Bull. Korean Math. Soc. **59** (2022), No. 1, pp. 167–177
https://doi.org/10.4134/BKMS.b210171
pISSN: 1015-8634 / eISSN: 2234-3016

DOUBLE VERTEX-EDGE DOMINATION IN TREES

Xue-Gang Chen and Moo Young Sohn

Abstract. A vertex v of a graph $G = (V, E)$ is said to ve-dominate every edge incident to v, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is called a double vertex-edge dominating set if every edge of E is ve-dominated by at least two vertices of S. The minimum cardinality of a double vertex-edge dominating set of G is the double vertex-edge domination number $\gamma_{dve}(G)$. In this paper, we provide an upper bound on the double vertex-edge domination number of trees in terms of the order n, the number of leaves and support vertices, and we characterize the trees attaining the upper bound. Finally, we design a polynomial time algorithm for computing the value of $\gamma_{dve}(T)$ for any trees. This gives an answer of an open problem posed in [4].

1. Introduction

Let G be a simple and undirected graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Let $n(G) = |V(G)|$. By an open neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood, $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. If the graph G is clear from context, we simply write $N(v)$, $N[v]$ and $d(v)$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

For any $e = vu \in E(G)$, we define $V(e) = \{v, u\}$. Let $N(e) = \{f \in E(G) : f$ is adjacent to $e\}$, $N[e] = N(e) \cup \{e\}$, $N_2(e) = \{f \in E(G) : V(e) \cap V(f) = \emptyset\}$ and one vertex in $V(f)$ is adjacent to one vertex in $V(e)$ and $N_{\leq 2}(e) = N[e] \cup N_2(e)$.

Let $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $G[S]$. The *diameter* of G, denoted by $diam(G)$, is the maximum distance among pairs of vertices in G.

A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. An edge incident with a leaf is called a pendant edge. A support vertex is said to be strong (weak, respectively) if it is adjacent to at least two leaves

Received March 3, 2021; Revised August 31, 2021; Accepted October 14, 2021.
2010 *Mathematics Subject Classification.* 05C69, 05C35.
Key words and phrases. Double vertex-edge dominating set, trees.

(exactly one leaf, respectively). A star of order $n \geq 2$, denoted by $K_{1,n-1}$, is a tree with $n-1$ leaves. A tree with $diam(T) = 3$ is called a double star.

A vertex v *ve*-dominates every edge uv incident to v, as well as every edge adjacent to these incident edges. That is, a vertex v *ve*-dominates every edge incident to a vertex in $N[v]$.

A set $S \subseteq V$ in a graph G is called a *vertex-edge dominating set* if every edge in $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The *vertex-edge domination number* $\gamma_{ve}(G)$ is defined to be the minimum cardinality of a vertex-edge dominating set in G. The concept of vertex-edge domination was introduced by Peters [7] in 1986 and studied further in [1, 5, 6].

A set $S \subseteq V$ in a graph G is said to be a *double vertex-edge dominating set* (or simply, a double *ve*-dominating set) if every edge in E is *ve*-dominated by at least two vertices of S.

The *double vertex-edge domination number* $\gamma_{dve}(G)$ is defined to be the minimum cardinality of a double *ve*-dominating set in G. The concept of double vertex-edge domination was introduced by Balakrishna et al. [4].

Balakrishna et al. showed that the problem of computing the double vertex-edge domination number is in the NP-complete class even when restricted to bipartite graphs. They provided a lower bound on the double vertex-edge domination number of trees in terms of the order n, the number of leaves and support vertices.

Proposition 1 ([4]). *If T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{dve}(T) \geq \frac{n-l-s+4}{2}$.*

Furthermore, they gave the following open problem.

Problem 1 ([4]). *Design an algorithm for computing the value of $\gamma_{dve}(T)$ for any tree T.*

In this paper, we provide an upper bound on the double vertex-edge domination number of trees in terms of the order n, the number of leaves and support vertices, and we characterize the trees attaining the upper bound. Finally, we design a polynomial time algorithm for computing the value of $\gamma_{dve}(T)$ for any trees. This gives an answer of Problem 1.

