Commun. Korean Math. Soc. **37** (2022), No. 1, pp. 57–64
https://doi.org/10.4134/CKMS.c200480
pISSN: 1225-1763 / eISSN: 2234-3024

b-GENERALIZED DERIVATIONS ON BANACH ALGEBRAS

Husain Alhazmi

Abstract. In this paper, we show, among others, that if A is a Banach algebra satisfying a functional identity involving a b-generalized derivation F on A, under some mild conditions, is of the form $F(x) = ax$ for all $x \in R$, where $a \in Q_r$, a right Martindale quotient ring of A.

1. Introduction and results

Throughout this paper, we let A denote a prime Banach algebra over a real or complex field with identity e, $Z(A)$ denote center of A, M be a closed linear subspace of A and Q_r right Martindale quotient ring of A. "*A linear mapping* $d : A \longrightarrow A$ *is said to be a derivation on* A *if* $d(xy) = d(x)y + xd(y)$ *holds for all* $x, y \in A$". In [9, Theorem 2], Posner proved that "*if a prime ring* R *admits a nonzero derivation* d *such that* $[d(x), x] \in Z(R)$ *for all* $x \in R$*, then* R *is commutative*". Further, generalizations of Posner's result can be found in [4, 14–16]. "*An additive mapping* $F : R \longrightarrow R$ *is called a generalized derivation of* R *if there exists a derivation* d *of* R *such that* $F(xy) = F(x)y + xd(y)$ *for all* $x, y \in R$".

In [5, 6], Herstein established that "*a ring* R *is commutative if it has no nonzero nilpotent ideal and there is a fixed integer* $n > 1$ *such that* $(xy)^n = x^n y^n$ *for all* $x, y \in R$" (see also [3]). In the case of Banach algebra, Yood [17] sharpened these results. More precisely, he proved the following result: "*Suppose that there are non-empty open subsets* G_1 *and* G_2 *of* A *such that for each* $x \in G_1$ *and* $y \in G_2$ *there is an integer* $n = n(x, y) > 1$ *such that either* $(xy)^n - x^n y^n$ *or* $(xy)^n - y^n x^n$ *lies in* M*. Then* $[x, y] \in M$ *for all* $x, y \in A$".

Motivated by above results, very recently Ali and Khan[1] proved the following result:

Theorem 1.1. *Let* A *be a unital prime Banach algebra and* G_1*,* G_2 *be open subsets of* A *such that for each* $x \in G_1$*, and* $y \in G_2$ *there is an integer* $m = m(x, y) > 1$*. If* A *admits a nonzero continuous linear derivation* $d : A \to A$

Received December 28, 2020; Accepted May 6, 2021.
2010 *Mathematics Subject Classification.* 16W25, 46J10.
Key words and phrases. Banach algebra, b-generalized derivation, right Martindale quotient ring.

such that either $d((xy)^m) - x^m y^m \in Z(A)$ or $d((xy)^m) - y^m x^m \in Z(A)$, then A is commutative.

Many authors have extended above result for generalized derivations, generalized skew derivations (see [2, 7, 10–13] and references therein).

We shall study the analogue problem on Banach algebras involving some special class of derivations namely b-generalized derivations. We will now recall the definition of a b-generalized derivation of A. In a recent paper [8], Koşan and Lee proposed that "an additive map $F : R \to Q_r$ is called a left b-generalized derivation, with an associated additive mapping δ from R to Q_r, if $F(xy) = F(x)y + bx\delta(y)$ for all $x, y \in R$ and $b \in Q_r$, where R is a prime ring and Q_r is the right Martindale quotient ring of R". Also, they proved that, "if R is a prime ring, then δ is a derivation of R". Particularly, we say F is a b-generalized derivation with an associated pair (b, δ). Clearly, "any generalized derivation with an associated derivation δ is a b-generalized derivation with an associated pair $(1, \delta)$". Similarly, "the mapping $x \to ax + b[x, c]$, for $a, b, c \in Q_r$, is a b-generalized derivation with an associated pair $(b, ad(c))$, where $ad(c)(x) = [x, c]$ denotes the inner derivation of R induced by the element c". More generally, "the mapping $x \to ax + qxc$, for $a, c, q \in Q_r$, is a b-generalized derivation with an associated pair $(q, ad(c))$". This mapping is called an inner b-generalized derivation.

