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A NOTE ON MODULAR EQUATIONS OF SIGNATURE 2 AND THEIR EVALUATIONS

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Abstract. In his notebooks, Srinivasa Ramanujan recorded several modular equations that are useful in the computation of class invariants, continued fractions and the values of theta functions. In this paper, we prove some new modular equations of signature 2 by well-known and useful theta function identities of composite degrees. Further, as an application of this, we evaluate theta function identities.

1. Introduction

Throughout this paper, we use the standard q-series notation f_k and is defined as

$$f_k := (q^k; q^k)_\infty = \prod_{m=1}^{\infty} (1 - q^{mk}), \ |q| < 1.$$

Ramanujan [7] has defined theta function [2, p. 36] as follows:

$$\varphi(q) := \mathfrak{f}(q, q) = 1 + \sum_{i=1}^{\infty} q^{i^2} = \frac{(-q; -q)_\infty}{(q; -q)_\infty},$$

$$\psi(q) := \mathfrak{f}(q, q^3) = \sum_{i=0}^{\infty} q^{i(i+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty},$$

$$f(-q) := \mathfrak{f}(-q, -q^2) = \sum_{i=0}^{\infty} (-1)^i q^{i(3i-1)/2} + \sum_{i=1}^{\infty} (-1)^i q^{i(3i+1)/2} = (q; q)_\infty.$$

For convenience, we write $f(-q^n)$ by f_n. Ramanujan [7] begins his study on modular equations in Chapter 15 by defining

$$F(x) := (1 - x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} x^n = \ {}_1F_0\left(\frac{1}{2}; x\right), \ |x| < 1.$$

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He then states a trivial identity

$$F\left(\frac{2t}{1+t}\right) = (1+t)F(t^2). \tag{1.1}$$

After setting $\alpha = 2t/(1+t)$ and $\beta = 2t^2$ in (1.1), Ramanujan offers a modular equation of degree two,

$$\beta(2-\alpha)^2 = \alpha^2,$$

and the factor $(1+t)$ in (1.1) is called the multiplier. Further Ramanujan developed theory of elliptic functions in which q is replaced by one or the other functions for $n = 3, 4$ and 6.

$$q_n := q_n(x) := \exp\left(-\pi \csc(\pi/n)\frac{F(1-x)}{F(x)}\right) \; 0 < x < 1,$$

where $F(x) = {}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)$ and ${}_2F_1$ represent the classical hypergeometric function [6] defined as follows:

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!} z^m, \; |z| < 1,$$

where

$$(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1).$$

These theories are now known as the theory of signature n, where $n = 3, 4$ and 6. For $n = 3$ and 4, the theories are known as cubic and quartic theories, respectively. Let us now take up a modular equation as given in the literature. An n^{th} degree modular equation [2] in the theory of signature 2, is an equation that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}$$

relating α and β. Then, always we say that β is of degree n over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

Ramanujan [7, Vol II] in his notebooks documented some cubic modular equations. Further these are proved by B. C. Berndt [3], through parameterization and modular forms. Also N. D. Baruah and N. Saikia [1], M. S. M. Naika and S. Chandankumar [5], K. R. Vasuki and C. Chamaraju [8] also obtained some interesting results on modular equations of various degrees. We classify this paper as follows. In Section 2 we list some P-Q type theta function identities which will be utilized to demonstrate our main results. Further, in Section 3, we prove composite degrees of modular equations of various degrees and in Section 4, we evaluate theta function identities of two parameters.

2. Results required

In this section, we shall mention some known results in the form of lemmas that will be required in our present investigation and are included so that the paper may be self contained.

