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SOME FIXED POINT THEOREMS FOR WEAKLY PICARD OPERATORS IN COMPLETE METRIC SPACES AND APPLICATIONS

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Abstract. In this paper, we prove new fixed point theorems for single-valued and multi-valued weakly Picard operators in complete metric spaces and give several examples. As applications, we give several results to Fredholm integral equation.

1. Introduction

It is well known that the classical Banach fixed point principle plays an important role in applied mathematics. There are many generalizations of classical Banach fixed point principle, see for instance ([1–13]) and others. Recently, different authors proposed different types of formulations, all expressing different contractive type conditions and most of these contractions are Picard operator and therefore lead to the uniqueness of the fixed point. In this paper we prove some fixed point theorems for single-valued and multi-valued weakly Picard operators in complete metric spaces which that the uniqueness of the fixed point is not guaranteed, and give several examples. Finally, we give several results to Fredholm integral equation.

2. Fixed point theorems for single-valued weakly Picard operator

In this section, we present two fixed point theorems for single-valued weakly Picard operators.

Definition 2.1. Let (X,d) be a metric space and $T : X \to X$ be a single-valued operator from X to itself. We say that T is a single-valued operator weakly Picard operator if for all $x \in X$, there exists a sequence $\{x_n\}$ such that:
(i) $x_1 = x, x_{n+1} = Tx_n$ for all $n = 1, 2, \ldots$;
(ii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

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Theorem 2.1. *Let (X, d) be a complete metric space and T be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le M(x,y,\alpha)d(x,y),$$

for all $x, y \in X$, where

$$M(x,y,\alpha) = \frac{d(x,Ty) + d(y,Tx) + d(x,y)}{2d(x,Tx) + d(y,Ty) + \alpha}.$$

Then

(1) *T has at least one fixed point $\bar{x} \in X$;*
(2) *for any $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point;*
(3) *if $\bar{x}, \bar{y} \in X$ are two distinct fixed points, then*

$$d(\bar{x}, \bar{y}) \ge \frac{\alpha}{3}.$$

Proof. Let $x_0 \in X$ be a fixed. Consider sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. Set $d_n = d(x_n, x_{n+1})$ for all $n \ge 0$. Since

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \le d(x_n, x_{n+1}),$$

and by hypothesis, we have

$$\begin{aligned}
d_{n+1} &= d(Tx_n, Tx_{n+1}) \\
&\le M(x_n, x_{n+1}, \alpha)d(x_n, x_{n+1}) \\
&= \Big[\frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) + d(x_n, x_{n+1})}{2d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + \alpha}\Big] \cdot d(x_n, x_{n+1}) \\
&= \Big[\frac{d(x_n, x_{n+2}) + d(x_n, x_{n+1})}{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \alpha}\Big] \cdot d(x_n, x_{n+1}) \\
&\le \Big[\frac{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \alpha}\Big] \cdot d(x_n, x_{n+1}) \\
&= \Big[\frac{2d_n + d_{n+1}}{2d_n + d_{n+1} + \alpha}\Big] \cdot d_n \text{ for all } n \ge 0.
\end{aligned}$$

Set

$$c_n = \frac{2d_n + d_{n+1}}{2d_n + d_{n+1} + \alpha} \text{ for all } n \ge 0.$$

Then $0 \le c_n < 1$ and $d_{n+1} \le c_n d_n$ for all $n \ge 0$. It follows that

$$d_n \le d_{n-1} \text{ and } d_n \le c_n c_{n-1} \cdots c_1 d_0 \text{ for all } n \ge 1.$$

By the function $f(t) = \frac{t}{t+\alpha}$ is increasing on $[0, +\infty)$, $c_n \le c_{n-1}$ for all $n \ge 2$. Therefore

$$c_n c_{n-1} \cdots c_1 \le c_1^n \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} c_n c_{n-1} \cdots c_1 = \lim_{n\to\infty} d_n = 0.$$

On the other hand, for all $m > n$, we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&= d_n + d_{n+1} + \cdots + d_{m-1} \\
&\leq c_n c_{n-1} \cdots c_1 d_0 + c_{n+1} c_n \cdots c_1 d_0 + \cdots + c_{m-1} c_{m-2} \cdots c_1 d_0 \\
&= (1 + c_{n+1} + \cdots + c_{m-1} c_{m-2} \cdots c_{n+1}) c_n c_{n-1} \cdots c_1 d_0 \\
&\leq (1 + c_1 + \cdots + c_1^{m-n-1}) c_n c_{n-1} \cdots c_1 d_0 \\
&\leq (1 + c_1 + c_1^2 + \cdots) c_n c_{n-1} \cdots c_1 d_0 \\
&= \frac{1}{1 - c_1} c_n c_{n-1} \cdots c_1 d_0 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus

$$\lim_{n,m \to \infty} d(x_m, x_n) = 0.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some point $\bar{x} \in X$. Now, we show that for any $n \geq 0$, either

$$\frac{1}{2} d(x_n, Tx_n) \leq d(x_n, \bar{x}) \text{ or } \frac{1}{2} d(Tx_n, Tx_{n+1}) \leq d(Tx_n, \bar{x}). \tag{1}$$

