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WHEN IS $C(X)$ AN EM-RING?

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Abstract. A commutative ring with unity R is called an EM-ring if for any finitely generated ideal I there exist a in R and a finitely generated ideal J with $\mathrm{Ann}(J) = 0$ and $I = aJ$. In this article it is proved that $C(X)$ is an EM-ring if and only if for each $U \in Coz\,(X)$, and each $g \in C^*\,(U)$ there is $V \in Coz\,(X)$ such that $U \subseteq V$, $\overline{V} = X$, and g is continuously extendable on V. Such a space is called an EM-space. It is shown that EM-spaces include a large class of spaces as F-spaces and cozero complemented spaces. It is proved among other results that X is an EM-space if and only if the Stone-Čech compactification of X is.

1. Introduction

Let X be a topological space, $C(X)$ be the ring of all continuous real valued functions defined on X and $C^*(X)$ be its subring of bounded functions. For each $f \in C(X)$, let $Z(f) = f^{-1}(0)$, $\mathrm{coz}(f) = X - Z(f)$ and $\mathrm{supp}(f) = \overline{\mathrm{coz}(f)}$. Let $Z(X)$ be the set of all zero sets in X and $Coz(X)$ be the set of all cozero sets in X. For any undefined terms, the reader may refer to [9], and for a new survey and results on $C(X)$, see [4].

If X is any topological space, then there is a Tychonoff space Y such that $C(X)$ is isomorphic to $C(Y)$. Thus we will assume that all spaces X are Tychonoff spaces, and so we are able to extend X into the Stone-Čech compactification βX.

A lot of work is done in the literature to characterize algebraic properties of $C(X)$ using the topological properties of X and viceversa. In the following some of these characterizations that will be used in this article.

A space X is called basically disconnected if for each $f \in C(X)$, $\mathrm{supp}(f)$ is open. It is known that X is basically disconnected if and only if $C(X)$ is a PP-ring (every principal ideal is projective), see [5].

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A space X is called an F-space if for each $f \in C(X)$, $\mathrm{coz}(f)$ is C*-embedded in X. It is known that X is an F-space if and only if $C(X)$ is a Bezout ring (every finitely generated ideal is principal) if and only if $C(X)$ is a PF-ring (every principal ideal is flat), see [8, 9].

A space X is called cozero complemented if and only if for each $U \in Coz(X)$ there exists $V \in Coz(X)$ such that $U \cap V = \emptyset$ and $\overline{U \cup V} = X$. It is known that X is cozero complemented if and only if $\mathrm{Min}(C(X))$ (the space of all minimal prime ideals in $C(X)$ with the Zariski topology) is compact, see [12].

Let R be a commutative ring. In [1], the authors introduced the notion of an annihilating content for a polynomial: if $f(x) \in R[x]$ and there exist $a \in R$ and a regular (non-zero divisor) polynomial $g(x) \in R[x]$ such that $f(x) = ag(x)$, then a is called an annihilating content for $f(x)$. Annihilating content simplifies computing the annihilator of a polynomial, which is not always an easy task, also it is used to find the annihilator of a finitely generated ideal. In [2], the authors defined EM-rings as those rings for which any polynomial in $R[x]$ has an annihilating content. Among other things it was shown that a Bezout ring is an EM-ring, the class of EM-rings is closed under localization, direct products, polynomial adjunction and that a Noetherian ring is an EM-ring if and only if each minimal prime ideal is principal. More investigation was done in [3], and it was proved that R is an EM-ring if and only if for each finitely generated ideal I of R, there exist $a \in R$ and a finitely generated ideal J of R such that $I = aJ$ and $Ann(J) = 0$.

In [7], the authors proved that a ring R is a PP-ring if and only if $R(+)R$ is an EM-ring if and only if $R(+)R$ is a generalized morphic ring (for each $f \in R$, $\mathrm{Ann}(f) = \{g \in R : fg = 0\}$ is principal). They also showed in [2] that a Noetherian ring is an EM-ring if and only if it is a generalized morphic ring, however they showed that if $X = \beta\mathbb{N} - \mathbb{N}$, then $C(X)$ is an EM-ring that is not generalized morphic. This motivated us to study EM-rings and generalized morphic rings in $C(X)$.

The purpose of this article is to characterize those topological spaces X for which $C(X)$ is an EM-ring or a generalized morphic ring. Although we don't know yet the exact relation between EM-rings and generalized morphic rings, but we manage to show that if $C(X)$ is generalized morphic, then indeed it is an EM-ring.

In Section 2, we study the annihilating content of a finitely generated ideal in $C(X)$ and simplify some of its computations. It is shown (Theorem 2.3) that the finitely generated ideal $I = (f_1, f_2, \ldots, f_n)$ of $C(X)$ has an annihilating content $h \in C(X)$ if and only if there exist $g_1, \ldots, g_{n+1} \in C(X)$ such that $f_i = hg_i$ for $1 \leq i \leq n$ and $0 = hg_{n+1}$ with $\bigcup_{i=1}^{n+1} \mathrm{supp}(g_i) = X$. Also it is shown (Theorem 2.4) that we can pick the annihilating content h (which is not unique) to be bounded.