Lemma 2.1. *Let*

$$A_n := \frac{f_n}{q^{n/24} f_{2n}}. \tag{2.1}$$

Then, the following results hold true.

$$(A_1A_3)^3 + \frac{8}{(A_1A_3)^3} = \left(\frac{A_3}{A_1}\right)^6 - \left(\frac{A_1}{A_3}\right)^6 \tag{2.2}$$

and

$$(A_1A_5)^2 + \frac{4}{(A_1A_5)^2} = \left(\frac{A_5}{A_1}\right)^3 - \left(\frac{A_1}{A_5}\right)^3. \tag{2.3}$$

For the proof of the above results, one can refer [3, 4, 10].

Lemma 2.2. *Let*

$$B_n := \frac{\varphi(-q^n)}{\varphi(-q^{2n})}. \tag{2.4}$$

Then, the following results hold true.

$$\left(\frac{B_1}{B_3}\right)^4 + \left(\frac{B_3}{B_1}\right)^4 + 6 = 4\left[(B_1B_3)^2 + \frac{1}{(B_1B_3)^2}\right] \tag{2.5}$$

and

$$\left(\frac{B_1}{B_5}\right)^3 - \left(\frac{B_5}{B_1}\right)^3 + 5\left(\frac{B_1}{B_5} - \frac{B_5}{B_1}\right) = 4\left[(B_1B_5)^2 - \frac{1}{(B_1B_5)^2}\right]. \tag{2.6}$$

For the proof of the above results, one can refer [1, 9].

Lemma 2.3. *Let*

$$C_n := \frac{\psi(-q^n)}{q^{n/8}\psi(-q^{2n})}. \tag{2.7}$$

Then, the following result holds true.

$$C_1C_3 - \frac{4}{C_1C_3} = \left(\frac{C_3}{C_1}\right)^2 - \left(\frac{C_1}{C_3}\right)^2. \tag{2.8}$$

For the proof of the above result, one can refer [4].

3. Main results

In this section, we shall establish five interesting theorems in the form of P-Q identities in the following theorems.

Theorem 3.1. *Let*

$$P := q^{1/12}\frac{f_1 f_6}{f_2 f_3} \qquad \textit{and} \qquad Q := q^{1/4}\frac{f_3 f_{18}}{f_6 f_9}.$$

Then the following result holds true.

$$(PQ)^6 + \frac{1}{(PQ)^6} - 10\left((PQ)^3 + \frac{1}{(PQ)^3}\right)$$
$$-\left((PQ)^3 + \frac{1}{(PQ)^3} - 1\right)\left(\left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6\right) + 20 = 0.$$

Proof. From (2.1) and together with the definition of P and Q, we have

$$P = \frac{A_1}{A_3} \qquad \text{and} \qquad Q = \frac{A_3}{A_9}. \tag{3.1}$$

Also from (2.2) and (3.1), we find that

$$\frac{P^3 A_3^6}{2\sqrt{2}} + \frac{2\sqrt{2}}{P^3 A_3^6} = \frac{1}{2\sqrt{2}}\left\{\frac{1}{P^6} - P^6\right\}.$$

On solving this for $P^3A_3^6/2\sqrt{2}$, we obtain

$$\frac{P^3 A_3^6}{2\sqrt{2}} = \frac{K_1 \pm \sqrt{K_1^2 - 4}}{2}, \tag{3.2}$$

where

$$K_1 = \frac{1}{2\sqrt{2}}\left\{\frac{1}{P^6} - P^6\right\}.$$

Therefore the identity (3.2) implies that

$$\frac{2\sqrt{2}}{P^3 A_3^6} = \frac{K_1 \mp \sqrt{K_1^2 - 4}}{2}. \tag{3.3}$$

In the similar manner, we deduce that

$$\frac{Q^3 A_9}{2\sqrt{2}} = \frac{K_2 \pm \sqrt{K_2^2 - 4}}{2} \tag{3.4}$$

and

$$\frac{2\sqrt{2}}{Q^3 A_9} = \frac{K_2 \mp \sqrt{K_2^2 - 4}}{2}, \tag{3.5}$$

where

$$K_2 = \frac{1}{2\sqrt{2}}\left\{\frac{1}{Q^6} - Q^6\right\}.$$

Multiplying (3.2) and (3.5) and employing (3.1) and after a little simplification, we obtain

$$(3.6)\quad (PQ)^3=\frac{1}{4}\left\{K_1K_2\pm K_2\sqrt{K_1^2-4}\mp K_1\sqrt{K_2^2-4}-\sqrt{K_1^2-4}\sqrt{K_2^2-4}\right\}.$$