Arguing by contradiction, we suppose that for some $n \geq 0$ such that

$$d(x_n, \bar{x}) < \frac{1}{2} d(x_n, Tx_n) \text{ and } d(Tx_n, \bar{x}) < \frac{1}{2} d(Tx_n, Tx_{n+1}).$$

Then, by the triangle inequality, we have

$$\begin{aligned}
d_n = d(x_n, Tx_n) &\leq d(x_n, \bar{x}) + d(Tx_n, \bar{x}) \\
&< \frac{1}{2} d(x_n, Tx_n) + \frac{1}{2} d(Tx_n, Tx_{n+1}) \\
&= \frac{1}{2} d_n + \frac{1}{2} d_{n+1} \\
&\leq d_n.
\end{aligned}$$

This is a contradiction. Hence, from (1) for every $n \geq 0$ we have, either

$$d(x_{n+1}, T\bar{x}) \leq M(x_n, \bar{x}, \alpha) d(x_n, \bar{x}),$$

or

$$d(x_{n+2}, T\bar{x}) \leq M(x_{n+1}, \bar{x}, \alpha) d(x_{n+1}, \bar{x}).$$

This is equivalent with either

$$d(x_{n+1}, T\bar{x}) \leq \Big[\frac{d(x_n, T\bar{x}) + d(x_{n+1}, \bar{x}) + d(x_n, \bar{x})}{2d(x_n, x_{n+1}) + d(\bar{x}, T\bar{x}) + \alpha}\Big] \cdot d(x_n, \bar{x}), \tag{2}$$

or

$$d(x_{n+2}, T\bar{x}) \leq \Big[\frac{d(x_{n+1}, T\bar{x}) + d(x_{n+2}, \bar{x}) + d(x_{n+1}, \bar{x})}{2d(x_{n+1}, x_{n+2}) + d(\bar{x}, T\bar{x}) + \alpha}\Big] \cdot d(x_{n+1}, \bar{x}) \tag{3}$$

holds for every $n \geq 0$. Then, either (2) holds for infinity natural numbers n or (3) holds for infinity natural numbers n. Suppose (2) holds for infinity natural numbers n. We can choose in that infinity set the sequence $\{n_k\}$ is a monotone strictly increasing sequence of natural numbers. Therefore, the sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and

$$d(x_{n_k+1}, T\bar{x}) \leq \Big[\frac{d(x_{n_k}, T\bar{x}) + d(x_{n_k+1}, \bar{x}) + d(x_{n_k}, \bar{x})}{2d(x_{n_k}, x_{n_k+1}) + d(\bar{x}, T\bar{x}) + \alpha}\Big] \cdot d(x_{n_k}, \bar{x}).$$

Letting $k \to \infty$ and because $\{x_{n_k+1}\}$ converges to $\bar{x}$ we obtain $\lim\limits_{k\to\infty} x_{n_k+1} = T\bar{x}$ thus $T\bar{x} = \bar{x}$. So $\bar{x}$ is a fixed point of T. If (3) holds for infinity natural numbers n, by using an argument similar to that of above we have $\bar{x}$ is a fixed point of T. Suppose $\bar{y}$ is a fixed point of T with $\bar{x} \neq \bar{y}$, then

$$0 = \frac{1}{2} d(\bar{x}, T\bar{x}) \leq d(\bar{x}, \bar{y}).$$

By hypothesis, we have

$$d(\bar{x}, \bar{y}) = d(T\bar{x}, T\bar{y}) \leq M(\bar{x}, \bar{y}, \alpha) d(\bar{x}, \bar{y}).$$

This implies

$$\begin{aligned} d(\bar{x}, \bar{y}) &\leq \Big[\frac{d(\bar{x}, T\bar{y}) + d(\bar{y}, T\bar{x}) + d(\bar{x}, \bar{y})}{2d(\bar{x}, T\bar{x}) + d(\bar{y}, T\bar{y}) + \alpha}\Big] \cdot d(\bar{x}, \bar{y}) \\ &= \frac{3d^2(\bar{x}, \bar{y})}{\alpha}. \end{aligned}$$

Hence

$$d(\bar{x}, \bar{y}) \geq \frac{\alpha}{3}. \qquad \square$$

Remark 2.2. Note that in Theorem 2.1, the ration $M(x, y, \alpha)$ might be greater than 1 and the uniqueness of the fixed point is not guaranteed. The following example shows this note precisely.