2. Upper bound

Let $k \geq 2$ be an integer and H_k be the graph obtained from the star $K_{1,k}$ by subdividing every edge twice. The center of the star $K_{1,k}$ is called the center of H_k. Let H'_k be the tree obtained from H_k by attaching a new vertex x and joining x to the center of H_k.

For any tree T, let $L(T)$ and $S(T)$ denote the set of leaves and support vertices, respectively. The following lemmas are easy to prove. We omit their proves.

Lemma 1. *Let T be a tree with order at least three and $u \in S(T)$. If T_1 is the tree obtained from T by adding a new vertex v and joining v to u, then $\gamma_{dve}(T_1) = \gamma_{dve}(T)$.*

Lemma 2. *Let T be a tree and $u \in V(T)$ such that ux_1x_2 is a path in T in which $d_T(u) \geq 2$, $d_T(x_1) = 2$ and $d_T(x_2) = 1$. If T_1 is the tree obtained from T by adding a new vertex v and joining v to u, then $\gamma_{dve}(T_1) = \gamma_{dve}(T)$.*

Lemma 3. *Let T be a tree and $u \in V(T)$ such that ux_1x_2 is a path in T in which $d_T(u) \geq 2$, $d_T(x_1) = 2$ and $d_T(x_2) = 1$. If T_1 is the tree obtained from T by attaching a path $P_2 = y_1y_2$ and joining y_1 to u, then $\gamma_{dve}(T_1) = \gamma_{dve}(T)+1$.*

Lemma 4. *Let T be a tree and $u \in V(T)$ such that ux_1x_2 is a path in T in which $d_T(u) \geq 2$, $d_T(x_1) = 2$ and $d_T(x_2) = 1$. If T_1 is the tree obtained from T by adding a path $P_3 = y_1y_2y_3$ and joining y_1 to u, then $\gamma_{dve}(T_1) = \gamma_{dve}(T)+2$.*

Lemma 5. *Let T be a tree and $u \in V(T)$ such that $ux_1 \in E(T)$, $d_T(u) \geq 2$, $d_T(x_1) = 1$ and u is a weak support vertex. If T_1 is the tree obtained from T by adding a H_k and joining x_1 to the center of H_k, then $\gamma_{dve}(T_1) = \gamma_{dve}(T)+2k$.*

Lemma 6. *Let T be a tree and $u \in V(T)$ such that $ux_1 \in E(T)$, $d_T(u) \geq 2$, $d_T(x_1) = 1$ and u is a weak support vertex. If T_1 is the tree obtained from T by adding a path $P_4 = y_1y_2y_3y_4$ and joining y_1 to x_1, then $\gamma_{dve}(T_1) = \gamma_{dve}(T)+2$.*

In what follows, we provide an upper bound of the double vertex-edge domination number for trees in terms of the order n, number of leaves l and support vertices s. In order to characterize the trees attaining this bound, we introduce a family Γ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_6\}$. If $i \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations:

- Operation τ_1: Attach a vertex and join it to any support vertex of T_i.
- Operation τ_2: Suppose that ux_1x_2 is a path in T_i in which $d_{T_i}(u) \geq 2$, $d_{T_i}(x_1) = 2$ and $d_{T_i}(x_2) = 1$. Attach a path P_3 and join one of its leaves to u.
- Operation τ_3: Suppose that $ux_1 \in E(T_i)$ in T_i in which $d_{T_i}(u) \geq 2$, $d_{T_i}(x_1) = 1$ and u is a weak support vertex. Attach a H_k and join x_1 to the center of H_k.
- Operation τ_4: Suppose that $ux_1 \in E(T_i)$ in T_i in which $d_{T_i}(u) \geq 2$, $d_{T_i}(x_1) = 1$ and u is a weak support vertex. Attach a P_4 and join x_1 to one leaf of P_4.