In Section 3, we define EM-spaces as follows: X is an EM-space if and only if $C(X)$ is an EM-ring. It is shown that the set of all real numbers with the

Euclidean topology is an EM-space, while the one-point compactification of an uncountable discrete space is not. It is shown (Corollary 3.9) that X is an EM-space if and only if βX is. Theorem 3.11 characterizes Tychonoff EM-spaces: a Tychonoff space X is an EM-space if and only if for each $U \in Coz\,(X)$, and each $g \in C^*\,(U)$ there is $V \in Coz\,(X)$ such that $U \subseteq V$, $\overline{V} = X$, and g is continuously extendable on V.

In Section 4, we relate EM-spaces with other spaces. We observe that the ring $C(X)$ is generalized morphic if and only if X is basically disconnected, and so if $C(X)$ is generalized morphic, then it is an EM-ring. It is shown that EM-spaces include wide range of spaces such as F-spaces and cozero complemented spaces (Theorem 4.1). It is also deduced that a quazi F-space EM-space is an F-space. Finally it is shown that a locally connected EM-space is cozero complemented.

2. Annihilating content

The idea of an annihilating content was first defined in [1] to factor a polynomial into an element in the ring multiplied by a regular polynomial to simplify calculating the zero divisor graph of the polynomial ring $R[x]$. In this section we use topological characterization for the annihilating content in $C(X)$.

Definition 2.1. Let R be a commutative ring and let $f(x) \in R[x]$. If there exist $a \in R$ and a regular polynomial $g(x) \in R[x]$ such that $f(x) = ag(x)$, then a is called an *annihilating content for* $f(x)$.

The idea of annihilating content was more developed in [2] and [3] to be used for finitely generated ideals of R.

Definition 2.2. Let R be a commutative ring with unity, and let I be a finitely generated ideal of R such that there exist $a \in R$ and a finitely generated ideal J with $\mathrm{Ann}(J) = 0$ and $I = aJ$. In this case a is called an *annihilating content for* I.

It was shown in [3] that the annihilating content is not unique, and if a is an annihilating content for $f(x)$ (the finitely generated ideal I), then $\mathrm{Ann}(f(x)) = \mathrm{Ann}(a)$ ($\mathrm{Ann}(I) = \mathrm{Ann}(a)$), also if $aR = bR$, then b is an annihilating content for $f(x)$ (for I), but not conversely.

We now give the analogue definition of an annihilating content in the ring $C(X)$. But we recall first that for $f, h \in C(X)$, $\mathrm{supp}(f) = \mathrm{supp}(h)$ if and only if $\mathrm{Ann}(f) = \mathrm{Ann}(h)$ and that $\mathrm{Ann}(f) = 0$ if and only if $\mathrm{coz}(f)$ is dense in X.

Theorem 2.3. *In a Tychonoff space X, the finitely generated ideal $I = (f_1, f_2, \dots, f_n)$ has an annihilating content $h \in C\,(X)$ if and only if there exist $g_1, g_2, \dots, g_{n+1} \in C(X)$ such that*

$$\begin{aligned} f_i &= hg_i, \qquad i = 1, 2, \dots, n, \\ 0 &= hg_{n+1}, \end{aligned}$$

$$\bigcup_{i=1}^{n+1} \operatorname{supp}(g_i) = X.$$

In this case, $\operatorname{supp}(h) = \bigcup_{i=1}^{n} \operatorname{supp}(f_i)$.

Proof. Assume $I = (f_1, f_2, \ldots, f_n)$ has an annihilating content. Then there exist $h, g_1, g_2, \ldots, g_n, g'_{n+1}, \ldots, g'_m \in C(X)$ such that

$$\begin{aligned} &f_i = hg_i,\ i = 1, 2, \ldots, n, \\ &0 = hg'_i,\ i = n+1, \ldots m, \\ &\operatorname{Ann}(g_1, g_2, \ldots, g_n, g'_{n+1}, \ldots, g'_m) = 0. \end{aligned}$$

Let $g_{n+1} = \sum_{i=n+1}^{m} |g'_i|$. Then it is clear that $hg_{n+1} = 0$, and that $\operatorname{Ann}(g_1, g_2, \ldots, g_n, g_{n+1}) = 0$, and so $\bigcup_{i=1}^{n+1} \operatorname{supp}(g_i) = X$. The converse is straightforward.

Now to show that $\operatorname{supp}(h) = \bigcup_{i=1}^{n} \operatorname{supp}(f_i)$, it is sufficient to show that $\operatorname{Ann}(h) = \operatorname{Ann}(f_1, f_2, \ldots, f_n)$. If $\alpha f_i = 0$ for each i, then $(\alpha h) g_i = 0$ for each i, and so $\alpha h \in \operatorname{Ann}(g_1, g_2, \ldots, g_m) = 0$. Hence the result. □

In the following, we show that the annihilating content for an ideal in $C(X)$, if exists, can be chosen to be bounded.

