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RATIONAL HOMOTOPY TYPE OF MAPPING SPACES BETWEEN COMPLEX PROJECTIVE SPACES AND THEIR EVALUATION SUBGROUPS

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Abstract. We use L_∞ models to compute the rational homotopy type of the mapping space of the component of the natural inclusion $i_{n,k}$: $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$ between complex projective spaces and show that it has the rational homotopy type of a product of odd dimensional spheres and a complex projective space. We also characterize the mapping $\operatorname{aut}_1 \mathbb{C}P^n \to \operatorname{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ and the resulting G-sequence.

1. Introduction

Let $f : X \to Y$ be a map between simply connected CW-complexes of finite type. We denote by $\operatorname{map}(X, Y; f)$ the path component of f in the space of continuous mappings from X to Y. The study of the rational homotopy type of $\operatorname{map}(X, Y; f)$ was initiated by Haefliger [10] who describes its Sullivan model. Afterwards there were attempts to find a Quillen model of $\operatorname{map}(X, Y; f)$ from either a Sullivan or a Quillen model of f. Chain complexes of which the homology coincides with rational homotopy groups of function spaces were investigated [8,12,13]. Those chain complexes were later developed into models of function spaces [2–5].

Following [5] we describe in this paper an L_∞ model of the inclusion $i_{n,k}$: $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$. We shall use rational homotopy theory for which the standard reference is [6].

The notion of L_∞-algebra was introduced by Lada [11] and we remind here the definition.

Definition 1. A permutation $\sigma \in S_n$ is called an $(i, n-i)$-shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$, where $i = 1, \ldots, n$. For graded objects $x_1, \ldots, x_n$, the Koszul sign $\epsilon(\sigma)$ is determined by

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

It depends not only of the permutation σ but also of degrees of $x_1, \ldots, x_n$.

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We assume that all vector spaces are over the field of rational numbers $\mathbb{Q}$.

Definition 2. An L_∞-algebra or a strongly homotopy Lie algebra is a graded vector space $L = \oplus_{i\geq 0} L_i$ with maps $\ell_k : L^{\otimes^k} \to L$ of degree $k-2$ such that

(1) ℓ_k is graded skew symmetric, that is, for a k-permutation σ

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \mathrm{sgn}(\sigma)\epsilon(\sigma)\ell_k(x_1, \ldots, x_k),$$

where $\mathrm{sgn}(\sigma)$ is the sign of σ.

(2) There are some generalized Jacobi identities

$$\sum_{i+j=n+1} \sum_{\sigma} \epsilon(\sigma)(-1)^{i(j-1)} \ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0,$$

where the summation extends to all $(i, n-i)$ shuffles of the symmetric group S_n.

If $\ell_k = 0$ for $k \geq 3$, one retrieves the definition of a graded differential Lie algebra (L, d) where $d = \ell_1$ and ℓ_2 is the Lie bracket.

Let (L, ℓ_k) be an L_∞ algebra and sL the suspension of L, and $C_\infty(L) = (\wedge sL, d)$ the generalized Cartan-Chevalley-Eilenberg functor (see [6, §22]). One gets linear mappings $d_k : \wedge^k(sL) \to sL$ defined by

$$d_k(sx_1 \wedge \cdots \wedge sx_k) = (-1)^{\frac{k(k-1)}{2}} \ell_k(x_1, \ldots, x_k),$$

each of which extends into a codifferential on the coalgebra $\wedge sL$. This gives an equivalence between L_∞ structures on L and codifferentials on $\wedge sL$ [11]. Moreover if L is of finite type, then $C^\infty(L) = (\wedge(sL)^\#, d)$ is a commutative differential graded algebra (cdga for short). The differential $d = d_1 + \cdots + d_k + \cdots$ is defined by

$$\langle d_k v, sx_1 \wedge \cdots \wedge sx_k\rangle = (-1)^\epsilon \langle v, \ell_k(x_1, \ldots, x_k)\rangle,$$

where $v \in (sL)^\#$ and $\epsilon = \sum_{i=1}^{k-1}(k-i)|x_i|$.