Theorem 1. *If $T \in \Gamma$, then $\gamma_{dve}(T) = \frac{n-l}{2} + s$.*

Proof. We use an induction on the number k of operations performed to construct the tree T. If $T = P_6$, then $\gamma_{dve}(T) = 4 = \frac{n-l}{2} + s$. Suppose the property is true for all trees of Γ constructed with $k-1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, $T' = T_{k-1}$, and assume that T' has order n' with l' leaves and s' support vertices. We will discuss it from the following cases.

Case 1. T is obtained from T' by Operation τ_1. Clearly, $n = n' + 1$, $s = s'$ and $l = l' + 1$. By Lemma 1, $\gamma_{dve}(T) = \gamma_{dve}(T')$. By induction on T', we obtain that $\gamma_{dve}(T) = \gamma_{dve}(T') = \frac{n'-l'}{2} + s' = \frac{n-l}{2} + s$.

Case 2. T is obtained from T' by Operation τ_2. Clearly, $n = n' + 3$, $s = s' + 1$ and $l = l' + 1$. By Lemma 4, $\gamma_{dve}(T) = \gamma_{dve}(T') + 2$. By induction on T', we obtain that $\gamma_{dve}(T) = \gamma_{dve}(T') + 2 = \frac{n'-l'}{2} + s' + 2 = \frac{n-l}{2} + s$.

Case 3. T is obtained from T' by Operation τ_3. Clearly, $n = n' + 3k + 1$, $s = s' + k - 1$ and $l = l' + k - 1$. By Lemma 5, $\gamma_{dve}(T) = \gamma_{dve}(T') + 2k$. By induction on T', we obtain that $\gamma_{dve}(T) = \gamma_{dve}(T') + 2k = \frac{n'-l'}{2} + s' + 2k = \frac{n-l}{2} + s$.

Case 4. T is obtained from T' by Operation τ_4. Clearly, $n = n' + 4$, $s = s'$ and $l = l'$. By Lemma 6, $\gamma_{dve}(T) = \gamma_{dve}(T') + 2$. By induction on T', we obtain that $\gamma_{dve}(T) = \gamma_{dve}(T') + 2 = \frac{n'-l'}{2} + s' + 2 = \frac{n-l}{2} + s$. □

We now are ready to establish our upper bound.

Theorem 2. *Let T be a tree of order $n \geq 4$ with l leaves and s support vertices. If T is not a star, then $\gamma_{dve}(T) \leq \frac{n-l}{2} + s$ with equality if and only if $T \in \Gamma$.*

Proof. If $T \in \Gamma$, then by Theorem 1, $\gamma_{dve}(T) = \frac{n-l}{2} + s$. To prove that if T is a tree of order $n \geq 4$ with l leaves and s support vertices and T is not a star, then $\gamma_{dve}(T) \leq \frac{n-l}{2} + s$ with equality only if $T \in \Gamma$, we proceed by induction on the order n. Since $n \geq 4$ and T is not a star, $diam(T) \geq 3$. If $diam(T) = 3$, then T is a double star and $\gamma_{dve}(T) = 2 < \frac{n-l}{2} + s$. So assume that $diam(T) \geq 4$.

Assume that every tree T' of order $4 \leq n' < n$ and with l' leaves and s' support vertices satisfies $\gamma_{dve}(T') \leq \frac{n'-l'}{2} + s'$ with equality only if $T' \in \Gamma$. Let T be a tree of order n and with l leaves and s support vertices and T is not a star.

