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ON SOME NEW TYPE OF GENERATING FUNCTIONS OF GENERALIZED POISSON-CHARLIER POLYNOMIALS

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Abstract. The present paper concerns with a study of certain generating functions and summation formulas of generalized Poisson-Charlier polynomials. Some special cases are also discussed.

1. Introduction

Recently, Khan and Ahmed [3] studied several variables analogue of Poisson-Charlier polynomials. The results obtained in this paper are analogous to those obtained in [3]. Let the sequence of functions $\{S_n(x)\,|\,n=0,1,2,\ldots\}$ be generated by ([5])

$$\sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = \frac{f(x,t)}{[g(x,t)]^m} S_m[h(x,t)], \tag{1.1}$$

where m is a non negative integer, the $A_{m,n}$ are arbitrary constants and f, g, h are suitable functions of x and t. The importance of a generating function of the form (1.1) in obtaining the bilateral and trilateral generating relations for the functions $S_n(x)$ was realized by several authors. For instance, using the generating functions of the type (1.1) for Hermite, Laguerre and Gegenbauer polynomials, Rainville [4], derived some bilateral and bilinear generating functions due to Mehler Rainville [4] (1960, p. 198, Eq. (2)). Brafman [1] (Rainville [4], 1960, pp. 198, 213). Hardy-Hille (Rainville [4], 1960, p. 212, Theorem 69) and Meixner (Rainville [4], 1960, p. 281, Eq. (24)).

The following results are required in this paper.

Lagrange's Expansion Formula ([6], p. 355). If $\phi(z)$ is holomorphic at $z = z_0$ and $\phi(z_0) \neq 0$, and if

$$z = z_0 + w\phi(z), \tag{1.2}$$

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then an analytic function $f(z)$, which is holomorphic at $z = z_0$, can be expanded as a power series in w by the Lagrange formula [Whittaker and Watson (1927), p. 133]

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D_z^{n-1}\{f'(z)[\phi(z)]^n\} \mid_{z=z_0}, \tag{1.3}$$

where $D_z = d/dz$.

If we differentiate both sides of (1.3) with respect to w, using the relationship (1.2), and replace $f'(z)\phi(z)$ in the resulting equation by $f(z)$, we can write (1.3) in the form [cf. Polya and Szego (1972), p. 146, Problem 207]:

$$\frac{f(z)}{1 - w\phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^{n}\{f(z)[\phi(z)]^n\} \mid_{z=z_0}, \tag{1.4}$$

which is usually more suitable to apply than (1.3).

For$\phi(z) \equiv 1$, both (1.3) and (1.4) evidently yield Taylor's expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0), \tag{1.5}$$

where

$$f^{(n)}(z_0) = D_z^n\{f(z)\} \mid_{z=z_0} . \tag{1.6}$$

The following basic lemmas (see Rainville [4, pp. 56–58]) given below are useful.

Lemma 1.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{1.7}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k). \tag{1.8}$$

Lemma 2.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=[\frac{n}{2}]}^{n} A(k,n-2k) \tag{1.9}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+2k). \tag{1.10}$$

Lemma 3.

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1). \tag{1.11}$$

Poisson-Charlier polynomials: The Poisson-Charlier polynomials $C_n(x;\alpha)$ are defined by ([6], p. 425)

$$C_n(x;\alpha) \ =\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}k!\ \alpha^{-k}, \tag{1.12}$$

where $\alpha > 0$, $x \in N_0$.

The following generating relations holds for (1.7):

$$\sum_{n=0}^{\infty} C_n(x;\alpha)\frac{t^n}{n!} = \left(1-\frac{t}{\alpha}\right)^x e^t, \tag{1.13}$$

and

$$\sum_{n=0}^{\infty}(\lambda)_n C_n(x;\alpha)\frac{t^n}{n!} = (1-t)^{-\lambda}{}_2F_0\left[\begin{array}{ll}\lambda,\ -x; & \\ & \dfrac{t}{\alpha(1-t)} \\ -; & \end{array}\right]. \tag{1.14}$$

2. Generalized Poisson-Charlier polynomials

In this section, we define and study the generalized Poisson-Charlier polynomials, where $j,k \in \mathbb{N}$ $(0 \le j < k)$ as

$$\sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} = \left(1-\frac{t}{\alpha}\right)^x E_j(t,k), \tag{2.1}$$

where

$$E_j(t,k) = \sum_{n=0}^{\infty}\frac{t^{kn+j}}{(kn+j)!} \tag{2.2}$$

is the Pseudo-Bessel function introduced in [2].