Theorem 2.4. *In a Tychonoff space X, if the finitely generated ideal $I = (f_1, f_2, \ldots, f_n)$ in $C(X)$ has an annihilating content, then I has a bounded annihilating content.*

Proof. Let $h_0 \in C(X)$ be an annihilating content for $I = (f_1, f_2, \ldots, f_n)$. Then there exist $g_1, g_2, \ldots, g_{n+1} \in C(X)$ such that

$$\begin{aligned} &f_i = h_0 g_i,\ i = 1, 2, \ldots, n, \\ &0 = h_0 g_{n+1}, \\ &\bigcup_{i=1}^{n+1} \operatorname{supp}(g_i) = X. \end{aligned}$$

Let

$$q(x) = \begin{cases} -h_0(x), & x \in h_0^{-1}\left((-\infty, -1]\right), \\ 1, & x \in h_0^{-1}\left(-1, 1\right), \\ h_0(x), & x \in h_0^{-1}([1, \infty)). \end{cases}$$

Then $q \in C(X)$. Since $Z(q) = \emptyset$ and $\left|\frac{h_0}{q}\right| \leq 1$, we have $h = \frac{h_0}{q} \in C^*(X)$. If $g_i^* = g_i q$, then

$$hg_i^* = hg_i q = h_0 g_i = \begin{cases} f_i, & i = 1, 2, \ldots, n, \\ 0, & i = n+1. \end{cases}$$

Actually, for every $i \in \{1, 2, \ldots, n+1\}$, $\operatorname{supp}(g_i^*) = \operatorname{supp}(g_i)$.

Therefore, $\bigcup_{i=1}^{n+1} \operatorname{supp}(g_i^*) = X$, and h is a bounded annihilating content for I. □

3. EM-spaces

In this section, we will give a topological characterization for EM-spaces, give examples and counter examples, and show that X is an EM-space if and only if βX is.

The idea of EM-rings was firstly defined in [2], then further investigations were carried out in [3].

Definition 3.1. A commutative ring R is called an *EM-ring* if every polynomial in $R[x]$ has an annihilating content.

The following theorem can be found in [3], it gives more equivalent conditions to EM-rings and simplifies computations.

Theorem 3.2. *For a commutative ring R, the following are equivalent.*

(1) *R is an EM-ring.*

(2) *For $a+bx \in R[x]$, there exist an element $c \in R$ and a regular polynomial $f_1 \in R[x]$ with $a+bx = cf_1$.*

(3) *For $a, b \in R$, there are an element $c \in R$ and a finitely generated ideal J of R with $(a, b) = cJ$ and $\operatorname{Ann}(J) = 0$.*

Now we define EM-spaces.

Definition 3.3. A Tychonoff spaces X is called an *EM-space* if $C(X)$ is an EM-ring.

In view of Definition 3.3, Theorem 2.3, and Theorem 3.2, we give a simple formula for being an EM-space.

Corollary 3.4. *A Tychonoff spaces X is an EM-space if and only if whenever $f_1, f_2 \in C(X)$, there exist $h, g_1, g_2, g_3 \in C(X)$ such that $f_i = hg_i$ for $i = 1, 2$, $hg_3 = 0$, and $\bigcup_{i=1}^{3} \operatorname{supp}(g_i) = X$.*

Example 3.5. It is proved in [2] that any Bezout ring is an EM-ring, and it is proved in [8] that $C(X)$ is Bezout if and only if X is an F-space. Now we give a direct proof that if X is an F-space, then X is an EM-space. For an ideal $I = (f_1, f_2) \subseteq C(X)$, take $h = |f_1| + |f_2|$, and let $g_i \in C(X)$ such that $g_i|_{\bigcup_{j=1,2} \operatorname{coz}(f_j)} = \frac{f_i}{h}$ for $i = 1, 2$, and $g_3 = 1 - |g_1| - |g_2|$. Then $f_i = hg_i$ for $i = 1, 2$, $hg_3 = 0$ and $\bigcup_{i=1}^{3} \operatorname{supp}(g_i) = X$. This implies that h is an annihilating content for I.