Similarly, multiplying (3.3) and (3.4) and employing (3.1), we obtain

$$(3.7)\quad \frac{1}{(PQ)^3}=\frac{1}{4}\left\{K_1K_2\mp K_2\sqrt{K_1^2-4}\pm K_1\sqrt{K_2^2-4}-\sqrt{K_1^2-4}\sqrt{K_2^2-4}\right\}.$$

Adding (3.6) and (3.7) and then simplifying, we get

$$2\left((PQ)^3+\frac{1}{(PQ)^3}\right)-K_1K_2=-\sqrt{K_1^2-4}\sqrt{K_2^2-4}.$$

Squaring on both sides and then simplifying, we obtain

$$(PQ)^6+\frac{1}{(PQ)^6}-K_1K_2\left((PQ)^3+\frac{1}{(PQ)^3}\right)+K_1^2+K_2^2-2=0.$$

Finally, substituting the values of K_1 and K_2 in the above identity and then simplifying, we obtain

$$\begin{aligned}&\big(P^{18}Q^6-P^{15}Q^{15}-P^{15}Q^3+10P^{12}Q^{12}+P^{12}-20P^9Q^9+P^6Q^{18}\\&+10P^6Q^6-P^3Q^{15}-P^3Q^3+Q^{12}\big)(1+PQ)^2(1-PQ+P^2Q^2)^2=0.\end{aligned}$$

Since $PQ\neq -1$ and can not be imaginary, we have

$$\begin{aligned}&P^{18}Q^6-P^{15}Q^{15}-P^{15}Q^3+10P^{12}Q^{12}+P^{12}-20P^9Q^9\\&+P^6Q^{18}+10P^6Q^6-P^3Q^{15}-P^3Q^3+Q^{12}=0.\end{aligned}$$

Now on dividing throughout by $(PQ)^9$, we obtain the required result. This theorem was also proved in [8, Theorem 2.3]. □

Theorem 3.2. *Let*

$$P:=q^{1/6}\frac{f_1f_{10}}{f_2f_5}\qquad and \qquad Q:=q^{5/6}\frac{f_5f_{50}}{f_{10}f_{25}}.$$

Then the following result holds true.

$$\begin{aligned}&(PQ)^5+\frac{1}{(PQ)^5}-4\left((PQ)^4+\frac{1}{(PQ)^4}\right)+\left(PQ+\frac{1}{PQ}\right)-\left((PQ)^3+\frac{1}{(PQ)^3}\right)\\&\times\left(\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right)-\left((PQ)^2+\frac{1}{(PQ)^2}\right)\left(\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right)+12=0.\end{aligned}$$

Proof. From (2.1) and the definition of P and Q, we have

$$(3.8)\qquad P=\frac{A_1}{A_5}\qquad\text{and}\qquad Q=\frac{A_5}{A_{25}}.$$

Also, from (2.3) and (3.8), we find that

$$\frac{P^2A_5^4}{2}+\frac{2}{P^2A_5^4}=\frac{1}{2}\left(\frac{1}{P^3}-P^3\right).$$