Example 2.3. Let $X = \{0, 1, 2\}$ and let $d : X \times X \to [0, +\infty)$ by

$$\begin{aligned} &d(0,0) = d(1,1) = d(2,2) = 0, \\ &d(0,1) = d(1,0) = \frac{1}{2}, \\ &d(0,2) = d(2,0) = \frac{3}{2}, \\ &d(1,2) = d(2,1) = 2. \end{aligned}$$

Then (X, d) is a complete metric space.

Let $T : X \to X$ by $T0 = 0, T1 = 1$ and $T2 = 1$. For $\alpha = 1$, we have

$$\begin{aligned} M(0,1,1) &= \frac{d(0,T1) + d(T0,1) + d(0,1)}{2d(0,T0) + d(1,T1) + 1} = \frac{3}{2}, \\ M(1,2,1) &= \frac{d(1,T2) + d(T1,2) + d(1,2)}{2d(1,T1) + d(2,T2) + 1} = \frac{4}{3}, \\ M(0,2,1) &= \frac{d(0,T2) + d(T0,2) + d(0,2)}{2d(0,T0) + d(2,T2) + 1} = \frac{5}{3}. \end{aligned}$$

Since $0 = \frac{1}{2}d(0, T0) \leq d(0, y)$ holds for any $y \in X$ and

$$0 = d(T0, T0) \leq M(0, 0, 1)d(0, 0) = 0,$$
$$\frac{1}{2} = d(T0, T1) \leq M(0, 1, 1)d(0, 1) = \frac{3}{4},$$
$$\frac{1}{2} = d(T0, T2) \leq M(0, 2, 1)d(0, 2) = \frac{15}{4},$$

then

$$\frac{1}{2}d(0, T0) \leq d(0, y) \text{ implies } d(T0, Ty) \leq M(0, y, 1)d(0, y) \text{ for all } y \in X.$$

Again, since $0 = \frac{1}{2}d(1, T1) \leq d(1, y)$ holds for any $y \in X$ and

$$\frac{1}{2} = d(T1, T0) \leq M(1, 0, 1)d(1, 0) = \frac{3}{4},$$
$$0 = d(T1, T1) \leq M(1, 1, 1)d(1, 1) = 0,$$
$$0 = d(T1, T2) \leq M(0, 2, 1)d(0, 2) = \frac{15}{6},$$

then

$$\frac{1}{2}d(1, T1) \leq d(1, y) \text{ implies } d(T1, Ty) \leq M(1, y, 1)d(1, y) \text{ for all } y \in X.$$

Finally, by $1 = \frac{1}{2}d(2, T2) \leq d(2, y)$ if and only if $y \in X \backslash \{2\}$ and

$$\frac{1}{2} = d(T2, T0) \leq M(2, 0, 1)d(2, 0) = \frac{5}{2},$$
$$0 = d(T2, T1) \leq M(2, 1, 1)d(2, 1) = \frac{8}{3},$$

then

$$\frac{1}{2}d(2, T2) \leq d(2, y) \text{ implies } d(T2, Ty) \leq M(2, y, 1)d(2, y) \text{ for all } y \in X.$$

Therefore T satisfies all the conditions of Theorem 2.1 for $\alpha = 1$. Also, T has two distinct fixed points $\{0, 1\}$ and

$$\frac{1}{2} = d(0, 1) \geq \frac{\alpha}{3} = \frac{1}{3}.$$

Corollary 2.4. *Let (X, d) be a complete metric space and T be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq M(x, y, \alpha)d(x, y),$$

for all $x, y \in X$, where

$$M(x, y, \alpha) = \frac{d(x, Ty) + d(y, Tx) + d(x, y)}{2d(x, Tx) + d(y, Ty) + \alpha}.$$

Then T has a unique fixed point if $M(x, y, \alpha) < 1$ for all $x, y \in X$.

Proof. From Theorem 2.1, T has a fixed point $\bar{x}$. If $\bar{y}$ is a fixed point of T, then

$$0 = \frac{1}{2}d(\bar{x}, T\bar{x}) \leq d(\bar{x}, \bar{y}).$$

By hypothesis, we have

$$d(\bar{x}, \bar{y}) = d(T\bar{x}, T\bar{y}) \leq M(\bar{x}, \bar{y}, \alpha)d(\bar{x}, \bar{y}).$$

This implies

$$[1 - M(\bar{x}, \bar{y}, \alpha)]d(\bar{x}, \bar{y}) \leq 0.$$

Since $0 \leq M(\bar{x}, \bar{y}, \alpha) < 1$, $d(\bar{x}, \bar{y}) = 0$. Hence $\bar{x} = \bar{y}$. □