If some support vertex of T, say u, is strong, then let T' be the tree obtained from T by removing a leaf adjacent to u. By Lemma 1, $\gamma_{dve}(T) = \gamma_{dve}(T')$. Clearly, $n' = n - 1$, $l' = l - 1$ and $s' = s$. Applying the induction hypothesis to T', $\gamma_{dve}(T) = \gamma_{dve}(T') \leq \frac{n'-l'}{2} + s' = \frac{n-l}{2} + s$. Further, if $\gamma_{dve}(T) = \frac{n-l}{2} + s$, then $\gamma_{dve}(T') = \frac{n'-l'}{2} + s'$ and $T' \in \Gamma$. Therefore, T is obtained from T' by using Operation τ_1 and $T \in \Gamma$. Henceforth, we can assume that every support vertex of T is weak.

Choose a vertex r as a root of T such that r is a leaf of path of maximum length in T. Let t be a leaf at maximum distance from r and v, u, w, z be the parents of t, v, u, w, respectively, in the rooted tree. By T_x, we denote the sub-tree induced by a vertex x and its descendants in the rooted tree T. Since T has no strong support vertex, $d_T(v) = 2$.

Suppose that $d_T(u) \geq 3$. If u is a support vertex, then let T' be the tree obtained from T by removing the leaf neighbor of u. By Lemma 2, $\gamma_{dve}(T) = \gamma_{dve}(T')$. Clearly, $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. By induction on T', $\gamma_{dve}(T) = \gamma_{dve}(T') \leq \frac{n'-l'}{2} + s' < \frac{n-l}{2} + s$. Now, if $d_T(u) \geq 3$, then every child of u is a support vertex. Let $T' = T - T_v$. By Lemma 3, $\gamma_{dve}(T) = \gamma_{dve}(T')+1$. Clearly, $n' = n-2$, $l' = l-1$ and $s' = s-1$. By induction on T', $\gamma_{dve}(T) = \gamma_{dve}(T')+1 \leq \frac{n'-l'}{2} + s' + 1 < \frac{n-l}{2} + s$. Therefore, in what follows, we may assume that $d_T(u) = 2$.

If $diam(T) = 4$, then $T = P_5$ and $\gamma_{dve}(T) = 3 < \frac{n-l}{2} + s$. If $diam(T) = 5$, then by exchanging the root of T to t, we have $T = P_6$ and $\gamma_{dve}(T) = 4 = \frac{n-l}{2} + s$. So, $P_6 \in \Gamma$. From now on, we may assume that T has diameter at least six.

If w has a child which is a support vertex, then $d_T(w) \geq 3$. Let $T' = T - T_u$. By Lemma 4, $\gamma_{dve}(T) = \gamma_{dve}(T') + 2$. Clearly, $n' = n-3$, $l' = l-1$ and $s' = s-1$. Applying the induction hypothesis to T', $\gamma_{dve}(T) = \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 = \frac{n-l}{2} + s$. Further, if $\gamma_{dve}(T) = \frac{n-l}{2} + s$, then $\gamma_{dve}(T') = \frac{n'-l'}{2} + s'$ and $T' \in \Gamma$. Therefore, T is obtained from T' by using Operation τ_2 and $T \in \Gamma$. Hence, we assume that w has no child which is a support vertex.

Let y be the parent of vertex z. In the following, we will discuss it from the following cases.

Case 1. $w \in S(T)$.

Case 1.1. $d_T(w) = d \geq 4$. Then $d-2 \geq 2$ and $T_w = H'_{d-2}$. If $d_T(z) \geq 3$, then let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + \gamma_{dve}(H'_{d-2}) = \gamma_{dve}(T') + 2(d-2)$. Clearly, $n' = n - 3(d-2) - 2$, $l' = l - (d-1)$ and $s' = s - (d-1)$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-2) \leq \frac{n'-l'}{2} + s' + 2(d-2) < \frac{n-l}{2} + s$.

Suppose that $d_T(z) = 2$. If $diam(T) = 6$, then $T = H'_{d-1}$. So $\gamma_{dve}(T) = 2(d-1) < \frac{n-l}{2} + s$. If $diam(T) \geq 7$, then let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-2)$. Clearly, $n' = n - 3(d-2) - 2$, $l' = l - (d-2)$ and $s' \leq s - (d-2)$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-2) \leq \frac{n'-l'}{2} + s' + 2(d-2) < \frac{n-l}{2} + s$.