On solving this for $P^2A_5^4/2$, we obtain

$$\frac{P^2A_5^4}{2} = \frac{K_1 \pm \sqrt{K_1^2-4}}{2}, \tag{3.9}$$

where

$$K_1 = \frac{1}{2}\left(\frac{1}{P^3} - P^3\right).$$

Therefore the identity (3.9) implies that

$$\frac{2}{P^2A_5^4} = \frac{K_1 \mp \sqrt{K_1^2-4}}{2}. \tag{3.10}$$

Proceeding on similar lines, it is not difficult to see that

$$\frac{Q^2A_{25}^4}{2} = \frac{K_2 \pm \sqrt{K_2^2-4}}{2} \tag{3.11}$$

and

$$\frac{2}{Q^2A_{25}^4} = \frac{K_2 \mp \sqrt{K_2^2-4}}{2}, \tag{3.12}$$

where

$$K_2 = \frac{1}{2}\left(\frac{1}{Q^3} - Q^3\right).$$

Multiplying (3.9) and (3.12) and employing (3.8), we obtain

$$(PQ)^2 = K_1K_2 \pm K_2\sqrt{K_1^2-4} \mp K_1\sqrt{K_2^2-4} - \sqrt{K_1^2-4}\sqrt{K_2^2-4}. \tag{3.13}$$

Multiplying (3.10) and (3.11) and then employing (3.8), we obtain

$$\frac{1}{(PQ)^2} = K_1K_2 \mp K_2\sqrt{K_1^2-4} \pm K_1\sqrt{K_2^2-4} - \sqrt{K_1^2-4}\sqrt{K_2^2-4}. \tag{3.14}$$

Adding (3.13) and (3.14) and then simplifying, we obtain

$$(PQ)^2 + \frac{1}{(PQ)^2} - 2K_1K_2 = -2\sqrt{K_1^2-4}\sqrt{K_2^2-4}.$$

Squaring on both sides and then simplifying, we obtain

$$(PQ)^4 + \frac{1}{(PQ)^4} - 4K_1K_2\left((PQ)^2 + \frac{1}{(PQ)^2}\right) + 16K_1^2 + 16K_2^2 - 62 = 0.$$

Finally, substituting the values of K_1 and K_2 in the above identity and then simplifying, we obtain the required result. □

Theorem 3.3. *If*

$$P := \frac{\varphi(-q)\varphi(-q^6)}{\varphi(-q^2)\varphi(-q^3)} \qquad \textit{and} \qquad Q := \frac{\varphi(-q^3)\varphi(-q^{18})}{\varphi(-q^6)\varphi(-q^9)},$$

then the following result holds true.

$$\left((PQ)^6 + \frac{1}{(PQ)^6}\right) - 16\left((PQ)^4 + \frac{1}{(PQ)^4}\right) + 37\left((PQ)^2 + \frac{1}{(PQ)^2}\right)$$
$$-\left(\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4\right)\left((PQ)^4 + \frac{1}{(PQ)^4} + (PQ)^2 + \frac{1}{(PQ)^2}\right)$$
$$-12\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right)\left(\frac{1}{2}\left((PQ)^4 + \frac{1}{(PQ)^4}\right) + (PQ)^2 + \frac{1}{(PQ)^2} - 1\right) - 44 = 0.$$

Proof. From (2.4) and the definition of P and Q, we can write

$$P = \frac{B_1}{B_3} \qquad \text{and} \qquad Q = \frac{B_3}{B_9}. \tag{3.15}$$

Also from (2.6) and (3.15), we find that

$$P^2B_3^4 + \frac{1}{P^2B_3^4} = \frac{1}{4}\left(P^4 + \frac{1}{P^4} + 6\right).$$