Example 2.5. Let $X = \{0, 1, 2\}$ and let $d : X \times X \to \mathbb{R}$ by

$$\begin{aligned}
&d(0,0) = d(1,1) = d(2,2) = 0,\\
&d(0,1) = d(1,0) = \frac{1}{2},\\
&d(0,2) = d(2,0) = 1,\\
&d(1,2) = d(2,1) = \frac{1}{2}.
\end{aligned}$$

Then (X, d) is a complete metric space. Let $T : X \to X$ by $T0 = 0, T1 = 0$ and $T2 = 0$. For $\alpha = 2$, we have

$$\begin{aligned}
M(0,0,2) &= \frac{d(0,T0) + d(0,T0) + d(0,0)}{2d(0,T0) + d(0,T0) + 2} = 0,\\
M(1,1,2) &= \frac{d(1,T1) + d(1,T1) + d(1,1)}{2d(1,T1) + d(1,T1) + 2} = \frac{2}{7},\\
M(2,2,2) &= \frac{d(2,T2) + d(2,T2) + d(2,2)}{2d(2,T2) + d(2,T2) + 2} = \frac{2}{5},\\
M(0,1,2) &= \frac{d(0,T1) + d(1,T0) + d(0,1)}{2d(0,T0) + d(1,T1) + 2} = \frac{2}{5},\\
M(1,0,2) &= \frac{d(1,T0) + d(0,T1) + d(1,0)}{2d(1,T1) + d(0,T0) + 2} = \frac{1}{3},\\
M(1,2,2) &= \frac{d(1,T2) + d(2,T1) + d(1,2)}{2d(1,T1) + d(2,T2) + 2} = \frac{2}{5},\\
M(2,1,2) &= \frac{d(2,T1) + d(1,T2) + d(2,1)}{2d(2,T2) + d(1,T1) + 2} = \frac{4}{9},\\
M(0,2,2) &= \frac{d(0,T2) + d(2,T0) + d(0,2)}{2d(0,T0) + d(2,T2) + 2} = \frac{2}{3},\\
M(2,0,2) &= \frac{d(2,T0) + d(0,T2) + d(2,0)}{2d(2,T2) + d(0,T0) + 2} = \frac{1}{2}.
\end{aligned}$$

Then $M(x, y, 2) < 1$ for all $x, y \in X$. Moreover, since $d(Tx, Ty) = 0$ for all $x, y \in X$, then

$$d(Tx, Ty) \leq M(x, y, 2)d(x, y) \quad \text{for all } x, y \in X.$$

Therefore T satisfies all the conditions of Corollary 2.4 for $\alpha = 2$. Also, T has a unique fixed points $\bar{x} = 0$.

Theorem 2.6. *Let (X, d) be a complete metric space and T be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(Tx, x) \le d(x, y) \text{ implies } d(Tx, Ty) \le N(x, y, \alpha)d(x, y),$$

for all $x, y \in X$, where

$$N(x, y, \alpha) = \frac{d(x, Ty) + d(y, Tx) + d(x, Tx) + d(y, Ty) + d(x, y)}{3d(x, Tx) + 2d(y, Ty) + \alpha}.$$

Then

(1) *T has at least one fixed point $\bar{x} \in X$;*
(2) *for any $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point;*
(3) *if $\bar{x}, \bar{y} \in X$ are two distinct fixed points, then*

$$d(\bar{x}, \bar{y}) \ge \frac{\alpha}{3}.$$

Proof. Let $x_0 \in X$ be a fixed. Consider the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. Set $d_n = d(x_n, x_{n+1})$ for all $n \ge 0$. Then we have

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \le d(x_n, x_{n+1}).$$

By hypothesis, we have

$$\begin{aligned} d_{n+1} &= d(Tx_n, Tx_{n+1}) \\ &\le N(x_n, x_{n+1}, \alpha)d(x_n, x_{n+1}) \\ &= \Big[\frac{d(x_n, x_{n+2}) + 2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{3d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2}) + \alpha}\Big] \cdot d(x_n, x_{n+1}) \\ &\le \Big[\frac{3d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2})}{3d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2}) + \alpha}\Big] \cdot d(x_n, x_{n+1}) \\ &= \Big[\frac{3d_n + 2d_{n+1}}{3d_n + 2d_{n+1} + \alpha}\Big] \cdot d_n \text{ for all } n \ge 1. \end{aligned}$$

Set

$$c_n = \frac{3d_n + 2d_{n+1}}{3d_n + 2d_{n+1} + \alpha} \text{ for all } n \ge 0.$$

Then $0 \le c_n < 1$ and $d_{n+1} \le c_n d_n$ for all $n \ge 0$. It follows that

$$d_n \le d_{n-1} \text{ and } d_n \le c_n c_{n-1} \cdots c_1 d_0 \text{ for all } n \ge 1.$$

By using an argument similar to that of the proof of Theorem 2.1, we have completes the proof. □

Remark 2.7. Since

$$M(x, y, \alpha) \le N(x, y, \alpha),$$

for all $x, y \in X$ and $\alpha > 0$, then Theorem 2.6 implies Theorem 2.1.