Case 1.2. $d_T(w) = 3$. Then $T_w = P_5$. If $d_T(z) \geq 3$, then let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 3$. Clearly, $n' = n-5$, $l' = l-2$ and $s' = s-2$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 3 \leq \frac{n'-l'}{2} + s' + 3 < \frac{n-l}{2} + s$.

If $d_T(z) = 2$, then let $T' = T - (T_w \setminus \{w\})$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. Clearly, $n' = n-4$, $l' = l-1$ and $s' = s-1$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 < \frac{n-l}{2} + s$.

Case 2. $w \notin S(T)$.

Case 2.1. $d_T(w) = d \geq 3$. Then $d-1 \geq 2$ and $T_w = H_{d-1}$.

If $d_T(z) \geq 3$, then let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-1)$. Clearly, $n' = n - 3(d-1) - 1$, $l' = l - (d-1)$ and $s' = s - (d-1)$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-1) \leq \frac{n'-l'}{2} + s' + 2(d-1) < \frac{n-l}{2} + s$.

Suppose that $d_T(z) = 2$. If $diam(T) = 6$, then $T = H_d$. So $\gamma_{dve}(T) = 2(d-1) < \frac{n-l}{2} + s$. Hence, we can assume that $diam(T) \geq 7$.

If $y \in S(T)$, then let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-1)$. Clearly, $n' = n - 3(d-1) - 1$, $l' = l - (d-2)$ and $s' = s - (d-1)$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-1) \leq \frac{n'-l'}{2} + s' + 2(d-1) < \frac{n-l}{2} + s$.

If $y \notin S(T)$, then let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-1)$. Clearly, $n' = n - 3(d-1) - 1$, $l' = l - (d-2)$ and $s' = s - (d-2)$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2(d-1) \leq \frac{n'-l'}{2} + s' + 2(d-1) \leq \frac{n-l}{2} + s$. Further, if $\gamma_{dve}(T) = \frac{n-l}{2} + s$, then $\gamma_{dve}(T') = \frac{n'-l'}{2} + s'$ and $T' \in \Gamma$. Therefore, T is obtained from T' by using Operation τ_3 and $T \in \Gamma$.

Case 2.2. $d_T(w) = 2$. If $diam(T) = 6, 7$, then by exchanging the root of T to t, we have $T = P_7$ and $T = P_8$, respectively. It is obvious that the result holds. So we can assume that $diam(T) \geq 8$. If $d_T(z) \geq 3$, then let $T' = T - T_w$. Clearly, $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$.

Suppose that there exists a leaf $y_4 \in L(T') \setminus \{r\}$ such that $d_{T'}(z, y_4) = 4$. Assume that $zy_1y_2y_3y_4$ is the path in T'. Since $d_T(r, y_4) = diam(T)$, we can assume that $d_{T'}(y_2) = d_{T'}(y_3) = 2$. Then there exists a γ_{dve}-set D' of T' such that $y_2, y_3 \in D'$. Since edge zy_1 is ve-dominated by at least two vertex, $N_{T'}[z] \cap D' \neq \emptyset$. Say $b \in N_{T'}[z] \cap D'$. Then edge zw is ve-dominated by b. So $D' \cup \{u, v\}$ is a double vertex-edge dominating set of T. Hence, $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 < \frac{n-l}{2} + s$. Hence we can assume that every leaf in $L(T') \setminus \{r\}$ has distance at most 3 from z.

Suppose that there exists a leaf $y_3 \in L(T')$ such that $d_{T'}(z, y_3) = 3$. Say $zy_1y_2y_3$ being the path in T'. Since $d_{T'}(y_2) = 2$, there exists a γ_{dve}-set D' of T' such that $y_1, y_2 \in D'$. Then edge zw is ve-dominated by y_1. So $D' \cup \{u, v\}$ is a double vertex-edge dominating set of T. Hence, $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 < \frac{n-l}{2} + s$.

