\psi_{2n}, \psi_{2n+1}\right)\right) M\left(\psi_{2n}, \psi_{2n+1}\right) \\
&\leq \frac{1}{l}\varphi_b(\psi_{2n}, \psi_{2n+1}),
\end{aligned}
$$
$$(10)$$

which in turn yields,

$$\varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right) \leq \varphi_b\left(\psi_{2n}, \psi_{2n+1}\right).$$

Similarly

$$\varphi_b\left(\psi_{2n+2}, \psi_{2n+3}\right) \leq \varphi_b\left(\psi_{2n+1}, \psi_{2n+2}\right).$$

So, we have $\varphi_b\left(\psi_n, \psi_{n+1}\right) \leq \varphi_b\left(\psi_{n-1}, \psi_n\right)$. Thus the sequence $\{\varphi_b\left(\psi_n, \psi_{n+1}\right)\}$ is non-increasing, so that one can find $q \geq 0$ such that $\varphi_b\left(\psi_n, \psi_{n+1}\right) \to q$ as $n \to \infty$. Now, we assert that, $q = 0$. Suppose on contrary $q > 0$, then making $n \to \infty$ in (10), we have

$$q \leq \limsup_{n \to \infty} \rho\left(M(\psi_{2n}, \psi_{2n+1})\right) q,$$

so that,

$$\frac{1}{l} \leq 1 \leq \limsup_{n \to \infty} \rho\left(M(\psi_{2n}, \psi_{2n+1})\right) \leq \frac{1}{l}.$$

Since, $\rho \in k$, $\lim_{n\to\infty} M(\psi_{2n}, \psi_{2n+1}) = 0$, which amounts to saying that $\lim_{n\to\infty} \varphi_b(M(\psi_{2n}, \psi_{2n+1})) = 0$, which is a contradiction. Hence $q = 0$ which implies that $\varphi_b(\psi_n, \psi_{n+1}) \to 0$ as $n \to \infty$.

Next, we intent to show that $\{\psi_{2n}\}$ is a Cauchy sequence. To do so, let on contrary that $\{\psi_{2n}\}$ is not a Cauchy sequence. Then for $\varepsilon > 0$ we have subsequences $\{\psi_{2n(a)}\}$ and $\{\psi_{2m(a)}\}$ of $\{\psi_{2n}\}$ for which $n(a)$ is the smallest index for $n(a) > m(a) > a$ such that

$$(11) \qquad \varphi_b(\psi_{2n(a)}, \psi_{2m(a)}) \geq \varepsilon,$$

and

$$(12) \qquad \varphi_b(\psi_{2n(a)}, \psi_{2m(a)-2}) < \varepsilon.$$

Using triangular inequality and condition (8) and (11), we have

$$(13) \qquad l(\varphi_b(\psi_{2n(a)}, \psi_{2n(a)+1}) \leq \rho(M(\psi_{2n(a)}, \psi_{2m(a)-1}))M(\psi_{2n(a)}, \psi_{2m(a)-1})$$

where

$$\begin{aligned} M(\psi_{2n(a)}, \psi_{2m(a)-1}) &= \max\{\varphi_b(\psi_{2n(a)}, \psi_{2m(a)-1}), \varphi_b(\psi_{2n(a)}, G\psi_{2n(a)}), \\ &\qquad \varphi_b(\psi_{2m(a)-1}, H\psi_{2m(a)-1})\}. \end{aligned}$$

Letting $a \to \infty$, we have

$$\limsup_{a\to\infty} M(\psi_{2n(a)}, \psi_{2m(a)-1}) = \limsup_{a\to\infty} \varphi_b(\psi_{2n(a)}, \psi_{2m(a)-1}).$$

Using triangular inequality, we have

$$\varphi_b(\psi_{2n(a)}, \psi_{2m(a)-1}) \leq l\left(\varphi_b(\psi_{2n(a)}, \psi_{2m(a)-2}) + \varphi_b(\psi_{2m(a)-2}, \psi_{2m(a)-1})\right),$$

which on letting $a \to \infty$, we obtain

$$(14) \qquad \limsup_{a\to\infty} \varphi_b(\psi_{2n(a)}, \psi_{2m(a)-1}) \leq l\varepsilon.$$