3. Fixed point theorems for multi-valued weakly Picard operator

Let (X, d) be a metric space. Let $CB(X)$ be the collection of all nonempty bounded closed subsets of X. Let $T : X \to CB(X)$ be a multivalued mapping on X. Let H be the Hausdorff metric on $CB(X)$ induced by d, that is,

$$H(A, B) := \max\{\sup_{x\in B} \rho(x, A); \sup_{x\in A} \rho(x, B)\},$$

where $A, B \in CB(X)$ and $\rho(x, A) := \inf_{y\in A} d(x, y)$. Denote

$$\delta(x, A) := \sup_{y\in A} d(x, y).$$

Definition 3.1. Let (X, d) be a metric space and $T : X \to CB(X)$ be a multivalued operator. We say that T is a multivalued operator weakly Picard operator if for all $x \in X$ and $y \in Tx$, there exists a sequence $\{x_n\}$ such that:
(i) $x_1 = x, x_2 = y$;
(ii) $x_{n+1} \in Tx_n$ for all $n = 1, 2, \ldots$;
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

Theorem 3.1. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be an multivalued mapping. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}\rho(x, Tx) \le d(x, y) \text{ implies } H(Tx, Ty) \le P(x, y, \alpha)d(x, y),$$

for all $x, y \in X$, where

$$P(x, y, \alpha) = \frac{\rho(x, Ty) + \rho(y, Tx) + d(x, y)}{2\delta(x, Tx) + \delta(y, Ty) + \alpha}.$$

Then
(1) *T has at least one fixed point $\bar{x} \in X$;*
(2) *if $\bar{x}, \bar{y} \in X$ are two fixed points, then*

$$d^2(\bar{x}, \bar{y}) \ge \frac{\alpha}{3} H(T\bar{x}, T\bar{y}).$$

Proof. Let $x_0 \in X$ and choose $x_1 \in Tx_0$.
Step 1. If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$. Thus, x_1 is a fixed point of T. If $H(Tx_0, Tx_1) > 0$, then for each $h_1 > 1$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < h_1 H(Tx_0, Tx_1).$$

Step 2. Similarly, if $H(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$. Thus, x_2 is a fixed point of T. If $H(Tx_1, Tx_2) > 0$, then for each $h_2 > 1$, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < h_2 H(Tx_1, Tx_2).$$

$$\vdots$$

Step n. Continuing in this manner, if $H(Tx_{n-1}, Tx_n) = 0$, then $Tx_{n-1} = Tx_n$. Thus, x_n is a fixed point of T. If $H(Tx_{n-1}, Tx_n) > 0$, then for each $h_n > 1$, there exists $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) < h_n H(Tx_{n-1}, Tx_n).$$

The above process continues, if at step k satisfy $H(Tx_{k-1}, Tx_k) = 0$, then x_k is a fixed point of T. If not, we get obtain two sequences $\{x_n\}$ and $\{h_n\}_{n\geq 1}$ such that $x_n \in Tx_{n-1}, h_n > 1$ and

$$d(x_n, x_{n+1}) < h_n H(Tx_{n-1}, Tx_n) \text{ for all } n \geq 1.$$

Since $\frac{1}{2}\rho(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n)$ and by hypothesis, we have

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leq P(x_{n-1}, x_n, \alpha) d(x_{n-1}, x_n) \\ &= \Big[\frac{\rho(x_{n-1}, Tx_n) + \rho(x_n, Tx_{n-1}) + d(x_{n-1}, x_n)}{2\delta(x_{n-1}, Tx_{n-1}) + \delta(x_n, Tx_n) + \alpha}\Big] \cdot d(x_{n-1}, x_n) \\ &= \Big[\frac{\rho(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)}{2\delta(x_{n-1}, Tx_{n-1}) + \delta(x_n, Tx_n) + \alpha}\Big] \cdot d(x_{n-1}, x_n). \end{aligned} \tag{4}$$

On the other hand, for some $y_n \in T(x_n)$, we have

$$\begin{aligned} \frac{\rho(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)}{2\delta(x_{n-1}, Tx_{n-1}) + \delta(x_n, Tx_n) + \alpha} &\leq \frac{d(x_{n-1}, y_n) + d(x_{n-1}, x_n)}{2d(x_{n-1}, x_n) + d(x_n, y_n) + \alpha} \\ &\leq \frac{2d(x_{n-1}, x_n) + d(x_n, y_n)}{2d(x_{n-1}, x_n) + d(x_n, y_n) + \alpha}. \end{aligned} \tag{5}$$