Hence we can assume that every leaf in $L(T') \setminus \{r\}$ has distance at most 2 from z. So z is a support vertex or is adjacent to a support vertex in T'. Then there exists a γ_{dve}-set D' of T' such that $z \in D'$. Then edge zw is ve-dominated by z. So $D' \cup \{u, v\}$ is a double vertex-edge dominating set of T. Hence, $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 < \frac{n-l}{2} + s$.

Suppose that $d_T(z) = 2$ and $d_T(y) = 2$. Let $T' = T - T_w$. Then $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. Clearly, $n' = n - 4$, $l' = l$ and $s' = s$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 \leq \frac{n-l}{2} + s$. Further, if $\gamma_{dve}(T) = \frac{n-l}{2} + s$, then $\gamma_{dve}(T') = \frac{n'-l'}{2} + s'$ and $T' \in \Gamma$. Therefore, T is obtained from T' by using Operation τ_4 and $T \in \Gamma$.

Suppose that $d_T(z) = 2$ and $d_T(y) \geq 3$. Let $T' = T - T_w$. It is obvious that $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2$. If y is a support vertex in T, then $n' = n - 4$, $l' = l$ and $s' = s - 1$. Applying the induction hypothesis to T', $\gamma_{dve}(T) \leq \gamma_{dve}(T') + 2 \leq$

$\frac{n'-l'}{2} + s' + 2 < \frac{n-l}{2} + s$. If y is not a support vertex in T, then $n' = n - 4$, $l' = l$ and $s' = s$. By Lemma 6, $\gamma_{dve}(T) = \gamma_{dve}(T')+2$. Applying the induction hypothesis to T', $\gamma_{dve}(T) = \gamma_{dve}(T') + 2 \leq \frac{n'-l'}{2} + s' + 2 \leq \frac{n-l}{2} + s$. Further, if $\gamma_{dve}(T) = \frac{n-l}{2} + s$, then $\gamma_{dve}(T') = \frac{n'-l'}{2} + s'$ and $T' \in \Gamma$. Therefore, T is obtained from T' by using Operation τ_4 and $T \in \Gamma$. □

3. Algorithms

Now, we work on an algorithm for finding a minimum double *ve*-dominating set of a tree. For technical reasons, we actually consider a slightly more general problem, which can be formulated as follows. Let the edge set of a tree $T = (V, E)$ be partitioned into three subsets, $E = W \cup B \cup Y$, each consisting of edges labeled W, B, Y, respectively. Let the vertex set of the tree T be partitioned into two subsets, $V = F \cup R$, each consisting of vertices labeled F and R, respectively.

For simplicity, the terms F, R, W, B, Y represent sets and labels interchangeably. A one-two *ve*-dominating set of T is a set D which satisfies the following three conditions:

(1) $R \subseteq D$;

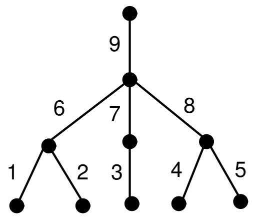
(2) For any edge $e \in Y$, e is *ve*-dominated by at least one vertex in $D \setminus R$.

(3) For any edge $e \in W$, e is *ve*-dominated by at least two vertices in $D \setminus R$.

The one-two *ve*-domination number $\gamma_{12}(T)$ is the minimum cardinality among all one-two *ve*-dominating sets of T. A one-two *ve*-dominating set of T with cardinality $\gamma_{12}(T)$ is also called a γ_{12}-set.

Note that the double vertex-edge domination problem is just the one-two *ve*-domination problem with $F = V(T)$, $W = E(T)$ and $R = Y = B = \emptyset$. This generalization can be viewed as a labeling algorithm, which appears in [2] for the first time, and is afterwards widely used in the literature for solving the domination-related problem in [3] and [8]. However, unlike the ordinary practice to partition the vertex set, we partition both the vertex set and the edge set, as can be seen in above definitions. To obtain a polynomial time algorithm for finding a minimum one-two *ve*-dominating set of a tree, we should design an edge data structure as follows.