Making use of (13) and (14), we obtain

$$\begin{aligned} \varepsilon &\leq \limsup_{a\to\infty} \left(\rho(M(\psi_{2n(a)}, \psi_{2m(a)-1})) M(\psi_{2n(a)}, \psi_{2m(a)-1})\right) \\ &\leq l\varepsilon \limsup_{a\to\infty} \rho(M(\psi_{2n(a)}, \psi_{2m(a)-1})), \end{aligned}$$

so that

$$\frac{1}{l} \leq \limsup_{a\to\infty} \rho(M(\psi_{2n(a)}, \psi_{2m(a)-1})) \leq \frac{1}{l}.$$

Since, $\rho \in$ k, we have $\lim_{a\to\infty} M(q_{2n(a)}, q_{2m(a)-1}) = 0$, Consequently,

$$(15) \qquad \lim_{a\to\infty} \varphi_b(\psi_{2n(a)}, \psi_{2m(a)-1}) = 0.$$

From (11) and triangular inequality, we have

$$\varepsilon \leq \varphi_b(\psi_{2n(a)}, \psi_{2m(a)}) \leq l\left(\varphi_b(\psi_{2n(a)}, \psi_{2m(a)-1}) + \varphi_b(\psi_{2m(a)-1}, \psi_{2m(a)})\right),$$

which on letting $a \to \infty$ besides using (15), we have

$$\limsup_{a\to\infty} \varphi_b(\psi_{2n(a)}, \psi_{2m(a)}) = 0,$$

which is again a contradiction. Thus $\{\psi_{2n}\}$ is a Cauchy and so is $\{\psi_n\}$. Since (Q, φ_b) is a complete $b-$metric-like space then there exists $\psi^* \in Q$ such that $\lim\limits_{n \to \infty} \psi_n = \psi^*$. If G is continuous, we have

$$G\psi^* = \lim_{n \to \infty} G\psi_{2n} = \lim_{n \to \infty} \psi_{2n+1} = \psi^*.$$

From (8), we have

$$l(\varphi_b(\psi^*, H\psi^*)) = l(\varphi_b(G\psi^*, H\psi^*)) \leq \rho(M(\psi^*, \psi^*)) M(\psi^*, \psi^*),$$

where

$$M(\psi^*, \psi^*) = \max\{\varphi_b(\psi^*, \psi^*), \varphi_b(\psi^*, G\psi^*), \varphi_b(\psi^*, H\psi^*)\}$$
$$= \varphi_b(\psi^*, H\psi^*).$$

Since, $\rho \in$ k, we obtain

$$l\varphi_b(\psi^*, H\psi^*) \leq \rho(M(\psi^*, \psi^*)) M(\psi^*, H\psi^*) \leq \frac{1}{l}\varphi_b(\psi^*, H\psi^*).$$

a contradiction. Therefore, $H\psi^* = \psi^*$. If H is continuous, then similarly we show G and H have a common fixed point.

To prove the uniqueness, let σ be the another common fixed point of G and H, then using (8), we get

$$l\varphi_b(\psi^*, \sigma) = l\varphi_b(G\psi^*, H\sigma) \leq \rho(M(\psi^*, \sigma)) M(\psi^*, \sigma),$$

where

$$M(\psi^*, \sigma) = \max\{\varphi_b(\psi^*, \sigma), \varphi_b(\psi^*, G\psi^*), \varphi_b(\sigma, H\sigma)\} = \varphi_b(\psi^*, \sigma),$$

which implies $\psi^* = \sigma$ and a unique common fixed point of G and H is available in Q. $\qquad\square$

COROLLARY 2.5. *Let (Q, φ_b) be a complete $b-$metric-like space with constant $l \geq 1$. Let $G : Q \to Q$ be a self mapping satisfying*

(16) $$l\varphi_b(G\psi, G\sigma) \leq \rho(M(\psi, \sigma)) M(\psi, \sigma) \ \forall \ \psi, \sigma \in Q$$

where

$$M(\psi, \sigma) = \max\{\varphi_b(\psi, \sigma), \varphi_b(\psi, G\psi), \varphi_b(\sigma, G\sigma)\},$$

and $\rho \in$ k. If G is continuous, then there is a unique fixed point of G available in Q.