Note that

$$E_0(t,1) = e^t \tag{2.3}$$

and

$$C_n^{(0,1)}(x;\alpha) = C_n(x;\alpha). \tag{2.4}$$

3. Generating functions of generalized Poisson-Charlier polynomials

The following generating functions hold for generalized Poisson-Charlier Polynomials given by (2.1)

$$C_m^{(j,k)}(x;\alpha) = m!\sum_{n=0}^{\frac{[m-j]}{k}}\frac{(-x)_{m-kn-j}\alpha^{kn-m+j}}{(m-kn-j)!(kn+j)!}. \tag{3.1}$$

To prove (3.1), we proceed as follows.

First we write,

$$\sum_{m=0}^{\infty} C_m^{(j,k)}(x;\alpha)\frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{(-x)_m t^m}{m!\alpha^m} \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}. \tag{3.2}$$

Setting $m \to m - kn - j$, so that $n \leq \frac{m-j}{k}$, we get

$$\sum_{m=0}^{\infty} C_m^{(j,k)}(x;\alpha)\frac{t^m}{m!} = \sum_{m=0}^{\infty} t^m \sum_{n=0}^{\frac{[m-j]}{k}} \frac{(-x)_{m-kn-j}\alpha^{kn-m+j}}{(m-kn-j)!(kn+j)!}. \tag{3.3}$$

Finally, comparing the coefficients of t, we get the result (3.1).
For $j = 0, k = 1$, (3.1) reduces to a known result:

$$C_m^{(0,1)}(x;\alpha) = C_m(x;\alpha) = m! \sum_{n=0}^{m} \frac{(-1)^n x! \alpha^{n-m}}{n!(m-n)!(x-n)!}. \tag{3.4}$$

For $j = 0, k = 2$, (3.1) gives

$$C_m^{(0,2)}(x;\alpha) = m! \sum_{n=0}^{[\frac{m}{2}]} \frac{(-x)_{m-2n}\alpha^{2n-m}}{(m-2n)!(2n)!}. \tag{3.5}$$

Setting $x \to x - 1$ in (2.1), we have

$$\left(1 - \frac{t}{\alpha}\right) \sum_{n=0}^{\infty} C_n^{(j,k)}(x-1;\alpha)\frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!}$$

or

$$\sum_{n=0}^{\infty} C_n^{(j,k)}(x-1;\alpha)\frac{t^n}{n!} - \frac{1}{\alpha}\sum_{n=1}^{\infty} C_{n-1}^{(j,k)}(x-1;\alpha)\frac{t^n}{(n-1)!} = \sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!},$$

comparing the coefficients of t^n, we get

$$C_n^{(j,k)}(x-1;\alpha) - \frac{n}{\alpha} C_{n-1}^{(j,k)}(x-1;\alpha) = C_n^{(j,k)}(x;\alpha). \tag{3.6}$$

Again from (2.1), we have

$$\left(1 - \frac{t}{\alpha}\right)^{-x} \sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}$$

or

$$\sum_{m=0}^{\infty} \frac{(x)_m t^m}{m!\alpha^m} \sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!}.$$

Setting $n \to nk + j - m \geqq 0 \to m \leq nk + j$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{nk+j} \frac{(x)_m t^m}{m!\alpha^m} C_{nk+j-m}^{(j,k)}(x;\alpha) \frac{t^{nk+j-m}}{(nk+j-m)!} = \sum_{n=0}^{\infty} \frac{t^{kn+j}}{(kn+j)!},$$

comparing the coefficients of t^{nk+j}, we have

$$\sum_{m=0}^{nk+j} \frac{(x)_m C_{nk+j-m}^{(j,k)}(x;\alpha)}{m!\alpha^m (nk+j-m)!} = \frac{1}{(kn+j)!}. \tag{3.7}$$