First root the tree T at any leaf, say, r. The height of T is the maximum distance between r and all other vertices. Let h be the height of T. The i-th level A_i $(0 \leq i \leq h)$ be the set of vertices of T which are at distance i from the root. For such a rooted tree T with order n we can number the edges of T with $1, 2, \ldots, n-1$ as follows. We go on every level starting with level h to level 1. For each i $(1 \leq i \leq h)$, we traverse the edges connecting the vertices on level i and $i-1$ in arbitrary order, say from left to right. Finally, we list out the fathers of all edges of T (we mean that the edge numbered $n-1$ has no father by writing $father(n-1) = 0$), and thus we can represent T by a data structure called an edge father array. Fig. 1 shows an example of a tree and its edge father array. The edge numbered $n-1$ is called the edge root of T.



edge	1	2	3	4	5	6	7	8	9
father	6	6	7	8	8	9	9	9	0

FIGURE 1. A rooted tree with its edge father array.

In order to obtain the $\gamma_{dve}(T)$, we will design an algorithm for the $\gamma_{12}(T)$ with $F = V(T)$, $W = E(T)$ and $R = Y = B = \emptyset$.

Algorithm 1 Computes the one-two ve-domination number of a tree T

Input: an edge rooted tree T represented by its edge father array $[1, 2, \ldots, n-1]$. An edge assignment L_1 and a vertex assignment L_2 such that every edge is labeled W and every vertex is labeled by F, respectively.
Output: a minimum one-two ve-dominating set of T.
$T' \leftarrow T$; $D \leftarrow \emptyset$; $R \leftarrow \emptyset$; $Y \leftarrow \emptyset$; $B \leftarrow \emptyset$;
for $e = 1$ to $n-2$ **do**
let $e = vu$, where v is a leaf of T', u is the parent vertex of v;
$father(e) \leftarrow$ the father edge of e;
$p(e) \leftarrow$ the parent vertex u of v;
if $v \in F$, $u \in F$ and $e \in W$ **then**
$L_2(u) \leftarrow R$; $L_2(p(father(e))) \leftarrow R$;
$L_1(e') \leftarrow B$ for every edge $e' \in N[father(e)]$;
$L_1(e') \leftarrow Y$ for every edge $e' \in N_2(father(e)) \cap W$;
$L_1(e') \leftarrow B$ for every edge $e' \in N_2(father(e)) \cap Y$;
$T' \leftarrow T' - \{v\}$;
if $v \in F$, $u \in F$ and $e \in Y$ **then**
if $p(father(e)) \in R$ **then**
$L_2(u) \leftarrow R$;
$L_1(e') \leftarrow B$ for every edge $e' \in N[father(e)]$;
$T' \leftarrow T' - \{v\}$;
else
$L_2(p(father(e))) \leftarrow R$;
$L_1(e') \leftarrow Y$ for every edge $e' \in N_{\leq 2}(father(e)) \cap W$;
$L_1(e') \leftarrow B$ for every edge $e' \in N_{\leq 2}(father(e)) \cap Y$;