We deal with the following:

Theorem 1.2. *Let A be a noncommutative unital prime Banach algebra and G_1, G_2 be open subsets of A such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If A admits a continuous linear b-generalized derivation $F : A \to A$ such that either $F((xy)^m) - x^m y^m \in Z(A)$ or $F((xy)^m) - y^m x^m \in Z(A)$, then $F(x) = ax$ for all $x \in A$, where $a \in Q_r$.*

Theorem 1.3. *Let A be a noncommutative unital prime Banach algebra and G_1, G_2 be open subsets of A such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If A admits a continuous linear b-generalized derivation $F : A \to A$ such that either $F((xy)^m) + x^m y^m \in Z(A)$ or $F((xy)^m) - y^m x^m \in Z(A)$, then $F(x) = ax$ for all $x \in A$, where $a \in Q_r$.*

The following are immediate consequences of Theorem 1.2 and Theorem 1.3.

Corollary 1.4. *Let A be a noncommutative unital prime Banach algebra and G_1, G_2 be open subsets of A such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If A admits a continuous linear generalized derivation $F : A \to A$ such that either $F((xy)^m) - x^m y^m \in Z(A)$ or $F((xy)^m) - y^m x^m \in Z(A)$, then $F(x) = ax$ for all $x \in A$, where $a \in Q_r$.*

Corollary 1.5. *Let A be a unital noncommutative prime Banach algebra and G_1, G_2 be open subsets of A such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If A admits a nonzero continuous linear derivation $d : A \to A$ such that either $d((xy)^m) - x^m y^m \in Z(A)$ or $d((xy)^m) - y^m x^m \in Z(A)$, then $d = 0$.*

Corollary 1.6. *Let A be a noncommutative unital prime Banach algebra and G_1, G_2 be open subsets of A such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If A admits a continuous linear generalized derivation $F : A \to A$ such that either $F((xy)^m) + x^m y^m \in Z(A)$ or $F((xy)^m) - y^m x^m \in Z(A)$, then $F(x) = ax$ for all $x \in A$, where $a \in Q_r$.*

Corollary 1.7. *Let A be a unital noncommutative prime Banach algebra and G_1, G_2 be open subsets of A such that for each $x \in G_1$, and $y \in G_2$ there is an integer $m = m(x, y) > 1$. If A admits a nonzero continuous linear derivation $d : A \to A$ such that either $d((xy)^m) + x^m y^m \in Z(A)$ or $d((xy)^m) - y^m x^m \in Z(A)$, then $d = 0$.*

Recall some prominent facts which we use to prove our results:

Fact 1. Let $p(t) = \sum_{r=0}^{n} b_r t^r$ be a polynomial in real variable t for infinite values of t and each $b_r \in A$. If $p(t) \in M$, then each b_r lies in M.

Fact 2. If F is a b-generalized derivation on A, then $G = F \pm nI_{id}$, where n is a positive integer and I_{id} is an identity map on A, is also a b-generalized derivation on A.

Now we are ready to prove our theorems:

Proof of Theorem 1.2. Fix $x \in G_1$, for each n we define the set $U_n = \{y \in A \mid F((xy)^n) - x^n y^n \notin Z(A)$ and $F((xy)^n) - y^n x^n \notin Z(A)\}$. It is easy to show that U_n is open. Applications of Baire category theorem yield there exists a positive integer r such that U_r is not dense. Thus, for a non empty open set G_3 in U_r^c such that either $F((xy)^r) - x^r y^r \in Z(A)$ or $F((xy)^r) - y^r x^r \in Z(A)$ for all $y \in G_3$. Then $v_0 + tw \in G_3$, where $v_0 \in G_3$, $w \in A$ and for adequately least real t. Thus, we have

$$F((x(v_0 + tw))^r) - x^r (v_0 + tw)^r \in Z(A) \tag{1.1}$$

or

$$F((x(v_0 + tw))^r) - (v_0 + tw)^r x^r \in Z(A). \tag{1.2}$$

Thus at least one of (1.1) and (1.2) is valid for infinitely many t. Suppose (1.1) holds for these t. Then the expression $F((x(v_0 + tw))^r) - x^r (v_0 + tw)^r$ can be written as