For $j=0, k=1$, it reduces to

$$\sum_{m=0}^{n} \frac{(x)_m C_{n-m}(x;\alpha)}{m!(n-m)!\alpha^m} = \frac{1}{n!}. \tag{3.8}$$

Multiplying both sides of (2.1) by $(1-\frac{t}{\alpha})^{-x+y}$, we have

$$\left(1-\frac{t}{\alpha}\right)^{-x+y} \sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} = \left(1-\frac{t}{\alpha}\right)^{y} E_j(t,k)$$

or

$$\sum_{m=0}^{n} \frac{(x-y)_m t^m}{m!\alpha^m} \sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n^{(j,k)}(y;\alpha)\frac{t^n}{n!}.$$

Replacing $n \to n-m$ and comparing the coefficient of t^n, we get

$$m!\sum_{m=0}^{n} \frac{(x-y)_m}{m!(n-m)!\alpha^m} C_{n-m}^{(j,k)}(x;\alpha) = C_n^{(j,k)}(y;\alpha). \tag{3.9}$$

For $y=0, j=0, k=1$, it reduces to

$$m!\sum_{m=0}^{n} \frac{(x)_m}{m!(n-m)!\alpha^m} C_{n-m}(x;\alpha) = C_n(\alpha). \tag{3.10}$$

Replacing x by $x+y$, we get from (3.9)

$$m!\sum_{m=0}^{n} \frac{(x)_m}{m!(n-m)!\alpha^m} C_{n-m}^{(j,k)}(x+y;\alpha) = C_n^{(j,k)}(y;\alpha). \tag{3.11}$$

4. Summation formulae

In this section we obtain the summation formula for generalized Poissson-Charlier polynomials $C_n^{(j,k)}(x,\alpha)$. We start with the definition of generalized Poisson-Charlier polynomials $C_n^{(j,k)}(x,\alpha)$ and write by using Lemma 1.1,

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x;\alpha)\frac{t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x;\alpha)\frac{t^{2n+1}}{(2n+1)!} \\ &= \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!}\left(\frac{t}{\alpha}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!}\left(\frac{t}{\alpha}\right)^{2k+1}\right] e^t.\end{aligned}$$

Replacing t by it, we get

$$\sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x;\alpha)\frac{t^{2n}}{2n!}(-1)^n + i\sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x;\alpha)\frac{t^{2n+1}}{(2n+1)!}(-1)^n$$

$$= \left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!}\left(\frac{t}{\alpha}\right)^{2k}(-1)^k + i\sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!}\left(\frac{t}{\alpha}\right)^{2k+1}(-1)^k\right](\cos t + i\sin t).$$

Comparing real and imaginary parts, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x;\alpha)\frac{t^{2n}}{2n!}(-1)^n \\
&= \cos t\left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!}\left(\frac{t}{\alpha}\right)^{2k}(-1)^k\right] - \sin t\left[\sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!}\left(\frac{t}{\alpha}\right)^{2k+1}(-1)^k\right] \\
&= \cos t\ {}_3F_2\left[\begin{matrix} -\frac{1-x}{2},\ \frac{2-x}{2},\ 1; \\ 1,\ \frac{3}{2}; \end{matrix}\ -\left(\frac{t}{\alpha}\right)^2\right] \\
&\quad + \frac{xt\sin t}{\alpha}\ {}_3F_2\left[\begin{matrix} -\frac{x}{2},\ \frac{1-x}{2},\ 1; \\ \frac{1}{2},\ \frac{3}{2}; \end{matrix}\ -\left(\frac{t}{\alpha}\right)^2\right]
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x;\alpha)\frac{t^{2n+1}}{(2n+1)!}(-1)^n \\
&= \cos t\left[\sum_{k=0}^{\infty} \frac{(-x)_{2k+1}}{(2k+1)!}\left(\frac{t}{\alpha}\right)^{2k+1}(-1)^k\right] + \sin t\left[\sum_{k=0}^{\infty} \frac{(-x)_{2k}}{(2k)!}\left(\frac{t}{\alpha}\right)^{2k}(-1)^k\right] \\
&= \sin t\ {}_3F_2\left[\begin{matrix} -\frac{x}{2},\ \frac{1-x}{2},\ 1; \\ \frac{1}{2},\ \frac{3}{2}; \end{matrix}\ -\left(\frac{t}{\alpha}\right)^2\right] \\
&\quad - \frac{xt\cos t}{\alpha}\ {}_3F_2\left[\begin{matrix} -\frac{1-x}{2},\ \frac{2-x}{2},\ 1; \\ 1,\ \frac{3}{2}; \end{matrix}\ -\left(\frac{t}{\alpha}\right)^2\right].
\end{aligned} \tag{4.2}$$