Example 3.6. Let $\mathbb{R}$ be the set of all real numbers with the Euclidean topology, and let $f_1, f_2 \in C(\mathbb{R})$. Define $h = (|f_1| + |f_2|)^{\frac{1}{2}}$. It is clear that $\operatorname{coz}(h) = \operatorname{coz}(f_1) \bigcup \operatorname{coz}(f_2)$. For $i = 1, 2$, define

$$\alpha_i(x) = \begin{cases} \frac{f_i}{h}(x), & x \in \operatorname{coz}(h), \\ 0, & \text{otherwise}. \end{cases}$$

Then $\alpha_i \in C(X)$, and $f_i = h\alpha_i$ for $i = 1, 2$. Now, define $\gamma(x) = \inf\{|x - a| : a \in \operatorname{supp}(h)\}$. It is clear that $h\gamma = 0$, $Z(\gamma) = \operatorname{supp}(h)$, and $\operatorname{coz}(\alpha_1) \cup \operatorname{coz}(\alpha_2) \cup$

$\text{coz}(\gamma) = \text{coz}(f_1) \cup \text{coz}(f_2) \cup \text{coz}(\gamma) = \text{coz}(h) \cup \text{coz}(\gamma)$ is dense in $\mathbb{R}$. Thus, $C(\mathbb{R})$ is an EM-ring.

In general if X is a metric space, then $C(X)$ is an EM-ring.

One can wonder if for any Tychonoff space X, the ring $C(X)$ is an EM-ring, since if $f_1, f_2 \in C(X)$, then using the same technique in the last example one shows that $(f_1, f_2) = h(\alpha_1, \alpha_2)$, but it is not always the case that we can find γ satisfying $\text{supp}(\alpha_1) \cup \text{supp}(\alpha_2) \cup \text{supp}(\gamma) = X$ with $h\gamma = 0$. This will be shown in the following example.

Example 3.7. Let $X = Y \cup \{\infty\}$ be the one-point compactification of an uncountable discrete space. For $i = 1, 2$, take $f_i \in C(X)$ such that $\text{coz}(f_i)$ are infinite disjoint countable sets. Suppose that $h, g_i \in C(X)$ for $i = 1, 2$, and $f_i = hg_i$. Then $\text{coz}(g_1) \cap \text{coz}(f_2) = \text{coz}(g_1) \cap \text{coz}(h) \cap \text{coz}(f_2) = \text{coz}(f_1) \cap \text{coz}(f_2) = \emptyset$. But $\text{coz}(f_2)$ is infinite, so $\infty \notin \text{coz}(g_1)$. Hence, $\text{coz}(g_1)$ is countable being an F_σ-set in X. Similar argument shows that $\text{coz}(g_2)$ is also countable.

Let g be any function in $C(X)$. If $\infty \in \text{coz}(g)$, then $\text{coz}(g) \cap \text{coz}(h) \supseteq \text{coz}(g) \cap \text{coz}(f_1) \neq \emptyset$, because $\text{coz}(f_1)$ is infinite, and thus $gh \neq 0$.

If $\infty \notin \text{coz}(g)$, then $\text{coz}(g)$ is countable, and $\bigcup_{i=1}^{2} \text{supp}(g_i) \cup \text{supp}(g) = \bigcup_{i=1}^{2} \text{coz}(g_i) \cup \text{coz}(g) \cup \{\infty\} \neq X$ since the left hand side is countable, but the right hand side is uncountable. Therefore, the ideal (f_1, f_2) does not have an annihilating content, and so $C(X)$ is not an EM-ring.

Theorem 3.8. *Let X be a Tychonoff space. Then $C^*(X)$ is an EM-ring if and only if $C(X)$ is an EM-ring.*

Proof. Let X be a space such that $C^*(X)$ is an EM-ring. Let $f, g \in C(X)$, and let $f^* = (f \wedge 1) \vee (-1)$, and $g^* = (g \wedge 1) \vee (-1)$. By assumption, there exist $h, \alpha^*, \beta^*, \gamma \in C^*(X)$, such that $f^* = h\alpha^*$, $g^* = h\beta^*$, $0 = h\gamma$, and $\text{supp}(\alpha^*) \cup \text{supp}(\beta^*) \cup \text{supp}(\gamma) = X$. Consider

$$q_1(x) = \begin{cases} \frac{f}{f^*}(x), & x \in f^{-1}((-\infty, -1] \cup [1, \infty)), \\ 1, & \text{elsewhere}, \end{cases}$$

$$q_2(x) = \begin{cases} \frac{g}{g^*}(x), & x \in g^{-1}((-\infty, -1] \cup [1, \infty)), \\ 1, & \text{elsewhere}, \end{cases}$$

and let $\alpha = \alpha^* q_1$, and $\beta = \beta^* q_2$. Then $\alpha h = \alpha^* q_1 h = f^* q_1 = f$, and $\beta h = \beta^* q_2 h = g^* q_2 = g$. Furthermore, $0 = h\gamma$, and

$$\text{supp}(\alpha) \cup \text{supp}(\beta) \cup \text{supp}(\gamma) = \text{supp}(\alpha^*) \cup \text{supp}(\beta^*) \cup \text{supp}(\gamma) = X.$$

This implies that h is an annihilating content for (f, g), and $C(X)$ is an EM-ring.