$$\begin{aligned}
&F(A_{r,0}(x, v_0, w)) - x^r B_{r,0}(v_0, w) \\
&+ F(A_{r-1,1}(x, v_0, w)) - x^r B_{r-1,1}(v_0, w)t + \cdots \\
&+ F(A_{1,r-1}(x, v_0, w)) - x^r B_{1,r-1}(v_0, w)t^{r-1} \\
&+ F(A_{0,r}(x, v_0, w)) - x^r B_{0,r}(v_0, w)t^r,
\end{aligned}$$

where $A_{i,j}(x, v_0, w)$ denotes the sum of all terms in which xv_0 appears exactly i times and xw appears exactly j times in the expansion of $F(x(v_0+tw)^r)$, where i and j are non-negative integers such that $i + j = r$. Similarly, $B_{i,j}(v_0, w)$ is sum of all terms in which v_0 appears exactly i times and w appears exactly j

times in the expansion of $(v_0 + tw)^r$, where i and j are non-negative integers such that $i+j = r$. The above expression is a polynomial in t and the coefficient of t^r in this polynomial is $F((xw)^r) - x^r w^r$. Therefore in view of Fact 1, we have $F((xw)^r) - x^r w^r \in Z(A)$. If (1.2) is holds for these t, then we are forced to conclude that $F((xw)^r) - w^r x^r \in Z(A)$. Thus, given $x \in G_1$ there is a positive integer r depending on w such that for each $w \in A$ either $F((xw)^r) - x^r w^r \in Z(A)$ or $F((xw)^r) - w^r x^r \in Z(A)$. Next, fix $y \in A$ and for each positive integer k, set $V_k = \{v \in A \mid F((vy)^k) - v^k y^k \notin Z(A)$ and $F((vy)^k) - y^k v^k \notin Z(A)\}$. Each V_k is open (as we shown above). If each V_k is dense then by the Baire category theorem so is the intersection also but this contrary to what was shown earlier concerning the open set G_1. Hence there is an integer $m > 1$ and a non empty open subset G_4 in the complement of V_m. If $x_0 \in G_4$ and $y \in A$, then $x_0 + tu \in G_4$ for all sufficiently small real t. Hence for positive integer $m > 1$ either

$$F(((x_0 + tu)y)^m) - (x_0 + tu)^m y^m \in Z(A)$$

or

$$F(((x_0 + tu)y)^m) - y^m (x_0 + tu)^m \in Z(A)$$

for each $u \in A$ and $x_0 \in G_4$. Arguing as above we see that either $F((uy)^m) - u^m y^m \in Z(A)$ or $F((uy)^m) - y^m u^m \in Z(A)$ for each $u \in A$.

Now let S_k, $k > 1$, be the set of $y \in A$ such that for each $w \in A$ either $F((wy)^k) - w^k y^k \in Z(A)$ or $F((wy)^k) - y^k w^k \in Z(A)$, then the union of S_k will be A. It can be easily prove that each S_k is closed. Hence again by Baire category theorem some S_l, $l > 1$, must have a non empty open subset G_5. Let $y_0 \in G_5$, for all sufficiently small real t and each $z \in A$ either

$$F((w(y_0 + tz))^l) - w^l (y_0 + tz)^l \in Z(A)$$

or

$$F((w(y_0 + tz))^l) - (y_0 + tz)^l w^l \in Z(A).$$

By earlier arguments, we have for each $w, z \in A$ either $F((wz)^l) - w^l z^l \in Z(A)$ or $F((wz)^l) - z^l w^l \in Z(A)$. Next, since A is unital then, for all real t either

$$F(((e + tx)y)^n) - (e + tx)^n y^n \in Z(A)$$

or

$$F(((e + tx)y)^n) - y^n (e + tx)^n \in Z(A)$$

for all $x, y \in A$. Hence taking coefficient of t in the expansion of above equations and using Fact 1, we get either

$$F(xy^n + \sum_{k=1}^{n-1} y^k x y^{n-k}) - nxy^n \in Z(A) \tag{1.3}$$

or

$$F(xy^n + \sum_{k=1}^{n-1} y^k x y^{n-k}) - ny^n x \in Z(A) \tag{1.4}$$

for all $x, y \in A$. Now, taking $F[(y(e+tx))^n]$ in instead of $F[((e+tx)y)^n]$, we see that either

$$F(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - nxy^n \in Z(A) \tag{1.5}$$