From (2.1), we have

$$\sum_{n=0}^{\infty} C_n^{(j,k)}(x;\alpha)\frac{t^n}{n!} = \left(1-\frac{t}{\alpha}\right)^x E_j(t,k).$$

Therefore,

$$\sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x;\alpha)\frac{t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x;\alpha)\frac{t^{2n+1}}{(2n+1)!}$$

$$= \left(1-\frac{t}{\alpha}\right)^x \left[\sum_{n=0}^{\infty} \frac{t^{2nk+j}}{(2nk+j)!} + \sum_{n=0}^{\infty} \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-x)_m}{\alpha^m m!} t^m \left[\sum_{n=0}^{\infty} \frac{t^{2nk+j}}{(2nk+j)!} + \sum_{n=0}^{\infty} \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!}\right].$$

Replacing t by it, we get

$$\sum_{n=0}^{\infty} C_{2n}^{(j,k)}(x;\alpha)\frac{t^{2n}}{2n!}(-1)^n + i\sum_{n=0}^{\infty} C_{2n+1}^{(j,k)}(x;\alpha)\frac{t^{2n+1}}{(2n+1)!}(-1)^n \tag{4.3}$$

$$= \left[\sum_{m=0}^{\infty} a_{2m}(-1)^m t^{2m} + i\sum_{m=0}^{\infty} a_{2m+1} t^{2m+1}\right]$$

$$\times \left[\sum_{n=0}^{\infty} \frac{t^{2nk+j}(-1)^n}{(2nk+j)!} + i\sum_{n=0}^{\infty} \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!}\right],$$

where $a_m = \frac{(-x)_m}{a^m m!}$.

Comparing real and imaginary parts, we get

$$C_{2n}^{(j,k)}(x;\alpha)\frac{t^{2n}}{(2n)!}(-1)^n = AC - BD \tag{4.4}$$

and

$$C_{2n+1}^{(j,k)}(x;\alpha)\frac{t^{2n+1}}{(2n+1)!}(-1)^n = BC + AD, \tag{4.5}$$

where

$$A = \sum_m a_{2m}(-1)^m t^{2m},\ B = i\sum_m a_{2m+1} t^{2m+1},$$

$$C = \sum_n \frac{t^{2nk+j}(-1)^n}{(2nk+j)!} \text{ and } D = i\sum_n \frac{t^{(2n+1)k+j}}{(2n+1)k+j)!}.$$

5. The generalized Poisson-Charlier polynomials of two variables

The generalized Poisson-Charlier polynomials of two variables $C_n^{(j,k)}(x,y;\alpha,\beta)$ are defined as follows:

$$C_n^{(j,k)}(x,y;\alpha,\beta) = \sum_{r=0}^{[\frac{n-j}{k}]} \sum_{s=0}^{[\frac{n-r-j}{k}]} (-1)^{r+s}(r+s)!\alpha^{-r}\beta^{-s} \binom{\frac{n}{k}}{r+s}\binom{x}{r}\binom{y}{s}, \tag{5.1}$$

where $\alpha,\beta > 0$, $x,y \in N_0$.

The following generating relations hold for (5.1):

$$\sum_{n=0}^{\infty} C_n^{(j,k)}(x,y;\alpha,\beta)\frac{t^n}{n!} = \left(1-\frac{t}{\alpha}\right)^x \left(1-\frac{t}{\beta}\right)^y E_j(t,k) \tag{5.2}$$

and

$$\sum_{n=0}^{\infty}(\lambda)_n C_n^{(j,k)}(x,y;\alpha,\beta)\frac{t^n}{n!} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda+j)_{r+s}(\lambda)_j}{r!s!}\left(\frac{t}{\alpha}\right)^r\left(\frac{t}{\beta}\right)^s(x)_r(-y)_s\phi_{r,s}, \tag{5.3}$$

where

$$\phi_{r,s} = \sum_{n=0}^{\infty}\frac{(\lambda+r+s+j)_{nk}t^{nk+j}}{(nk+j)!}. \tag{5.4}$$