Definition 3. Two cdga's (A, d) and (B, d) have the same homotopy type if they are linked by a sequence of quasi-isomorphisms

$$(A, d) = A_0 \to A_1 \leftarrow A_2 \cdots \to A_{n-1} \leftarrow A_n = (B, d).$$

Let V be a graded vector space. A Sullivan algebra $(\wedge V, d)$ is the free graded commutative algebra generated by V together with a filtration $V(0) \subset V(1) \subset \cdots \subset V$ such that $dV(i) \subset \wedge V(i-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$. A Sullivan model of a simply connected space X is a Sullivan algebra $(\wedge V, d)$ such that there exists a quasi-isomorphism $\varphi : (\wedge V, d) \to A_{PL}(X)$, where $A_{PL}(X)$ denotes the cdga of piecewise linear forms of X [16]. A cdga model of X is a cdga (A, d) which has the same homotopy type as $A_{PL}(X)$.

Definition 4. If $f : X \to Y$ is a map between simply connected spaces of finite type, then there is a cdga map $\phi : (\wedge V, d) \to (B, d)$, called a model of f, where (B, d) and $(\wedge V, d)$ are respective cdga models of X and Y, respectively.

Definition 5. Let L be an L_∞-algebra of finite type. Then L is called an L_∞ model of a topological space X if $C^\infty(L)$ is a Sullivan model of X. It is minimal if $\ell_1 = 0$. In this case $\pi_*(\Omega X) \otimes \mathbb{Q} \cong L$.

In this note, we give another proof of the following result using L_∞ models of function spaces (see [15], Example 3.4).

Theorem 6. *The function space* $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ *has the rational homotopy type of* $\mathbb{C}P^k \times S^{2k+3} \times \cdots \times S^{2(n+k)+1}$.

Moreover we study evaluation subgroups of the mapping $\mathrm{aut}_1\, \mathbb{C}P^n \to \mathbb{C}P^{n+k}$ and prove the following result.

Theorem 7. *The G-sequence associated with the inclusion*

$$\mathrm{aut}_1\, \mathbb{C}P^n \to \mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$$

is not exact.

2. L_∞-models of function spaces

Definition 8. Let $\phi : (\wedge V, d) \to (B, d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta : (\wedge V)^n \to B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We denote by $\mathrm{Der}(\wedge V, B; \phi)$ the $\mathbb{Z}$-graded vector space of all ϕ-derivations. The differential on $\mathrm{Der}(\wedge V, B; \phi)$ is defined by $\delta\theta = d\theta - (-1)^k \theta d$.

Define $\widetilde{\mathrm{Der}}(\wedge V, B; \phi)$ as

$$\widetilde{\mathrm{Der}}(\wedge V, B, \phi)_i = \begin{cases} \mathrm{Der}(\wedge V, B; \phi)_i, & i > 1, \\ \{\theta \in \mathrm{Der}(\wedge V, B; \phi)_1 : \delta\theta = 0\}, & i = 1. \end{cases}$$

If $\varphi_1, \ldots, \varphi_k \in \widetilde{\mathrm{Der}}(\wedge V, B; \phi)$ are ϕ-derivations of respective degrees $n_1, \ldots, n_k$, define

$$\begin{aligned} &[\varphi_1, \ldots, \varphi_k](v) \\ &= (-1)^{n_1 + \cdots + n_k - 1} \sum \Big(\sum_{i_1, \ldots, i_k} \epsilon \phi(v_1 \cdots \hat{v}_{i_1} \cdots \hat{v}_{i_k} \cdots v_m) \varphi_1(v_{i_1}) \cdots \varphi_k(v_{i_k}) \Big), \end{aligned}$$

where $dv = \sum v_1 \cdots v_m$ and ϵ is the corresponding Koszul sign of the permutation

$$(\varphi_1, \ldots, \varphi_k, v_1, \ldots, v_m) \to (v_1, \ldots, \hat{v}_{i_1}, \ldots, \hat{v}_{i_k}, \ldots, v_m, \varphi_1, v_{i_1}, \ldots, \varphi_k, v_{i_k}).$$