or

$$F(y^n x + \sum_{k=1}^{n-1} y^k x y^{n-k}) - ny^n x \in Z(A) \tag{1.6}$$

for all $x, y \in A$. Then at least one of pair of equations {(1.3), (1.5)}, {(1.3), (1.6)}, {(1.4), (1.5)} and {(1.4), (1.6)} must hold. On combining the equations in these pairs, we get either

$$F([x, y^n]) \in Z(A) \text{ for all } x, y \in A.$$

Or

$$F([x, y^n]) \pm n[x, y^n] \in Z(A) \text{ for all } x, y \in A.$$

Replacing y by $e + ty$ in above equation and using same arguments as we have used above, we obtain either

$$F([x, y]) \in Z(A) \text{ for all } x, y \in A. \tag{1.7}$$

Or

$$F([x, y]) \pm n[x, y] \in Z(A) \text{ for all } x, y \in A. \tag{1.8}$$

First we consider the case

$$F([x, y]) \in Z(A) \text{ for all } x, y \in A.$$

This can be written as

$$[F([x, y]), w] = 0 \text{ for all } x, y, w \in A.$$

Replacing y by yx in above relation, we obtain

$$[F([x, y])x + b[x, y]d(x), w] = 0 \text{ for all } x, y, w \in A.$$

This implies that

$$F([x, y])[x, w] + b[x, y][d(x), w] + b[[x, y], w]d(x) + [b, w][x, y]d(x) = 0 \tag{1.9}$$

for all $x, y, w \in A$. Replacing x by $x + z$, where $z \in Z(A)$, we get

$$F([x, y])[x, w] + b[x, y][d(x), w] + b[[x, y], w]d(z) + [b, w][x, y]d(z) = 0 \tag{1.10}$$

for all $x, y, w \in A$ and for all $z \in Z(A)$. In view of last two relations, we get

$$(b[[x, y], w] + [b, w][x, y])d(z) = 0 \text{ for all } x, y, w \in A \text{ and } z \in Z(A).$$

If $d(z) \neq 0$, then

$$b[[x, y], w] + [b, w][x, y] = 0 \text{ for all } x, y, w \in A. \tag{1.11}$$

For $w = b$, above relations reduce into

$$b[[x, y], b] = 0 \text{ for all } x, y \in A.$$

Since $b \neq 0$, so we have $[[x,y],b] = 0$ for all $x, y \in A$, and hence $b \in Z(A)$. Using this in (1.11), we get $b[[x,y],w] = 0$ for all $x, y, w \in A$. This implies A is commutative. If $d(z) = 0$, then in view of (1.10), we have

$$F([x,y])[x,w] + b[x,y][d(x),w] = 0$$

for all $x, y, w \in A$. Taking $x = w$ in above expression gives

$$b[w,y][d(w),w] = 0$$

for all $x, w \in A$. This further implies that either A is commutative or $[d(w),w] = 0$ for all $w \in A$. In view of [9, Theorem 2], either A is commutative or $d = 0$. Since A is noncommutative, so we have $d = 0$, i.e., $F(x) = ax$ for all $x \in A$.

Now we consider the case

$$F([x,y]) \pm n[x,y] \in Z(A) \text{ for all } x, y \in A.$$

In view of Fact 2, it follows that $G([x,y]) \in Z(A)$ for all $x, y \in A$. Proceeding as above we get the required result. This completes the proof. □

Proof of Theorem 1.3. Proceeding same as above, we arrive at

$$F(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) + nxy^n \in Z(A) \tag{1.12}$$

or

$$F(xy^n + \sum_{k=1}^{n-1} y^k xy^{n-k}) - ny^n x \in Z(A) \tag{1.13}$$

for all $x, y \in A$. Now, taking $F[(y(e+tx))^n]$ in instead of $F[((e+tx)y)^n]$, we see that either

$$F(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) + nxy^n \in Z(A) \tag{1.14}$$

or

$$F(y^n x + \sum_{k=1}^{n-1} y^k xy^{n-k}) - ny^n x \in Z(A) \tag{1.15}$$

for all $x, y \in A$. Then at least one of pair of equations {(1.12), (1.14)}, {(1.12), (1.15)}, {(1.13), (1.14)} and {(1.13), (1.6)} must hold. On combining the equations in these pairs, we get either