Proof of (5.2). We have from L.H.S,

$$\begin{aligned}
&\left(1-\frac{t}{\alpha}\right)^x\left(1-\frac{t}{\beta}\right)^y E_j(t,k) \\
&= \sum_{r=0}^{\infty}\frac{(-x)_r}{r!}\left(\frac{t}{\alpha}\right)^r\sum_{s=0}^{\infty}\frac{(-y)_s}{s!}\left(\frac{t}{\beta}\right)^s\sum_{n=0}^{\infty}\frac{t^{kn+j}}{(kn+j)!} \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^r x!}{(x-r)!}\frac{(-1)^s y!}{(y-s)!}\frac{(\alpha)^{-r}(\beta)^{-s}}{r!s!}\frac{t^{kn+j+r+s}}{(kn+j)!}.
\end{aligned}$$

Replacing $n \to \frac{n-r-j}{k}$, then $n \to n-s$, we get

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n-j}{k}]}\sum_{s=0}^{\frac{n-r-j}{k}}\frac{(-1)^{r+s}x!}{(x-r)!}\frac{y!}{(y-s)!}\frac{(\alpha)^{-r}(\beta)^{-s}}{r!s!}\frac{t^n}{(\frac{n}{k}-r-s-j)!},$$

from which the result follows. □

Proof of (5.3). We have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\lambda)_n C_n^{(j,k)}(x,y;\alpha,\beta)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n-j}{k}]}\sum_{s=0}^{[\frac{n-r-j}{k}]}(\lambda)_n(-1)^{r+s}\binom{n}{r+s}\binom{x}{r}\binom{y}{s}(r+s)!\ \alpha^{-r}\beta^{-s}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n-j}{k}]}\sum_{s=0}^{[\frac{n-r-j}{k}]}(\lambda)_n\frac{(-x)r(-y)_s}{r!s!(n-r-s)!(\alpha)^r(\beta)^s}t^n.
\end{aligned}$$

Setting $n \to nk+r+j+s$, we have

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(\lambda)_{nk+r+j+s}\frac{(-x)r(-y)_s}{r!s!(n+j)!(\alpha)^r(\beta)^s}t^{nk+r+j+s} \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(\lambda)_{nk+r+j+s}\left[\left(\frac{t}{\alpha}\right)^r\left(\frac{t}{\beta}\right)^s\frac{(-x)r(-y)_s}{r!s!}\frac{t^{nk+j}}{(nk+j)!}\right]
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(\lambda+r+s+j)_{nk}(\lambda+j)_{r+s}(\lambda)_j\left[\left(\frac{t}{\alpha}\right)^r\left(\frac{t}{\beta}\right)^s\frac{(-x)r(-y)_s}{r!s!}\frac{t^{nk+j}}{(nk+j)!}\right]$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda+j)_{r+s}(\lambda)_j}{r!s!}\left(\frac{t}{\alpha}\right)^r\left(\frac{t}{\beta}\right)^s(x)_r(-y)_s\phi_{r,s},$$

which proves (5.3), where, $\phi_{r,s}$ is given by (5.4). ☐

Special Case: For $j = 0$, and $k = 1$, (5.2) reduces to (3.2).

Generalized Poisson-Charlier polynomials of n-variables $C_n^{(j,k)}(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_n)$ are defined as follows:

$$\begin{aligned}(5.5)\quad & C_n^{(j,k)}(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_k=0}^{n-r-s}(-1)^{r_1+r_2+\cdots+r_k}\binom{\frac{n}{k}}{r_1+r_2+\cdots+r_k}\binom{x_1}{r_1}\binom{x_2}{r_2}\cdots\binom{x_n}{r_k}\\ &\quad\times(r_1+r_2+\cdots+r_k)!\,\alpha_1^{-r_1}\alpha_2^{-r_2}\cdots\alpha_n^{-r_k},\end{aligned}$$

where $j, k \in \mathbb{N}$, $0 \le j < k$, $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$, $x_1, x_2, \ldots, x_n \in N_0$.

Results for three and n-variables will follow as in case of two variables.

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