We note that $[\varphi_1, \ldots, \varphi_k]$ is of degree $n_1 + \cdots + n_k - 1$. Now define linear maps ℓ_k of degree $k-2$ on $s^{-1}\widetilde{\mathrm{Der}}(\wedge V, B, \phi)$ by

$$\ell_1(s^{-1}\varphi) = -s^{-1}\delta\varphi, \quad \ell_k(s^{-1}\varphi_1, \ldots, s^{-1}\varphi_k) = (-1)^{\epsilon_k} s^{-1}[\varphi_1, \ldots, \varphi_k],$$

where $\epsilon_k = \sum_{i=1}^{k-1} (k-i)|\varphi_i|$.

Proposition 9 (Lemma 3.3,[5])**.** *If* $\phi : (\wedge V, d) \to (B, d)$ *is a Sullivan model of a mapping* $f : X \to Y$ *between simply connected spaces and* V *is finite dimensional, then* $(s^{-1}\widetilde{\mathrm{Der}}(\wedge V, B; \phi), \ell_k)$ *is an* L_∞ *model of* $\mathrm{map}(X, Y; f)$.

3. Component of the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+k}$

Recall that the minimal Sullivan model of $\mathbb{C}P^n$ is given by $(\wedge(x_2, x_{2n+1}), d)$ where $dx_2 = 0$, $dx_{2n+1} = x_2^{n+1}$. Our objective is to compute an L_∞ model of the component of the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+k}$. For $k = 0$, one gets a model of $\mathrm{aut}_1 \mathbb{C}P^n = \mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^n; \mathrm{Id})$ from the differential Lie algebra (L, δ) of derivations of $(\wedge(x_2, x_{2n+1}), d)$, of which $H_*(L, \delta)$ is spanned by $\{z_3, z_5, \ldots, z_{2n+1}\}$ [7, §3]. Therefore $\mathrm{aut}_1 \mathbb{C}P^n$ has the rational homotopy type of the product $S^3 \times S^5 \times \cdots \times S^{2n+1}$. This result was also proved by Møller and Raussen using another method [15, Example 3.4].

Let $f : (\wedge V, d) \to (B, d)$ be a morphism of differential graded algebras. For $v \in V$ and $b \in B$ we denote by (v, b) the unique f-derivation θ such that $\theta(v) = b$ and zero on the remaining generators of $\wedge V$.

From now on we assume that $k \geq 1$. A model of the inclusion

$$i_{n,k} : \mathbb{C}P^n \to \mathbb{C}P^{n+k}$$

is given by

$$\psi : (A, d) = (\wedge(x_2, x_{2n+2k+1}) \to (\wedge(y_2, y_{2n+1}), d) = (B, d),$$

where $\psi(x_2) = y_2$, $\psi(x_{2n+2k+1}) = y_2^k y_{2n+1}$. We consider the composition

$$\phi : A = (\wedge(x_2, x_{2n+2k+1}) \xrightarrow{\psi} (\wedge(y_2, y_{2n+1}), d) = B \simeq (\wedge(y_2)/(y_2^{n+1}), 0).$$

Hence $\phi(x_2) = y_2$ and $\phi(x_{2n+2k+1}) = 0$. The induced map

$$(\mathrm{Der}(A, B; \psi), \delta) \to (\mathrm{Der}(A, H^*(B); \phi), \delta)$$

is a quasi-isomorphism [1]. In the sequel we compute

$$\widetilde{\mathrm{Der}}(\wedge(x_2, x_{2n+2k+1}), \wedge(y_2)/(y_2^{n+1}); \phi)$$

and determine its brackets. As a vector space

$$\widetilde{\mathrm{Der}}(\wedge(x_2, x_{2n+2k+1}), \wedge(y_2)/(y_2^{n+1}); \phi)$$

is spanned by

$$\{\beta_2, \alpha_{2k+2i-1},\ i = 1, \ldots, n+1\},$$

where $\alpha_{2k+2i-1} = (x_{2n+2k+1}, y_2^{n-i+1})$ and $\beta_2 = (x_2, 1)$. Note that $|\beta_2| = 2$ and $|\alpha_{2k+2i-1}| = 2k + 2i - 1$. Computations show that the only non zero brackets are given by $\underbrace{[\beta_2, \ldots, \beta_2]}_{k+i} = \alpha_{2k+2i-1}$ for $i = 1, \ldots, n+1$.