From (4) and (5), we have

$$H(Tx_{n-1}, Tx_n) \leq \Big[\frac{2d(x_{n-1}, x_n) + d(x_n, y_n)}{2d(x_{n-1}, x_n) + d(x_n, y_n) + \alpha}\Big] \cdot d(x_{n-1}, x_n).$$

Set

$$c_n = \frac{2d(x_{n-1}, x_n) + d(x_n, y_n)}{2d(x_{n-1}, x_n) + d(x_n, y_n) + \alpha}.$$

Then $0 < c_n < 1$ and

$$d_n < h_n c_n d_{n-1}, \text{ where } d_n = d(x_n, x_{n+1}), d_{n-1} = d(x_{n-1}, x_n).$$

We choose $h_n = \frac{1}{\sqrt{c_n}}$. Then we have

$$d_n < \sqrt{c_n} d_{n-1}.$$

This implies

$$d_n < \sqrt{c_n c_{n-1} \cdots c_1} d_0.$$

By using an argument similar to that of the proof of Theorem 2.1, there exists $\bar{x} \in X$ such that $\lim\limits_{n\to\infty} x_n = \bar{x}$. We show that for any $n \geq 0$, either

$$\frac{1}{2}\rho(x_n, Tx_n) \leq d(x_n, \bar{x}) \text{ or } \frac{1}{2}\rho(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, \bar{x}). \tag{6}$$

Arguing by contradiction, we suppose that for some $n \geq 0$ such that

$$d(x_n, \bar{x}) < \frac{1}{2}\rho(x_n, Tx_n) \text{ and } d(x_{n+1}, \bar{x}) < \frac{1}{2}\rho(x_{n+1}, Tx_{n+1}).$$

Then, by the triangle inequality, we have

$$\begin{aligned} d_n = d(x_n, x_{n+1}) &\leq d(x_n, \bar{x}) + d(x_{n+1}, \bar{x}) \\ &< \frac{1}{2}\rho(x_n, Tx_n) + \frac{1}{2}\rho(x_{n+1}, Tx_{n+1}) \\ &\leq \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}) \\ &\leq d_n. \end{aligned}$$

This is a contradiction. Hence, from (6) and by hypotheses for each $n \geq 0$, either

$$H(Tx_n, T\bar{x}) \leq P(x_n, \bar{x}, \alpha)d(x_n, \bar{x}), \tag{7}$$

or

$$H(Tx_{n+1}, T\bar{x}) \leq P(x_{n+1}, \bar{x}, \alpha)d(x_{n+1}, \bar{x}). \tag{8}$$

Then, either (7) holds for infinity natural numbers n or (8) holds for infinity natural numbers n. Suppose (7) holds for infinity natural numbers n. We can choose in that infinity set the sequence $\{n_k\}$ is a monotone strictly increasing sequence of natural numbers. Therefore, sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and

$$\begin{aligned} \rho(\bar{x}, T\bar{x}) &\leq d(x_{n_k+1}, \bar{x}) + H(Tx_{n_k}, T\bar{x}) \\ &\leq d(x_{n_k+1}, \bar{x}) + \Big[\frac{\rho(x_{n_k}, T\bar{x}) + \rho(\bar{x}, Tx_{n_k}) + d(x_{n_k}, \bar{x})}{2\delta(x_{n_k}, Tx_{n_k}) + \delta(\bar{x}, T\bar{x}) + \alpha}\Big] \cdot d(x_{n_k}, \bar{x}) \\ &\leq d(x_{n_k+1}, \bar{x}) + \Big[\frac{2d(x_{n_k}, \bar{x}) + \rho(\bar{x}, T\bar{x}) + d(x_{n_k+1}, \bar{x})}{2d(x_{n_k}, x_{n_k+1}) + \delta(\bar{x}, T\bar{x}) + \alpha}\Big] \cdot d(x_{n_k}, \bar{x}). \end{aligned}$$

On taking limit on both sides of above inequality, we have $\rho(\bar{x}, T\bar{x}) = 0$. It means that $\bar{x} \in T\bar{x}$. If (8) holds for infinity natural numbers n, by using an argument similar to that of above we have $\bar{x}$ is a fixed point of T. Now, let $\bar{y}$ is a fixed point of T. Since $0 = \frac{1}{2}\rho(\bar{x}, T\bar{x}) \leq d(\bar{x}, \bar{y})$ and by hypothesis, we have

$$\begin{aligned} H(T\bar{x}, T\bar{y}) &\leq P(\bar{x}, \bar{y}, \alpha)d(\bar{x}, \bar{y}) \\ &= \Big[\frac{\rho(\bar{x}, T\bar{y}) + \rho(\bar{y}, T\bar{x}) + d(\bar{x}, \bar{y})}{2\delta(\bar{x}, T\bar{x}) + \delta(y, T\bar{y}) + \alpha}\Big] \cdot d(\bar{x}, \bar{y}) \\ &\leq \Big[\frac{d(\bar{x}, \bar{y}) + d(\bar{x}, \bar{y}) + d(\bar{x}, \bar{y})}{\alpha}\Big] \cdot d(\bar{x}, \bar{y}). \end{aligned}$$