Conversely, let $C(X)$ be an EM-ring. Let $f, g \in C^*(X) \subseteq C(X)$. Then, by assumption, and by Theorem 2.4, we can find $h_0 \in C^*(X)$, and $\alpha_0, \beta_0, \gamma_0 \in C(X)$ such that $f = \alpha_0 h_0$, $g = \beta_0 h_0$, $0 = \gamma_0 h_0$, and $\text{supp}(\alpha_0) \cup \text{supp}(\beta_0) \cup$

$\operatorname{supp}(\gamma_0) = X$. Evidently, there is a bound $M > 0$, with $|f|, |g|, |h_0| < M$. Take

$$\gamma = (\gamma_0 \wedge M) \vee (-M)$$

and let $\alpha_0^* = (\alpha_0 \wedge M) \vee (-M)$. For $A = \alpha_0^{-1}((-\infty, -M] \cup [M, \infty))$, take

$$q_1(x) = \begin{cases} \frac{\alpha_0}{\alpha_0^*}(x), & x \in A, \\ 1, & \text{elsewhere,} \end{cases}$$

and then let $\beta_1 = \frac{\beta_0}{q_1} \in C(X)$ since $Z(q_1) = \emptyset$, and $\beta = (\beta_1 \wedge M) \vee (-M)$. For $B = \beta_1^{-1}((-\infty, -M] \cup [M, \infty))$, take

$$q_2(x) = \begin{cases} \frac{\beta_1}{\beta}(x), & x \in B, \\ 1, & \text{elsewhere,} \end{cases}$$

and then let $\alpha = \frac{\alpha_0^*}{q_2} \in C(X), Z(q_2) = \emptyset$, and $h = q_1 q_2 h_0$. In fact,

$$|\alpha| = \left|\frac{\alpha_0^*}{q_2}\right| = \begin{cases} \left|\frac{\alpha_0^*\beta}{\beta_1}\right| \leq |\alpha_0^*| \leq M & \text{on } B, \\ |\alpha_0^*| \leq M & \text{on } X - B, \end{cases}$$

$$|h| = |q_1 q_2 h_0| = \begin{cases} |h_0| \leq M & \text{on } X - (A \cup B), \\ \left|\frac{\alpha_0}{\alpha_0^*} h_0\right| = \left|\frac{f}{\alpha_0^*}\right| = \frac{|f|}{M} \leq 1 & \text{on } A - B, \\ \left|\frac{\beta_1 h_0}{\beta}\right| = \left|\frac{\beta_0 h_0}{\beta}\right| = \left|\frac{g}{\beta}\right| = \frac{|g|}{M} \leq 1 & \text{on } B - A, \\ \left|\frac{\alpha_0 \beta_1 h_0}{\alpha_0^* \beta}\right| = \left|\frac{\beta_0 h_0}{\beta}\right| = \frac{|g|}{M} \leq 1 & \text{on } A \cap B. \end{cases}$$

By this we have $|h| \leq \max\{M, 1\}$, $\alpha, \beta, \gamma, h \in C^*(X)$, and having same support as $\alpha_0, \beta_0, \gamma_0, h_0$, respectively, since q_1, q_2 are units.

Additionally,

$$\alpha h = \alpha_0^* q_1 h_0 = \alpha_0 h_0 = f,$$
$$\beta h = \beta q_1 q_2 h_0 = \beta_1 q_1 h_0 = \beta_0 h_0 = g,$$
$$\gamma h = 0,$$
$$\operatorname{supp}(\alpha) \cup \operatorname{supp}(\beta) \cup \operatorname{supp}(\gamma) = X.$$

Consequently, $C^*(X)$ is an EM-ring. □

Since $C^*(X)$ is isomorphic to $C(\beta X)$, see [9], we get the following result.

Corollary 3.9. *Let X be a Tychonoff space. Then $C(X)$ is an EM-ring if and only if $C(\beta X)$ is an EM-ring.*

We now go forward to give a topological characterization of an EM-space, but first we will need the following lemma.

Lemma 3.10. *Let X be a Tychonoff EM-space. Then for every $U \in Coz(X)$ and every $g \in C^*(U)$, there exists a zero set $Z \in Z(X)$ such that $U \subseteq X - Z$, $\operatorname{Int} Z = \emptyset$, and g is continuously extendable on $U \cup (\partial U - Z)$.*

Proof. Let $U = \operatorname{coz}(f)$ be a cozero set where $f \in C^*(X)$, and let $g \in C^*(U)$. Then

$$k = \begin{cases} fg & \text{on } U, \\ 0 & \text{on } X - U, \end{cases}$$

is continuous. As $C(X)$ is an EM-ring, there exist $\alpha, \beta, \gamma, h \in C(X)$ such that $f = \alpha h$, $k = \beta h$, $0 = \gamma h$, and $\operatorname{supp}(\alpha) \cup \operatorname{supp}(\beta) \cup \operatorname{supp}(\gamma) = X$. Let $Z = Z(|\alpha| + |\beta| + |\gamma|)$. Then $U \subseteq X - Z$, and $\operatorname{Int} Z = \emptyset$.