We deduce the following result (see [15] for a different proof).

Proposition 10. *The function space* $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ *has a Sullivan model of the form*

$$(\wedge(z_2, z_{2k+1}, \ldots, z_{2k+2n+1}), d),$$

where $dz_2 = 0$, $dz_{2k+1} = z_2^{k+1}, \ldots, dz_{2k+2n+1} = z_2^{k+n+1}$.

Proof. An L_∞ model (L, ℓ_k) of $\mathrm{map}(\mathbb{C}P(n), \mathbb{C}P(n+k); i_{n,k})$ is spanned by

$$\langle s^{-1}\beta_2, s^{-1}\alpha_{2k+2i-1},\ i = 1, \ldots, n+1\rangle.$$

Moreover $\ell_j = 0$ for $j = 1, \ldots, k$ and $\ell_{k+i}(s^{-1}\beta_2, \ldots, s^{-1}\beta_2) = s^{-1}\alpha_{2k+2i-1}$, for $i = 1, \ldots, n+1$. Therefore

$$C^\infty(L) = \wedge(z_2, z_{2k+1}, z_{2k+3}, \ldots, z_{2k+2n+1}), d), \quad dz_2 = 0,\ dz_{2k+2i+1} = z_2^{k+i+1},$$

where $0 \leq i \leq n$. □

Theorem 11. *The function space* $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ *has the rational homotopy type of* $\mathbb{C}P^k \times S^{2k+3} \times \cdots \times S^{2(n+k)+1}$.

Proof. By the above result, a Sullivan model of $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ is given by

$$(\wedge(x_2, x_{2k+1}, x_{2k+3}, \ldots, x_{2n+2k+1}),$$

where $dx_2 = 0$, $dx_{2i+1} = x_2^{i+1}$, $i = k, k+1, \ldots, k+n$. We consider the relative Sullivan model

$$(\wedge(x_2, x_{2k+1}), d) \to (\wedge(x_2, x_{2k+1}) \otimes \wedge x_{2k+3}, D),$$

where

$$dx_2 = 0, dx_{2k+} = x_2^{k+1}, Dx_2 = dx_2, Dx_{2k+1} = dx_{2k+1}, Dx_{2k+3} = x_2^{k+2}.$$

It is a Sullivan model of the fibration $S^{2k+3} \to E \xrightarrow{p} \mathbb{C}P^k$, where p is classified by a map $f : \mathbb{C}P^k \to B\,\mathrm{aut}_1\, S^{2k+3}$. Using the algebra of derivations on the minimal Sullivan model of S^{2k+3} [16], it is easily seen that $B\,\mathrm{aut}_1\, S^{2k+3}$ has the rational homotopy type of $K(\mathbb{Q}, 2k+4)$ [7, Proposition 2.1].

Moreover equivalence classes

$$[\mathbb{C}P^k,\ K(\mathbb{Q}, 2k+4)]$$

are in a bijective correspondence with $H^{2k+4}(\mathbb{C}P^k, \mathbb{Q}) = \{0\}$. Therefore the classifying map f is rationally trivial. So we deduce that the fibration is trivial. Hence the cdga

$$(A, d) = (\wedge(x_2, x_{2k+1}, x_{2k+3}), d), dx_2 = 0,\ dx_{2k+1} = x_2^{k+1},\ dx_{2k+3} = x_2^{k+2}$$

and

$$(\wedge(x_2, x_{2k+1}) \otimes \wedge z_{2k+3}, d), dx_2 = 0,\ dx_{2k+1} = x_2^{k+1},\ dz_{2k+3} = 0$$

are isomorphic. We deduce that the cdga (A, d) is a Sullivan model of $\mathbb{C}P^k \times S^{2k+3}$. It follows from an induction argument that $\mathrm{map}(\mathbb{C}P^k, \mathbb{C}P^{n+k}; i_{n,k})$ has the rational homotopy type of $\mathbb{C}P^k \times S^{2k+3} \times \cdots \times S^{2(n+k)+1}$. □