$$F([x,y^n]) \in Z(A) \text{ for all } x, y \in A, \text{ or}$$

$$F([x,y^n]) \pm n(x \circ y^n) \in Z(A) \text{ for all } x, y \in A.$$

Replacing y by $e + ty$ in above equation and using same arguments as we have used above, we obtain either

$$F([x,y]) \in Z(A) \text{ for all } x, y \in A, \text{ or} \tag{1.16}$$

$$F([x,y]) \pm n(x \circ y) \in Z(A) \text{ for all } x, y \in A. \tag{1.17}$$

Equation (1.16) is the same as (1.7). So we have the required conclusion from above. We deal only with (1.17). Taking $x = y$ in (1.17), we obtain $2nx \in Z(A)$ for all $x \in R$. This implies A is commutative. Hence the result. $\square$

Acknowledgment. The author would like to express his sincere thanks to the learned referee for his/her valuable comments to improve the paper.

References

[1] S. Ali and A. N. Khan, *On commutativity of Banach algebras with derivations*, Bull. Aust. Math. Soc. **91** (2015), no. 3, 419–425. https://doi.org/10.1017/S0004972715000118

[2] M. Ashraf and B. A. Wani, *On commutativity of rings and Banach algebras with generalized derivations*, Adv. Pure Appl. Math. **10** (2019), no. 2, 155–163. https://doi.org/10.1515/apam-2017-0024

[3] H. E. Bell, *On a commutativity theorem of Herstein*, Arch. Math. (Basel) **21** (1970), 265–267. https://doi.org/10.1007/BF01220913

[4] H. E. Bell, *On the commutativity of prime rings with derivation*, Quaest. Math. **22** (1999), no. 3, 329–335. https://doi.org/10.1080/16073606.1999.9632085

[5] I. N. Herstein, *Power maps in rings*, Michigan Math. J. **8** (1961), 29–32. http://projecteuclid.org/euclid.mmj/1028998511

[6] I. N. Herstein, *A remark on rings and algebras*, Michigan Math. J. **10** (1963), 269–272. http://projecteuclid.org/euclid.mmj/1028998910

[7] A. N. Khan, S. Ali, H. Alhazmi, and V. De Filippis, *On skew derivations and generalized skew derivations in Banach algebras*, Quaest. Math. **43** (2020), no. 9, 1259–1272. https://doi.org/10.2989/16073606.2019.1607604

[8] M. T. Koşan and T.-K. Lee, *b-generalized derivations of semiprime rings having nilpotent values*, J. Aust. Math. Soc. **96** (2014), no. 3, 326–337. https://doi.org/10.1017/S1446788713000670

[9] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100. https://doi.org/10.2307/2032686

[10] B. Prajapati and S. K. Tiwari, *Some commutativity theorems on Banach algebras*, Rend. del Circ. Mate. di Palermo Series 2 (2020) (preprint). https://doi.org/10.1007/s12215-020-00543-0

[11] N. Rehman, *On continuous linear generalized derivation in rings and Banach algebras*, Asian-Eur. J. Math. **14** (2021), no. 2, 2150020, 8 pp. https://doi.org/10.1142/S1793557121500200

[12] G. S. Sandhu, D. Kumar, D. K. Camci, and N. Aydin, *On derivations satisfying certain identities on rings and algebras*, Facta Univ. Ser. Math. Inform. **34** (2019), no. 1, 85–99.

[13] R. K. Sharma and B. Prajapati, *Generalized derivations and commutativity of prime Banach algebras*, Beitr. Algebra Geom. **58** (2017), no. 1, 179–187. https://doi.org/10.1007/s13366-016-0306-6

[14] J. Vukman, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** (1990), no. 1, 47–52. https://doi.org/10.2307/2048360

[15] J. Vukman, *A result concerning derivations in noncommutative Banach algebras*, Glas. Mat. Ser. III **26(46)** (1991), no. 1-2, 83–88.

[16] J. Vukman, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 877–884. https://doi.org/10.2307/2159463

[17] B. Yood, *On commutativity of unital Banach algebras*, Bull. London Math. Soc. **23** (1991), no. 3, 278–280. https://doi.org/10.1112/blms/23.3.278

Husain Alhazmi
Faculty of Science
Department of Mathematics
King Abdulaziz University
Jeddah, Saudi Arabia
Email address: hsalhazmi@kau.edu.sa