This implies

$$d^2(\bar{x}, \bar{y}) \geq \frac{\alpha}{3}H(T\bar{x}, T\bar{y}).$$

□

Example 3.2. Let $X = \{0, 1, 2\}$ and let $d : X \times X \to [0, +\infty)$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \in X, \\ 2, & \text{if } x \neq y \in X. \end{cases}$$

Then (X, d) is a complete metric space.

Let $T : X \to CB(X)$ by $T0 = \{0\}, T1 = \{1\}$ and $T2 = \{1, 2\}$. For $\alpha = 2$, we have

$$H(T0, T1) = H(T0, T2) = H(T1, T2) = 2,$$

and

$$P(0, 1, 2) = \frac{\rho(0, T1) + \rho(1, T0) + d(0, 1)}{2\delta(0, T0) + \delta(1, T1) + 2} = 3,$$

$$P(0, 2, 2) = \frac{\rho(0, T2) + \rho(2, T0) + d(0, 2)}{2\delta(0, T0) + \delta(2, T2) + 2} = \frac{3}{2},$$

$$P(1, 2, 2) = \frac{\rho(1, T2) + \rho(2, T1) + d(1, 2)}{2\delta(1, T1) + \delta(2, T2) + 2} = 1.$$

We early check that $\frac{1}{2}\rho(x, Tx) \leq d(x, y)$ for all $x, y \in X$. On the other hand

$$2 = H(T0, T1) \leq P(0, 1, 2)d(0, 1) = 6,$$

$$2 = H(T0, T2) \leq P(0, 2, 2)d(0, 2) = 3,$$

$$2 = H(T1, T2) \leq P(1, 2, 2)d(1, 2) = 2.$$

Hence

$$H(Tx, Ty) \leq P(x, y, \alpha)d(x, y) \quad \text{for all } x, y \in X.$$

Therefore T satisfies all the conditions of Theorem 3.1 for $\alpha = 2$. Also, T has three distinct fixed points $\{0, 1, 2\}$. Moreover, we early check that

$$d^2(\bar{x}, \bar{y}) \geq \frac{2}{3}H(T\bar{x}, T\bar{y}) \quad \text{for all } \bar{x}, \bar{y} \in X.$$

Corollary 3.3. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be an multivalued mapping. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}\rho(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq P(x, y, \alpha)d(x, y),$$

for all $x, y \in X$, where

$$P(x, y, \alpha) = \frac{\rho(x, Ty) + \rho(y, Tx) + d(x, y)}{2\delta(x, Tx) + \delta(y, Ty) + \alpha}.$$

Then T has a unique fixed point if $P(x, y, \alpha) < 1$ for all $x, y \in X$.

Proof. From Theorem 3.1, T has a fixed point $\bar{x}$. If $\bar{y}$ is a fixed point of T, then

$$0 = \frac{1}{2}\rho(\bar{x}, T\bar{x}) \leq d(\bar{x}, \bar{y}).$$

By hypothesis, we have

$$H(T\bar{x}, T\bar{y}) \leq P(\bar{x}, \bar{y}, \alpha)d(\bar{x}, \bar{y}).$$

Since $d(\bar{x},\bar{y}) \leq H(T\bar{x},T\bar{y})$,

$$[1-P(\bar{x},\bar{y},\alpha)]d(\bar{x},\bar{y}) \leq 0.$$

Thus, $d(\bar{x},\bar{y})=0$. Hence $\bar{x}=\bar{y}$. □

Theorem 3.4. *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be an multivalued mapping. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}\rho(x,Tx) \leq d(x,y) \text{ implies } H(Tx,Ty) \leq Q(x,y,\alpha)d(x,y),$$

for all $x, y \in X$, where

$$Q(x,y,\alpha) = \frac{\rho(x,Ty)+\rho(y,Tx)+\rho(x,Tx)+\rho(y,Ty)+d(x,y)}{3\delta(x,Tx)+2\delta(y,Ty)+\alpha}.$$