Recall that a Sullivan algebra $(\wedge V, d)$ is called formal if there is a quasi-isomorphism $(\wedge V, d) \to H^*(\wedge V, d)$. Spheres and complex projective spaces are formal. Moreover a product of formal spaces is also formal. We deduce that:

Corollary 12. *The function space* $\mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ *is formal.*

4. Evaluation subgroups of the inclusion $i_{n,k}:\mathbb{C}P^n\to\mathbb{C}P^{n+k}$

We consider the inclusion $i_{n,k}:\mathbb{C}P^n\to\mathbb{C}P^{n+k}$ and the corresponding Sullivan model ϕ of the previous section given by the composition

$$\phi : A=(\wedge(x_2,x_{2n+2k+1}),d)\xrightarrow{\psi}\wedge(y_2,y_{2n+1}),d)=B\xrightarrow[\simeq]{\gamma}H^*(B).$$

Forgetting the desuspension, a model of the inclusion $(i_{n,k})_*:\mathrm{aut}_1\,\mathbb{C}P^n\to\mathrm{map}(\mathbb{C}P^n,\mathbb{C}P^{n+k};i_{n,k})$ is given by

$$\phi^*:(\mathrm{Der}(B,H^*(B);\gamma),\delta)\to(\mathrm{Der}(A,H^*(B);\phi),\delta).$$

We now characterize the map ϕ^* when $k>n$.

Theorem 13. *If $k>n$, then the induced map*

$$\phi^*:(\mathrm{Der}(B,H^*(B);\gamma),\delta)\longrightarrow(\mathrm{Der}(A,H^*(B),\phi),\delta))$$

is homotopy trivial.

Proof. We note that $L=\mathrm{Der}(B,H^*(B);\gamma)$ is spanned by

$$\{\delta_2,\theta_1,\theta_3,\dots,\theta_{2n+1}\},$$

where $\delta_2=(y_2,1)$, $\theta_{2i+1}=(y_{2n+1},y_2^{n-i})$, $i=0,\dots,n$. The differential is given by $\delta\delta_2=(n+1)\theta_1$ and zero otherwise. Therefore

$$\pi_*(\mathrm{aut}_1\,\mathbb{C}P^n)\otimes\mathbb{Q}=H_*(L,\delta)=\langle[\theta_3],\dots,[\theta_{2n+1}]\rangle.$$

Hence $\mathrm{aut}_1\,\mathbb{C}P^n$ has the rational homotopy type of $S^3\times S^5\times\cdots\times S^{2n+1}$. Let

$$L'=(\mathrm{Der}(A,H^*(B),\phi),\delta)=(\langle\beta_2,\alpha_{2k+1},\dots,\alpha_{2n+2k+1}\rangle,\delta).$$

The mapping $\phi^*:L\to L'$ is defined by $\phi^*(\delta_2)$, $\phi^*(\theta_{2i+1})=0$ for $i<k$, and $\phi^*(\theta_{2i+1})=\alpha_{2i+1}$ for $i\ge k$. If $k>n$, then $\phi^*(\delta_2)=\beta_2$ and zero otherwise. Moreover

$$C^\infty(s^{-1}L)=(\wedge(x_2,y_1,\dots,y_{2i-1},\dots,y_{2n+1}),d),$$

where $dx_2=0$ and $dy_{2i-1}=x_2^i$. In particular $dy_1=x_2$. In the same way

$$C^\infty(s^{-1}L')=(\wedge(u_2,v_{2k+1},\dots,v_{2n+2k+1}),d),$$

where $du_2=0$, $dv_{2i+1}=u_2^{i+1}$. Hence

$$\Phi=C^\infty(\phi^*):C^\infty(s^{-1}L')\to C^\infty(s^{-1}L)$$

is defined by $\Phi(u_2)=x_2$ and vanishes on other generators. As $C^\infty(s^{-1}L')$ is quasi-isomorphic to