Notice that $U \subseteq \operatorname{coz}(\alpha) \cap \operatorname{coz}(f) \cap \operatorname{coz}(h)$. Thus, on U,

$$\left|\frac{\beta}{\alpha}\right| = \left|\frac{k}{f}\right| = |g| < M$$

for some $M > 0$. Thus for all $x \in U$,

$$(*) \qquad \frac{|\beta(x)| + |\alpha(x)|}{M+1} < |\alpha(x)|.$$

Let $x_0 \in \partial U - Z$. Then $\gamma(x_0) = 0$, since $\overline{U} = \operatorname{supp}(f) \subseteq \operatorname{supp}(h) \subseteq Z(\gamma)$. But $x_0 \notin Z$, thus $x_0 \in \operatorname{coz}(|\alpha| + |\beta| + |\gamma|)$, and so $(|\alpha| + |\beta|)(x_0) > 0$. By continuity of $|\alpha| + |\beta|$, there is a neighborhood U_1 of x_0 such that for each $x \in U_1$,

$$(**) \qquad \frac{3}{4}(|\alpha| + |\beta|)(x_0) < (|\alpha| + |\beta|)(x) < \frac{5}{4}(|\alpha| + |\beta|)(x_0).$$

If $\alpha(x_0) = 0$, then by continuity of α, there is a neighborhood U_2 of x_0 such that for each $x \in U_2$,

$$|\alpha(x)| < \frac{1}{4}\left(\frac{(|\alpha| + |\beta|)(x_0)}{1+M}\right)$$

and by $(*)$, for each $x \in U \cap U_2$,

$$(|\alpha| + |\beta|)(x) < \frac{1}{4}(|\alpha| + |\beta|)(x_0).$$

Clearly, this contradicts $(**)$ as $U \cap U_1 \cap U_2 \neq \emptyset$. Therefore, $\alpha(x_0) \neq 0$. In other words, $\partial U - Z \subseteq \operatorname{coz}(\alpha)$. Evidently,

$$g^* = \begin{cases} g & \text{on } U \\ \frac{\beta}{\alpha} & \text{on } \partial U - Z \end{cases} \in C^*(U \cup (\partial U - Z))$$

is the desired extension. ☐

We now give a topological characterization for EM-spaces.

Theorem 3.11. *Let X be a Tychonoff space. Then X is an EM-space if and only if for each $U \in Coz(X)$, and each $g \in C^*(U)$ there is $V \in Coz(X)$ such that $U \subseteq V$, $\overline{V} = X$, and g is continuously extendable on V.*

Proof. ($\Rightarrow$) Assume X is an EM-space. Let $U \in Coz(X)$ and $f \in C^*(X)$ such that $U = \operatorname{coz}(f)$. Let $g \in C^*(U)$, and then

$$k = \begin{cases} fg & \text{on } U \\ 0 & \text{on } X - U \end{cases} \in C^*(X).$$

Let f^{β},$k^{\beta} \in C(\beta X)$ be the extensions of f and k in βX, respectively. If $\tilde{U} = \text{coz}\left(f^{\beta}\right)$, then $\tilde{g} = \frac{k^{\beta}}{f^{\beta}} \in C^{*}\left(\tilde{U}\right)$, and $\tilde{g}|_{U} = g$. On the other hand, $C(\beta X)$ is an EM-ring by Corollary 3.9. Hence, by Lemma 3.10, there are $\tilde{\sigma} \in C(\beta X)$ with $\tilde{U} \subseteq \beta X - Z(\tilde{\sigma})$, $\text{Int}_{\beta X} Z(\tilde{\sigma}) = \emptyset$, and $\tilde{g}_1 \in C^{*}\left(\tilde{U} \cup \left(\partial\tilde{U} - Z(\tilde{\sigma})\right)\right)$ such that $\tilde{g}_1$ is an extension of $\tilde{g}$. Now,

$$\tilde{g}_{\sigma} = \begin{cases} \tilde{g}_1\tilde{\sigma} & \text{on } \tilde{U} \cup \left(\partial\tilde{U} - Z(\tilde{\sigma})\right) \\ 0 & \text{on } \partial\tilde{U} \cap Z(\tilde{\sigma}) \end{cases} \in C^{*}\left(\overline{\tilde{U}}\right).$$

By Teitze-Urysohn Theorem, $\tilde{g}_{\sigma}$ has an extension on βX, say $\tilde{g}_{\tilde{\sigma}}$. Moreover, on U, $\tilde{g}_{\tilde{\sigma}} = g\sigma$ for $\sigma = \tilde{\sigma}|_{X}$. Take $V = \text{coz}(\sigma)$. Then $\overline{V} = X$ (since $\text{Int}_{\beta X} Z(\tilde{\sigma}) = \emptyset$), and

$$g^{*} = \frac{\tilde{g}_{\tilde{\sigma}}|_{V}}{\sigma} \in C^{*}(V)$$

is the desired extension of g.