On solving this for $P^2B_3^4$, we obtain

$$P^2B_3^4 = \frac{K_1 \pm \sqrt{K_1^2 - 4}}{2}, \tag{3.16}$$

where

$$K_1 = \frac{1}{4}\left(P^4 + \frac{1}{P^4} + 6\right).$$

Therefore the identity (3.16) implies that

$$\frac{1}{P^2B_3^4} = \frac{K_1 \mp \sqrt{K_1^2 - 4}}{2}. \tag{3.17}$$

In the similar manner, we deduce that

$$Q^2B_9^4 = \frac{K_2 \pm \sqrt{K_2^2 - 4}}{2} \tag{3.18}$$

and

$$\frac{1}{Q^2B_9^4} = \frac{K_2 \mp \sqrt{K_2^2 - 4}}{2}, \tag{3.19}$$

where

$$K_2 = \frac{1}{4}\left(Q^4 + \frac{1}{Q^4} + 6\right).$$

Multiplying (3.16) and (3.19) and employing (3.15), we obtain

$$4(PQ)^2 = K_1K_2 \pm K_2\sqrt{K_1^2 - 4} \mp K_1\sqrt{K_2^2 - 4} - \sqrt{K_1^2 - 4}\sqrt{K_2^2 - 4}. \tag{3.20}$$

Multiplying (3.17) and (3.18) and then employing (3.15), we obtain

$$\frac{4}{(PQ)^2} = K_1K_2 \mp K_2\sqrt{K_1^2 - 4} \pm K_1\sqrt{K_2^2 - 4} - \sqrt{K_1^2 - 4}\sqrt{K_2^2 - 4}. \tag{3.21}$$

Adding (3.20) and (3.21) and then simplifying, we deduce

$$2\left((PQ)^2+\frac{1}{(PQ)^2}\right)-K_1K_2=-\sqrt{K_1^2-4}\sqrt{K_2^2-4}.$$

Squaring on both sides and then simplifying, we obtain

$$\left((PQ)^2+\frac{1}{(PQ)^2}\right)^2-K_1K_2\left((PQ)^2+\frac{1}{(PQ)^2}\right)+K_1^2+K_2^2-4=0.$$

Finally, substituting the values of K_1 and K_2 in the above identity and then simplifying, we obtain the result. □

Theorem 3.4. *Let*

$$P:=\frac{\varphi(-q)\varphi(-q^{10})}{\varphi(-q^2)\varphi(-q^5)}\qquad \textit{and}\qquad Q:=\frac{\varphi(-q^5)\varphi(-q^{50})}{\varphi(-q^{10})\varphi(-q^{25})}.$$

Then the following result holds true.

$$\left((PQ)^5+\frac{1}{(PQ)^5}\right)-16\left((PQ)^4+\frac{1}{(PQ)^4}\right)+25\left((PQ)^3+\frac{1}{(PQ)^3}\right)$$
$$+26\left(PQ+\frac{1}{PQ}\right)+\left((PQ)^3+\frac{1}{(PQ)^3}\right)\left(\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right)$$
$$+10\left((PQ)^2+\frac{1}{(PQ)^2}\right)\left(\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right)-\left((PQ)^2+\frac{1}{(PQ)^2}\right)\left(\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right)$$
$$+15\left(PQ+\frac{1}{PQ}\right)\left(\frac{P}{Q}+\frac{Q}{P}\right)+10\left(\frac{P}{Q}+\frac{Q}{P}\right)$$
$$-5\left((PQ)^3+\frac{1}{(PQ)^3}+PQ+\frac{1}{PQ}\right)\left(\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right)$$
$$-25\left((PQ)^2+\frac{1}{(PQ)^2}\right)\left(\frac{P}{Q}+\frac{Q}{P}\right)+5\left((PQ)^4+\frac{1}{(PQ)^4}\right)\left(\frac{P}{Q}+\frac{Q}{P}\right)-72=0.$$