$$(\wedge(w_2,w_{2k+1}),d)\otimes(\wedge(w_{2k+3},\dots,w_{2n+2k+1}),0),$$

where $dw_2=0$, $dw_{2k+1}=w_2^{k+1}$ and, $C^\infty(s^{-1}L)$ is quasi-isomorphic to

$$(\wedge(z_3,\dots,z_{2n+1}),0),$$

then induced map

$$\tilde{\Phi}:(\wedge(w_2,w_{2k+1},w_{2k+3},\dots,w_{2n+2k+1}),d)\to(\wedge(z_3,\dots,z_{2n+1}),0)$$

between minimal Sullivan models is zero. □

Definition 14. Let X be a topological space. We say $\alpha \in \pi_n(X)$ is a Gottlieb element if the map: $f \vee 1_X : S^n \vee X \to X$ extends to $S^n \times X$, where f represents the homotopy class α [9].

Gottlieb elements form a subgroup of $\pi_*(X)$ which will be denoted by $G_*(X)$. It comes from the definition that $G_*(X)$ is the image of $\pi_*(\mathrm{ev}) : \pi_*(\mathrm{aut}_1 X, 1_X) \to \pi_*(X, x_0)$, where ev is the evaluation map at x_0. If $f : X \to Y$, then $G_*(Y, X; f)$ is the image of $\pi_*(ev)$ where $ev : \mathrm{map}(X, Y; f) \to Y$ is the evaluation map at the base point.

Let $(\wedge V, d)$ be the minimal Sullivan model of a simply connected space X. Define the Gottlieb group of $(\wedge V, d)$

$$G_n(\wedge V, d) = \{[\theta] \in H_n(\mathrm{Der} \wedge V, \delta) : \theta(v) = 1,\ v \in V^n\}.$$

Hence $G_*(\wedge V, d) \cong \mathrm{im}\, H_*(\epsilon_*)$, where $\epsilon_* : \mathrm{Der} \wedge V \to \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon)$ is the post composition with the augmentation map $\epsilon : \wedge V \to \mathbb{Q}$. Then $G_n(\wedge V) \cong G_n(X_{\mathbb{Q}})$, where $h : X \to X_{\mathbb{Q}}$ is the rationalization [6, Propostion 29.8]. There are also relative Gottlieb groups $G_*^{rel}(Y, X; f)$ and a G-sequence

$$\cdots \to G_{n+1}^{rel}(Y, X; f) \to G_n(X) \to G_n(Y, X; f) \to \cdots$$

which was introduced by Lee and Woo. The sequence is exact in some cases, for instance if f has a left homotopy inverse [17]. We follow the description of rational evaluation homotopy groups as given by Lupton and Smith [12].

Using augmentation maps we obtain the commutative diagram.

$$\begin{array}{ccc} \mathrm{Der}(B, H^*(B); \gamma) & \xrightarrow{\phi^*} & \mathrm{Der}(A, H^*(B); \phi) \\ \downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle \epsilon_*} \\ \mathrm{Der}(B, \mathbb{Q}; \epsilon) & \xrightarrow{\widehat{\phi}^*} & \mathrm{Der}(A, \mathbb{Q}; \epsilon) \end{array}$$

In the same way we define $G_*(A, H^*(B); \phi)$ as the image of $H_*(\epsilon_*)$ in $H_*(\mathrm{Der}(A, \mathbb{Q}, \epsilon))$.