($\Leftarrow$) Let X be the prescribed space. Let $f_1, f_2 \in C(X)$, and $U = \text{coz}(f_1) \cup \text{coz}(f_2)$. Then $\frac{f_1}{|f_1|+|f_2|}, \frac{f_2}{|f_1|+|f_2|} \in C^{*}(U)$. So by assumption, there exist dense $V_1, V_2 \in Coz(X)$, $f_1^{*} \in C^{*}(V_1)$, and $f_2^{*} \in C^{*}(V_2)$, such that f_1^{*}, f_2^{*} are extensions of $\frac{f_1}{|f_1|+|f_2|}, \frac{f_2}{|f_1|+|f_2|}$, respectively. Consider $V = V_1 \cap V_2$, and let $\sigma_1 \in C^{*}(X)$ be such that $V = \text{coz}(\sigma_1)$. Let $\sigma = |\sigma_1| + (|f_1| + |f_2|)^{\frac{1}{2}}$. Then $\text{coz}(\sigma) = V$. Define

$$h = \begin{cases} \frac{|f_1|+|f_2|}{\sigma} & \text{on } V, \\ 0 & \text{on } X - V. \end{cases}$$

For continuity of h, it is sufficient to show that it is continuous on ∂V. Let $x_0 \in \partial V$, and $\epsilon > 0$. Then $h(x_0) = 0$, and there is a neighborhood U_0 of x_0, such that for each $x \in U_0$, $(|f_1| + |f_2|)^{\frac{1}{2}}(x) < (|f_1| + |f_2|)^{\frac{1}{2}}(x_0) + \epsilon = \epsilon$. In fact, for each $x \in U_0 \cap V \cap U$,

$$|h(x)| = \frac{|f_1| + |f_2|}{\sigma}(x) < \frac{|f_1| + |f_2|}{(|f_1| + |f_2|)^{\frac{1}{2}}}(x) = (|f_1| + |f_2|)^{\frac{1}{2}}(x) < \epsilon.$$

So, $h \in C(X)$.

Now let

$$\alpha = \begin{cases} f_1^{*}\sigma & \text{on } V, \\ 0 & \text{on } X - V, \end{cases} \quad \beta = \begin{cases} f_2^{*}\sigma & \text{on } V, \\ 0 & \text{on } X - V. \end{cases}$$

Then $\alpha, \beta \in C(X)$. By letting $\gamma = \sigma - |\alpha| - |\beta|$, we get $X = \text{supp}(\sigma) = \text{supp}(\gamma + |\alpha| + |\beta|) \subseteq \text{supp}(|\gamma| + |\alpha| + |\beta|) = \text{supp}(\alpha) \cup \text{supp}(\beta) \cup \text{supp}(\gamma) \subseteq X$.

Since $f_1 = \alpha h$, $f_2 = \beta h$, and $0 = \gamma h$, we conclude that $C(X)$ is an EM-ring. □

Corollary 3.12. *For a Tychonoff space X, the following statements are equivalent*:

(1) $C(X)$ *is an EM-ring*;
(2) $C^*(X)$ *is an EM-ring*;
(3) X *is an EM-space*;
(4) βX *is an EM-space*;
(5) *For each* $f, g \in C(X)$*, there exist* $h, \alpha, \beta, \gamma \in C(X)$ *such that* $f = h\alpha, g = h\beta, 0 = h\gamma$ *and* $\mathrm{Ann}(\alpha, \beta, \gamma) = 0$;
(6) *For each* $U \in Coz\,(X)$*, and each* $g \in C^*\,(U)$ *there is* $V \in Coz\,(X)$ *such that* $U \subseteq V$*,* $\overline{V} = X$*, and* g *is continuously extendable on* V*.*

Someone may wonder whether being an EM-space is preserved by continuous functions. Actually this is not true. The following is an example of a continuous function that maps an EM-space to a space that is not an EM-space.

Example 3.13. Let W^* be the space of ordinals that are less than or equal to ω_1, the first uncountable ordinal. It was proved in [12] that every ordinal space is cozero complemented, and it is shown in Theorem 4.1 in this paper that a cozero complemented space is an EM-space.

Let S be the subspace of W^* that results after removing all limit ordinals, and let $Y = S \cup \{\infty\}$ be the one point compactification of S. It is shown in Example 3.7 that Y is not an EM-space.

Consider the function $f : W^* \longrightarrow Y$ defined by

$$f\,(x) = \begin{cases} x & x \text{ is a non-limit ordinal,} \\ \infty & x \text{ is a limit ordinal.} \end{cases}$$

Let V be open in Y. If $\infty \notin V$, then $f^{-1}\,(V) = V$, and V is a union of isolated points thus open. If $\infty \in V$, then V is cofinite, and again $f^{-1}\,(V)$ is open. So, f is continuous.

4. Relation with other spaces

In this section, we relate EM-spaces to some other well known spaces, and show that EM-spaces include a wide class of spaces.