Proof. From (2.4) and the definition of P and Q, we can write P and Q as

$$P=\frac{B_1}{B_5}\qquad \text{and}\qquad Q=\frac{B_5}{B_{25}}. \tag{3.22}$$

Also from (2.4) and (3.22), we find that

$$P^2B_5^4-\frac{1}{P^2B_5^4}=\frac{1}{4}\left(t^3+8t\right),$$

where

$$t=P-\frac{1}{P}.$$

On solving this for $P^2B_5^4$, we obtain

$$P^2B_5^4=\frac{K_1\pm\sqrt{K_1^2+4}}{2}, \tag{3.23}$$

where

$$K_1 = \frac{1}{4}\left(t^3 + 8t\right).$$

The identity (3.23) implies that

$$\frac{1}{P^2 B_5^4} = \frac{K_1 \mp \sqrt{K_1^2 + 4}}{2}. \tag{3.24}$$

In the similar manner, we deduce that

$$Q^2 B_{25}^4 = \frac{K_2 \pm \sqrt{K_2^2 + 4}}{2} \tag{3.25}$$

and

$$\frac{1}{Q^2 B_{25}^4} = \frac{K_2 \mp \sqrt{K_2^2 + 4}}{2}, \tag{3.26}$$

where

$$K_2 = \frac{1}{4}\left(s^3 + 8s\right) \qquad \text{and} \qquad s = Q - \frac{1}{Q}.$$

Multiplying (3.23) and (3.26) and employing (3.22), we obtain

$$4(PQ)^2 = K_1K_2 \pm K_2\sqrt{K_1^2 + 4} \mp K_1\sqrt{K_2^2 + 4} - \sqrt{K_1^2 + 4}\sqrt{K_2^2 + 4}. \tag{3.27}$$

Multiplying (3.24) and (3.25) and then employing (3.22), we obtain

$$\frac{4}{(PQ)^2} = K_1K_2 \mp K_2\sqrt{K_1^2 + 4} \pm K_1\sqrt{K_2^2 + 4} - \sqrt{K_1^2 + 4}\sqrt{K_2^2 + 4}. \tag{3.28}$$

Adding (3.27) and (3.28) and then simplifying, we deduce

$$2\left((PQ)^2 + \frac{1}{(PQ)^2}\right) - K_1K_2 = -\sqrt{K_1^2 + 4}\sqrt{K_2^2 + 4}.$$

Squaring on both sides and then simplifying, we obtain

$$\left((PQ)^2 + \frac{1}{(PQ)^2}\right)^2 - K_1K_2\left((PQ)^2 + \frac{1}{(PQ)^2}\right) = K_1^2 + K_2^2 + 4.$$

Finally, substituting the values of K_1 and K_2 in the above identity and then simplifying, we obtain Theorem 3.4. □

Theorem 3.5. *If*

$$P := q^{1/4}\frac{\psi(q)\psi(q^6)}{\psi(q^2)\psi(q^3)} \qquad \textit{and} \qquad Q := q^{3/4}\frac{\psi(q^3)\psi(q^{18})}{\psi(q^6)\psi(q^9)},$$

then the following result holds true.

$$(PQ)^3 + \frac{1}{(PQ)^3} - \left((PQ)^2 + \frac{1}{(PQ)^2}\right) + \left(PQ + \frac{1}{PQ}\right)$$
$$+ \left(4(PQ)^2 + \frac{4}{(PQ)^2} - \left(PQ + \frac{1}{PQ}\right)\right)\left(\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right) + 46 = 0.$$

Proof. From (2.7) and the definition of P and Q, we have

$$P = \frac{C_1}{C_3} \qquad \text{and} \qquad Q = \frac{C_3}{C_9}. \tag{3.29}$$

Also, from (2.8) and (3.29), we find that

$$\frac{PC_3^2}{2} - \frac{2}{PC_3^2} = \frac{1}{2}\left(\frac{1}{P^2} - P^2\right).$$

On solving this for $PC_3^2/2$, we obtain

$$PC_3^2 = K_1 \pm \sqrt{K_1^2 + 4}, \tag{3.30}$$

where

$$K_1 = \frac{1}{2}\left(\frac{1}{P^2} - P^2\right).$$

Therefore the identity (3.30) implies that

$$\frac{1}{PC_3^2} = K_1 \mp \sqrt{K_1^2 + 4}. \tag{3.31}$$

In the similar manner, we deduce that

$$QC_9^2 = K_2 \pm \sqrt{K_2^2 + 4} \tag{3.32}$$

and

$$\frac{1}{QC_9^2} = K_2 \mp \sqrt{K_2^2 + 4}, \tag{3.33}$$

where

$$K_2 = \frac{1}{2}\left(\frac{1}{Q^2} - Q^2\right).$$

Multiplying (3.30) and (3.33) and employing (3.29), we obtain

$$PQ = K_1K_2 \pm K_2\sqrt{K_1^2 + 4} \mp K_1\sqrt{K_2^2 + 4} - \sqrt{K_1^2 + 4}\sqrt{K_2^2 + 4}. \tag{3.34}$$