In order to define relative rational Gottlieb groups, we recall that if $\phi : (C, d_C) \to (C', d_{C'})$ is a map of chain complexes, the mapping cone of ϕ, denoted by $\mathrm{Rel}(\phi)$, is the complex of which the underlying graded vector space is $sC \oplus C'$ and the differential is given by $D(sx, y) = (-sd_C(x), \phi(x) + d_{C'} y)$ [12] or [14, p. 46]. Define chain maps $J : C'_n \to \mathrm{Rel}_n(\phi)$ and $P : \mathrm{Rel}_n(\phi) \to C_{n-1}$ by $J(y) = (0, y)$ and $P(sx, y) = x$. This yields an exact sequence of chain complexes

$$0 \to C'_* \xrightarrow{J} \mathrm{Rel}_*(\phi) \xrightarrow{P} C_{*-1} \to 0,$$

which induces a long exact sequence in homology [14, Proposition 4.3]. We consider the mapping cone $\mathrm{Rel}(\phi^*)$ of

$$\phi^* : (\mathrm{Der}(B, H^*(B), \gamma), \delta) \to (\mathrm{Der}(A, H^*(B), \phi), \delta),$$

$\mathrm{Rel}(\widehat{\phi}^*)$ the mapping cone of $\widehat{\phi}^* : \mathrm{Der}(B, \mathbb{Q}; \epsilon) \to \mathrm{Der}(A, \mathbb{Q}; \epsilon)$ and the induced map $(\epsilon_*, \epsilon_*) : \mathrm{Rel}(\phi^*) \to \mathrm{Rel}(\widehat{\phi}^*)$. The relative Gottlieb group $G_*^{rel}(A, B; \phi)$ is the image of $H_*(\epsilon_*, \epsilon_*)$. From the tower

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Der}(A, H^*(B); \phi) & \xrightarrow{J} & \mathrm{Rel}(\phi^*) & \xrightarrow{P} & \mathrm{Der}(B, H^*(B); \gamma) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle (\epsilon_*, \epsilon_*)} & & \downarrow{\scriptstyle \epsilon_*} & & \\ 0 & \longrightarrow & \mathrm{Der}(A, \mathbb{Q}; \epsilon) & \xrightarrow{\hat{J}} & \mathrm{Rel}(\widehat{\phi}^*) & \xrightarrow{\hat{P}} & \mathrm{Der}(B, \mathbb{Q}; \epsilon) & \longrightarrow & 0 \end{array}$$

one gets a sequence

$$\cdots \to G_{k+1}(B, H^*(B), \gamma) \to G_k(A, H^*(B), \phi^*) \to G_k^{rel}(A, H^*(B), \phi^*) \to \cdots$$

called G-sequence of ϕ.

Proposition 15. *The G-sequence associated to the inclusion* $\mathrm{aut}_1 \mathbb{C}P^n \to \mathrm{map}(\mathbb{C}P^n, \mathbb{C}P^{n+k}; i_{n,k})$ *is not exact.*

Proof. Clearly $G_*(B, H^*(B); \gamma) = \langle [(y_{2n+1}, 1)] \rangle$ and similarly

$$G_*(A, H^*(B), \phi) = \langle [(x_2, 1)], [(x_{2n+2k+1}, 1)] \rangle.$$

We consider first the case where $k > n$. Then the only non zero differential on $\mathrm{Rel}(\phi^*) = (sL \oplus L', d)$ is given by

$$d(s\delta_2, 0) = (-s\theta_1, 0) + (0, \phi^*(\delta_2)) = (-s\theta_1, 0) + (0, \beta_2).$$

Similarly the only non zero differential on

$$\mathrm{Rel}(\widehat{\phi}^*) = \langle (sy_2^*, 0), (sy_{2n+1}^*, 0), (0, x_2^*), (0, x_{2n+2k+1}^*) \rangle$$

is $d(sy_2^*, 0) = (0, x_2^*)$. We conclude that

$$\begin{aligned} G_*^{rel}(A, H^*(B), \phi) &= \langle [(sy_{2n+1}^*, 0)], (0, x_{2n+2k+1}^*) \rangle \\ &\cong sG_*(\mathbb{C}P^n) \oplus G_*(\mathbb{C}P^{n+k}). \end{aligned}$$

Hence in the G-sequence reduces to fragments

$$0 \to G_{2n+2}^{rel}(A, H^*(B); \phi^*) \xrightarrow{\cong} G_{2n+1}(B, H^*(B); \gamma) \to 0,$$

$$0 \to G_{2n+2k+1}(A, H^*(B); \phi^*) \xrightarrow{\cong} G_{2n+2k+1}^{rel}(A, H^*(B); \phi^*) \to 0$$

and terminates with

$$0 \to G_2(A, H^*(B); \phi^*) \to 0.$$

As $G_2(A, H^*(B); \phi^*) \cong \mathbb{Q}$, we conclude that the last fragment of the G-sequence is not exact.