Then

(1) *T has at least one fixed point $\bar{x} \in X$;*
(2) *if $\bar{x},\bar{y} \in X$ are two fixed points, then*

$$d^2(\bar{x},\bar{y}) \geq \frac{\alpha}{3}H(T\bar{x},T\bar{y}).$$

Proof. Let $x_0 \in X$. By using an argument similar to that of the proof of Theorem 3.1, for each $n \geq 1$, there exist $x_1, x_2, x_3, \dots, x_n$ with $x_n \in Tx_{n-1}$ for all $n \geq 1$. If $H(Tx_{n-1},Tx_n)=0$, then $Tx_{n-1}=Tx_n$. Thus, x_n is a fixed point of T. If $H(Tx_{n-1},Tx_n)>0$, then for each $h_n>1$, there exists $x_{n+1} \in Tx_n$ such that

$$d(x_n,x_{n+1}) < h_n H(Tx_{n-1},Tx_n).$$

Since $\frac{1}{2}\rho(x_{n-1},Tx_{n-1}) \leq d(x_{n-1},x_n)$ and by hypothesis, we have

$$\begin{aligned} H(Tx_{n-1},Tx_n) &\leq Q(x_{n-1},x_n,\alpha)d(x_{n-1},x_n) \\ &\leq \Big[\frac{3d(x_{n-1},x_n)+2d(x_n,y_n)}{3d(x_{n-1},x_n)+2d(x_n,y_n)+\alpha}\Big] \cdot d(x_{n-1},x_n) \end{aligned}$$

for some $y_n \in Tx_n$. Set

$$c_n = \frac{3d(x_{n-1},x_n)+2d(x_n,y_n)}{3d(x_{n-1},x_n)+2d(x_n,y_n)+\alpha}.$$

Then $0<c_n<1$ and

$$d_n < h_n c_n d_{n-1}, \text{ where } d_n = d(x_n,x_{n+1}), d_{n-1}=d(x_{n-1},x_n).$$

By using an argument similar to that of the proof of Theorem 3.1, we have completes the proof. □

Remark 3.5. If T is a single map, then Theorem 3.1 reduces to Theorem 2.1 and Theorem 3.4 reduces to Theorem 2.6.

4. Application to Fredholm integral equation

Consider the Fredholm integral equation

$$x(t) = \int_a^b K(t,s)x(s)ds,\ t \in [a,b],$$

where $K(t,s)$ is continuous on the square $[a,b]\times[a,b]$ and $x \in C_{[a,b]}$.

Theorem 4.1. *Let $K(t,s)$ be a continuous function on the square $[a,b]\times[a,b]$. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}\max_{t\in[a,b]}|x(t) - \int_a^b K(t,s)x(s)ds| \le \max_{t\in[a,b]}|x(t)-y(t)|$$

implies

$$\max_{t\in[a,b]}|\int_a^b K(t,s)[x(s)-y(s)]ds| \le \frac{F(x,y)}{G(x,y,\alpha)}\max_{t\in[a,b]}|x(t)-y(t)|$$

for all $x,y\in C_{[a,b]}$, where

$$\begin{aligned} F(x,y) = &\max_{t\in[a,b]}|x(t) - \int_a^b K(t,s)y(s)ds| \\ &+ \max_{t\in[a,b]}|y(t) - \int_a^b K(t,s)x(s)ds| + \max_{t\in[a,b]}|x(t)-y(t)|, \\ G(x,y,\alpha) = &\ 2\max_{t\in[a,b]}|x(t) - \int_a^b K(t,s)x(s)ds| \\ &+ \max_{t\in[a,b]}|y(t) - \int_a^b K(t,s)y(s)ds| + \alpha. \end{aligned}$$

Then

(1) *Fredholm integral equation has at least one solution $\bar{x}\in C_{[a,b]}$;*

(2) *if $\bar{x},\bar{y}\in C_{[a,b]}$ are two distinct solutions of Fredholm integral equation, then*

$$\max_{t\in[a,b]}|\bar{x}(t)-\bar{y}(t)| \ge \frac{\alpha}{3}.$$

Proof. Set $X = C_{[a,b]}$ and $d : X\times X \to [0,+\infty)$ by

$$d(x,y) = \max_{t\in[a,b]}|x(t)-y(t)|.$$

Then (X,d) is a complete metric space. Let $T : X \to X$ by

$$Tx(t) = \int_a^b K(t,s)x(s)ds.$$

Since hypothesis, we have

$$\frac{1}{2}d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le M(x,y,\alpha)d(x,y),$$

for all $x, y \in X$, where

$$M(x, y, \alpha) = \frac{F(x, y)}{G(x, y, \alpha)}.$$

By Theorem 2.1, there exists $\bar{x} \in X$ such that $\bar{x} = T\bar{x}$. This means that $\bar{x}$ is a solution of Fredholm integral equation. Moreover, if $\bar{x}, \bar{y} \in C_{[a,b]}$ are two distinct solutions of Fredholm integral equation, then

$$\max_{t \in [a,b]} |\bar{x}(t) - \bar{y}(t)| \geq \frac{\alpha}{3}.$$

This completes the proof. □

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