Concerning the relation between EM-rings and generalized morphic rings, which was the motivation to start this article, using the result in [6] that any countably generated z-ideal is generated by an idempotent, we get that for a Tychonoff space X, the ring $C(X)$ is generalized morphic if and only if X is basically disconnected if and only if $C(X)$ is a PP-ring. For more equivalent conditions, see [14]. Note that this result is not necessarily true outside $C(X)$, since there are commutative rings that are generalized morphic but not PP-rings. Thus, if $C(X)$ is a generalized morphic ring, then it is an EM-ring, since any PP-ring is an EM-ring, see [2]. It is still an open question to characterize the relation between EM-rings and generalized morphic rings outside Noetherian rings and $C(X)$.

It was shown in Example 3.5 that any F-space is an EM-space, and now using Corollary 3.12, we get an even clearer proof. Using the technique in Example 3.6, one deduce that any metric space is an EM-space, or more generally, if for each $f \in C(X)$, there exists $g \in C(X)$ with $\operatorname{supp}(f) = Z(g)$, then X is an EM-space.

Theorem 4.1. *A cozero complemented space is an EM-space.*

Proof. Let X be a cozero complemented space. Let $U \in Coz\,(X)$. Then there exists $U' \in Coz\,(X)$ such that $U \cap U' = \emptyset$ and $\overline{U \cup U'} = X$. Let $g \in C^*\,(U)$. Then $g^* = \begin{cases} g & \text{on } U \\ 1 & \text{on } U' \end{cases} \in C^*\,(U \cup U')$ is the desired extension. □

It is well known that a Tychonoff space X is basically disconnected if and only if X is a cozero complemented F-space. So, the set $\mathbb{R}$ of real numbers with Euclidean topology is an EM-space which is not an F-space, while $\beta\mathbb{N} - \mathbb{N}$ is an EM-space that is not cozero complemented, since both spaces are not basically disconnected. One may ask if there is an EM-space that is not an F-space nor cozero complemented! The answer is yes; let X_1 be a connected F-space, and X_2 be a connected cozero complemented space. Then the free union space $X = X_1 + X_2$ is an EM-space that is neither an F-space nor a cozero complemented space.

Recalling that a space X is a quasi F-space if every dense cozero set of X is C*-embedded, it is directly deduced that a space X is an F-space if and only if it is an EM-space and a quasi F-space. Quasi F-spaces have been studied in number of articles including [11].

Recall that a space X is called an almost P-space if every G_δ-set has dense interior, see [13]. An almost P-space which is an EM-space is an F-space, since an almost P-space is a quasi F-space.

Now we give an extra condition on an EM-space to get a cozero complemented space.

Theorem 4.2. *A locally connected EM-space is cozero complemented.*

Proof. Let X be an EM-space that is locally connected, $U \in Coz\,(X)$, $f \in C^*\,(X)$ such that $f \geqslant 0$, and $U = \operatorname{coz}(f)$. Consider $g = \cos\left(\frac{1}{f}\right) + 2 \in C^*\,(U)$. Then, by Theorem 3.11, there exists a dense cozero set V that contains U, and g is continuously extendable on V. We need to show that $\partial U \cap V = \emptyset$.

Let $b \in \partial U$, and let W be any neighborhood of b. Since X is locally connected, W can be considered to be connected. Then for some $a > 0$, $f\,(W \cap U) = (0, a)$. Moreover, for some $n \in \mathbb{N}$, $\frac{1}{2n\pi} < a$. Therefore,

$$\left[\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi}\right] \subseteq f\,(W \cap U)\,.$$

So,

$$(***) \qquad [1, 3] \subseteq g\,(W \cap U)\,.$$

If $b \in V$, then g has a continuous extension at b, and this contradicts $(***)$. This implies that $\partial U \cap V = \emptyset$.

Assume that $l \in C^*(X)$, with $V = \operatorname{coz}(l)$. Then $l(\partial U) = 0$. Furthermore, $l^* = \begin{cases} 0 & \text{on } \overline{U} \\ l & \text{elsewhere} \end{cases} \in C^*(X)$.

Eventually, $U \cap \operatorname{coz}(l^*) = \emptyset$, and $\overline{U \cup \operatorname{coz}(l^*)} = \overline{V} = X$. □

Corollary 4.3. *A locally connected space is*:

(1) *an EM-space if and only if it is cozero complemented.*

(2) *an F-space if and only if it is basically disconnected.*

The following diagram illustrates the relations just obtained, where all the implications are strict, see also [10].

$$\begin{array}{ccccc} & & \text{Basically disconnected} & & \text{Metric space} \\ & \swarrow & & \searrow & \downarrow \\ \text{F-space} & & & & \forall f, \exists g : \operatorname{supp}(f) = Z(g) \\ & \searrow & & & \downarrow \\ & & \text{EM-space} & \longleftarrow & \text{Cozero complemented} \end{array}$$

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