Multiplying (3.31) and (3.32) and then employing (3.29), we obtain

$$\frac{1}{PQ} = K_1K_2 \mp K_2\sqrt{K_1^2 + 4} \pm K_1\sqrt{K_2^2 + 4} - \sqrt{K_1^2 + 4}\sqrt{K_2^2 + 4}. \tag{3.35}$$

Adding (3.34) and (3.35) and then simplifying, we obtain

$$PQ + \frac{1}{PQ} - 2K_1K_2 = -2\sqrt{K_1^2 + 4}\sqrt{K_2^2 + 4}.$$

Squaring on both sides and then simplifying, we obtain

$$(PQ)^2 + \frac{1}{(PQ)^2} - 4K_1K_2\left(PQ + \frac{1}{PQ}\right) - 16K_1^2 - 16K_2^2 - 62 = 0.$$

Finally, substituting the values of K_1 and K_2 in the above identity and then simplifying, we complete the proof. □

4. Evaluations of $r_{k,n}$

Lemma 4.1. *For all positive real numbers k and n, define $r_{k,n}$ by*

$$r_{k,n} := \frac{f_1}{k^{1/4}q^{(k-1)/24}f_k},$$

where $q = e^{-2\pi\sqrt{n/k}}$. Then the following results hold true.
(i) $r_{k,n} = 1$ (ii) $r_{k,1/n} = r_{k,n}^{-1}$ (iii) $r_{k,n} = r_{n,k}$.

For the proof one can refer [10].

Theorem 4.2. *If $r_{k,n}$ is as defined as in Lemma 4.1, then the following result holds true.*

$$\left(\frac{r_{2,n}}{r_{2,81n}}\right)^6 + \left(\frac{r_{2,81n}}{r_{2,n}}\right)^6 - 10\left[\left(\frac{r_{2,n}}{r_{2,81n}}\right)^3 + \left(\frac{r_{2,81n}}{r_{2,n}}\right)^3\right] + 20$$
$$= \left[\left(\frac{r_{2,n}r_{2,81n}}{r_{2,9n}^2}\right)^6 + \left(\frac{r_{2,9n}^2}{r_{2,n}r_{2,81n}}\right)^6\right]\left[\left(\frac{r_{2,n}}{r_{2,81n}}\right)^3 + \left(\frac{r_{2,81n}}{r_{2,n}}\right)^3 - 1\right].$$

Proof. The proof follows directly from Theorem 3.1 with $P = \frac{r_{2,n}}{r_{2,9n}}$ and $Q = \frac{r_{2,9n}}{r_{2,81n}}$. □

Theorem 4.3 ([10]). *If $r_{k,n}$ is as defined as in Lemma 4.1, then the following result holds true.*

$$r_{2,1/9} = \sqrt[6]{5 - 2\sqrt{6}} = r_{2,9}^{-1}.$$

Proof. In order to prove the result asserted in Theorem 4.3, if we set $n = 1/9$ in Theorem 4.2 and upon using Lemma 4.1 we find that

$$\left(r_{2,1/9}\right)^{12} + \frac{1}{\left(r_{2,1/9}\right)^{12}} - 12\left((r_{2,1/9})^6 + \frac{1}{(r_{2,1/9})^6}\right) + 22 = 0.$$