If $k \leq n$, then $\phi^*(\theta_{2n+1}) = \alpha_{2n+1}$, hence $d(s\theta_{2n+1}, 0) = (0, \alpha_{2n+1})$, therefore $[(sy_{2n+1}^*, 0)] \in H_*(\mathrm{Rel}(\widehat{\phi}^*))$ is not in the image of $H_*(\epsilon_*, \epsilon_*)$. The only change in the G-sequence is the fragment

$$0 \to G_{2n+2}^{rel}(A, H^*(B); \phi^*) \to 0,$$

which in not exact as well, as $G_{2n+2}^{rel}(A, H^*(B)) \cong \mathbb{Q}$. □

References

[1] J. Block and A. Lazarev, *André-Quillen cohomology and rational homotopy of function spaces*, Adv. Math. **193** (2005), no. 1, 18–39. https://doi.org/10.1016/j.aim.2004.04.014

[2] E. H. Brown, Jr., and R. H. Szczarba, *On the rational homotopy type of function spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 12, 4931–4951. https://doi.org/10.1090/S0002-9947-97-01871-0

[3] U. Buijs, Y. Félix, and A. Murillo, *Lie models for the components of sections of a nilpotent fibration*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5601–5614. https://doi.org/10.1090/S0002-9947-09-04870-3

[4] U. Buijs, Y. Félix, and A. Murillo, *L_∞ models of based mapping spaces*, J. Math. Soc. Japan **63** (2011), no. 2, 503–524. http://projecteuclid.org/euclid.jmsj/1303737796

[5] U. Buijs, Y. Félix, and A. Murillo, *L_∞ rational homotopy of mapping spaces*, Rev. Mat. Complut. **26** (2013), no. 2, 573–588. https://doi.org/10.1007/s13163-012-0105-z

[6] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, 205, Springer-Verlag, New York, 2001. https://doi.org/10.1007/978-1-4613-0105-9

[7] J.-B. Gatsinzi, *On the genus of elliptic fibrations*, Proc. Amer. Math. Soc. **132** (2004), no. 2, 597–606. https://doi.org/10.1090/S0002-9939-03-07203-4

[8] J.-B. Gatsinzi and R. Kwashira, *Rational homotopy groups of function spaces*, in Homotopy theory of function spaces and related topics, 105–114, Contemp. Math., 519, Amer. Math. Soc., Providence, RI, 2010. https://doi.org/10.1090/conm/519/10235

[9] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756. https://doi.org/10.2307/2373349

[10] A. Haefliger, *Rational homotopy of the space of sections of a nilpotent bundle*, Trans. Amer. Math. Soc. **273** (1982), no. 2, 609–620. https://doi.org/10.2307/1999931

[11] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. Algebra **23** (1995), no. 6, 2147–2161. https://doi.org/10.1080/00927879508825335

[12] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G-sequences*, J. Pure Appl. Algebra **209** (2007), no. 1, 159–171. https://doi.org/10.1016/j.jpaa.2006.05.018

[13] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map. II. Quillen models and adjoint maps*, J. Pure Appl. Algebra **209** (2007), no. 1, 173–188. https://doi.org/10.1016/j.jpaa.2006.05.019

[14] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press, Inc., Publishers, New York, 1963.

[15] J. M. Møller and M. Raussen, *Rational homotopy of spaces of maps into spheres and complex projective spaces*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 721–732. https://doi.org/10.2307/2000242

[16] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. No. 47 (1977), 269–331 (1978).

[17] M. H. Woo and K. Y. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25** (1988), no. 1, 149–160.

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