EXAMPLE 2.6. In Example 2.3, we consider the mappings $G, H : Q \to Q$ defined by

$$G\psi = \frac{\psi}{2\sqrt{2}}, \ H\psi = \frac{\psi}{4\sqrt{2}} \text{ for all } \psi \in Q$$

and $\rho = \frac{1}{4}$ for all $t \geq 0$. Then,

$$
\begin{aligned}
l\varphi_b(G\psi, H\sigma) &= 2\left|\frac{\psi}{2\sqrt{2}} + \frac{\sigma}{4\sqrt{2}}\right|^2 \\
&= \left|\frac{\psi}{2} + \frac{\sigma}{4}\right|^2 \\
&\leq \frac{1}{4}\max\left\{|\psi + \sigma|^2, \left|\psi + \frac{\psi}{2\sqrt{2}}\right|^2, \left|\sigma + \frac{\sigma}{4\sqrt{2}}\right|^2\right\}, \\
&\leq \rho(M(\psi, \sigma)) M(\psi, \sigma).
\end{aligned}
$$

Thus all the requirements of Theorem 2.4 are met out. Observe that, 0 remains a unique common fixed point of G and H in Q.

3. Application

In this section, we apply Theorem 2.2 to solve an integral equation. To do this, consider the following integral equation (for all $s, t \in [0, 1]$):

$$a(s) = h(s) + \int_0^1 K(s, t, a(t)) dt$$

where $h : [0, 1] \to \mathbb{R}$, $F : [0, 1] \times [0, 1] \to \mathbb{R}$ and $K : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Consider $Q = C[0, 1]$ to be the set of all real continuous functions on $[0, 1]$ while $\varphi_b : Q \times Q \to [0, \infty)$ is defined by

$$\varphi_b(a, b) = \max \mid a(s) + b(s) \mid^2 \text{ for all } a, b \in Q.$$

Then (Q, φ_b) is a complete $b-$metric-like space with $l = 2$.

THEOREM 3.1. *Suppose that (for all $a, b \in Q$)*

1. there exists a continuous $\xi \in [0, 1] \times [0, 1] \to \mathbb{R}$ such that

(17) $$\mid K(s, t, a(t)) + K(s, t, b(t)) \mid \leq \lambda^{\frac{1}{2}} \mid \xi(x, y) \mid (\mid a(t) + b(t) \mid)$$

2. set $\rho(r) = \lambda \in [0, \frac{1}{l})$ for all $r \geq 0$
3. $\sup\limits_{s \in [0,1]} \int_0^1 \mid \xi(s, t) \mid dt \leq 1$.

Then the integral equation (17) has a unique solution.

Proof. Define $G : Q \to Q$ by

$$Ga(s) = h(s) + \int_0^1 K(s, t, a(t)) dt \text{ for all } s, t \in [0, 1].$$

Observe that, the point 'a' is a fixed point of the operator G if and only if it is a solution of the integral equation (17). Now, for all $a, b \in Q$, we have

$$
\begin{aligned}
|Ga(s) + Gb(s)|^2 &= \left| \int_0^1 K(s,t,a(t))dt + \int_0^1 K(s,t,b(t))dt \right|^2 \\
&\leq \left(\int_0^1 |K(s,t,a(t)) + K(s,t,b(t))| \, dt \right)^2 \\
&\leq \left(\int_0^1 \lambda^{\frac{1}{2}} \mid \xi(x,y) \mid (\mid a(t) + b(t) \mid) dt \right)^2 \\
&= \left(\int_0^1 \lambda^{\frac{1}{2}} \mid \xi(x,y) \mid (\mid a(t) + b(t) \mid^2)^{\frac{1}{2}} dt \right)^2 \\
&= \left(\int_0^1 \lambda^{\frac{1}{2}} \mid \xi(x,y) \mid (\varphi_b(a,b))^{\frac{1}{2}} dt \right)^2 \\
&\leq \lambda \varphi_b(a,b) \left(\int_0^1 \mid \xi(x,y) \mid dt \right)^2 \\
&\leq \lambda \varphi_b(a,b) \\
&\leq \rho(M(a,b))M(a,b).
\end{aligned}
$$

Therefore all the conditions of the Theorem 2.2 are satisfied and 'a' is the unique solution of the integral equation (17). Hence G has a unique fixed point.   □

4. Conclusion

The Fredholm integral equation can also be used to calculate the inertia forces created by the earthquake in a dam. Observe that an earth dam is a three-dimensional structure and earthquake time history is a time-varying occurrence with nonlinear elastic material properties. An earthquake strikes the dam's foundation rock, which is not rigid, creating radiation damping. To analyze the behavior of dams during earthquakes, an approximate solution is on card, which can be calculated using the above Fredholm integral equation. In this paper, it is proved that Geraghty contractions can be used to obtain a unique fixed point in $b-$metric-like space. Moreover, as an application, a solution to Fredholm integral equation is also provided which helps us to study the behavior of dams during earthquakes.

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