On solving, we obtain

$$r_{2,1/9}^6 + \frac{1}{r_{2,1/9}^6} = 10, 2.$$

Since $r_{k,n}$ is increasing in n, we choose

$$r_{2,1/9}^6 + \frac{1}{r_{2,1/9}^6} = 10.$$

Therefore on solving, we get

$$r_{2,1/9}^6 = \sqrt{5 \pm 2\sqrt{6}}.$$

Since $0 < r_{2,1/9} < 1$, we choose $r_{2,1/9}^6 = \sqrt{5 - 2\sqrt{6}}$ and it completes the proof. □

Theorem 4.4. *If $r_{k,n}$ is as defined in Lemma 4.1, then the following result holds true.*

$$\left(\frac{r_{2,n}}{r_{2,625n}}\right)^5+\left(\frac{r_{2,625n}}{r_{2,n}}\right)^5-4\left(\left(\frac{r_{2,n}}{r_{2,625n}}\right)^4+\left(\frac{r_{2,625n}}{r_{2,n}}\right)^4\right)+\left(\frac{r_{2,n}}{r_{2,625n}}+\frac{r_{2,625n}}{r_{2,n}}\right)$$
$$-\left(\left(\frac{r_{2,n}}{r_{2,625n}}\right)^3+\left(\frac{r_{2,625n}}{r_{2,n}}\right)^3\right)\left(\left(\frac{r_{2,n}r_{2,625n}}{r_{2,25n}^2}\right)^3+\left(\frac{r_{2,25n}^2}{r_{2,n}r_{2,625n}}\right)^3\right)$$
$$-\left(\left(\frac{r_{2,n}}{r_{2,625n}}\right)^2+\left(\frac{r_{2,625n}}{r_{2,n}}\right)^2\right)\left(\left(\frac{r_{2,n}r_{2,625n}}{r_{2,25n}^2}\right)^3+\left(\frac{r_{2,25n}^2}{r_{2,n}r_{2,625n}}\right)^3\right)+12=0.$$

Proof. The proof follows directly from Theorem 3.2 with $P=\frac{r_{2,n}}{r_{2,25n}}$ and $Q=\frac{r_{2,25n}}{r_{2,625n}}$. □

Theorem 4.5. *We have*

$$r_{2,25}=\sqrt{\frac{a^2+4a+25+\sqrt{a^4+30a^2+200a+5(357+48\sqrt{6})}}{6a}}=r_{2,1/25}^{-1},$$

where $a=\left(145+30\sqrt{6}\right)^{1/6}$.

Proof. Upon setting $n=1/25$ in Theorem 4.4 and upon using Lemma 4.1 we find that

$$\frac{1}{(r_{2,25})^{10}}+(r_{2,25})^{10}-4\left(\frac{1}{(r_{2,25})^8}+(r_{2,25})^8\right)-2\left(\frac{1}{(r_{2,25})^6}+(r_{2,25})^6\right)$$
$$-2\left(\frac{1}{(r_{2,25})^4}+(r_{2,25})^4\right)+\left(\frac{1}{(r_{2,25})^2}+(r_{2,25})^2\right)+\frac{1}{r_{2,25}}+r_{2,25}+12=0.$$

On letting $(r_{2,25})^2+(r_{2,25})^{-2}=t$, we obtain

$$t^5-4t^4-7t^3+14t^2+12t+8=0.$$

On solving the above equation, we obtain

$$t=2,-2,\frac{1}{3}\left(\left(145+30\sqrt{6}\right)^{1/3}+\frac{25}{\left(145+30\sqrt{6}\right)^{1/3}}+4\right)$$

and the remaining two roots are imaginary. Since $r_{2,n}>1$, we choose

$$(r_{2,25})^2+\frac{1}{(r_{2,25})^2}=\frac{1}{3}\left(\left(145+30\sqrt{6}\right)^{1/3}+\frac{25}{\left(145+30\sqrt{6}\right)^{1/3}}+4\right).$$

On solving the above for $r_{2,25}$, we obtain the required result. □

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