$T' \leftarrow T' - \{v\}$;
if $v \in F$, $u \in R$ and $e \in Y$ **then**
$L_2(p(father(e))) \leftarrow R$;
$L_1(e') \leftarrow Y$ for every edge $e' \in N_{\leq 2}(father(e)) \cap W$;
$L_1(e') \leftarrow B$ for every edge $e' \in N_{\leq 2}(father(e)) \cap Y$;
$T' \leftarrow T' - \{v\}$;
if $v \in R$, $u \in F$ and $e \in Y$ **then**
$L_2(p(father(e))) \leftarrow R$;
$L_1(e') \leftarrow Y$ for every edge $e' \in N_{\leq 2}(father(e)) \cap W$;
$L_1(e') \leftarrow B$ for every edge $e' \in N_{\leq 2}(father(e)) \cap Y$;
$D \leftarrow D \cup \{v\}$;
$T' \leftarrow T' - \{v\}$;
if $e \in B$ **then**
if $v \in F$ **then**
$T' \leftarrow T' - \{v\}$;
else
$D \leftarrow D \cup \{v\}$; $T' \leftarrow T' - \{v\}$;
end for
if $e \in W$ **then**
$D \leftarrow D \cup \{v, u\}$;
if $e \in Y$ **then**
$D \leftarrow D \cup (\{v, u\} \cap R) \cup \{w\}$, where $w \in (\{v, u\} \cap F)$;
if $e \in B$ **then**
$D \leftarrow D \cup (\{v, u\} \cap R)$.
return D

Theorem 3. *Algorithm* 1 *produces a minimum one-two ve-dominating set of a tree T in polynomial time.*

Proof. It is easy to see that the running time of algorithm 1 is polynomial time. For the correctness of the algorithm, it is sufficient to consider T with at least two edges, since the algorithm obviously produces a minimum one-two ve-dominating set of a tree with one edge correctly. Assume that $e = vu$ is a pending edge of T and v is a leaf in the lowest level of the edge rooted tree. Then the proof of Theorem 3 is followed by five simple claims.

Claim 1. If $v \in F$, $u \in F$ and $e \in W$, then there exists a minimum one-two ve-dominating set of T containing both u and $p(father(e))$.

Proof. Let D be a minimum one-two ve-dominating set of T. Since $e \in W$, then there exist at least two vertices in $N[u]$ to ve-dominate e. Let $D' = (D \backslash (D \cap N[u])) \cup \{u, p(father(e))\}$. It is easy to see that D' is another minimum one-two ve-dominating set of T containing both u and $p(father(e))$. □

Claim 2. Suppose that $v \in F$, $u \in F$ and $e \in Y$. If $p(father(e)) \notin R$, then there exists a minimum one-two ve-dominating set of T containing $p(father(e))$. If $p(father(e)) \in R$, then there exists a minimum one-two ve-dominating set of T containing u.

Proof. Let D be a minimum one-two ve-dominating set of T. Suppose that $e \in Y$ and $p(father(e)) \notin R$. Then there exists at least one vertex in $N[u]$ to ve-dominate e. Let $D' = (D \setminus (D \cap N[u])) \cup \{p(father(e))\}$. It is easy to see that D' is another minimum one-two ve-dominating set of T containing $p(father(e))$. Suppose that $e \in Y$ and $p(father(e)) \in R$. Then there exists at least one vertex $w \in N[u] \setminus \{p(father(e))\}$ to ve-dominate e. Let $D' = (D \setminus \{w\}) \cup \{u\}$. It is easy to see that D' is another minimum one-two ve-dominating set of T containing both u and $p(father(e))$. □

Claim 3. If $v \in F$, $u \in R$ and $e \in Y$, then there exists a minimum one-two ve-dominating set of T containing $p(father(e))$.

Proof. Let D be a minimum one-two ve-dominating set of T. Since $v \in F$, $u \in R$ and $e \in Y$, then there exists at least one vertex $w \in N(u)$ to ve-dominate e. Let $D' = (D \setminus \{w\}) \cup \{p(father(e))\}$. It is easy to see that D' is another minimum one-two ve-dominating set of T containing $p(father(e))$. □

Claim 4. If $v \in R$, $u \in F$ and $e \in Y$, then there exists a minimum one-two ve-dominating set of T containing $p(father(e))$.

Proof. Let D be a minimum one-two ve-dominating set of T. Since $v \in R$, $u \in F$ and $e \in Y$, then there exists at least one vertex $w \in N[u] \setminus \{v\}$ to ve-dominate e. Let $D' = (D \setminus \{w\}) \cup \{p(father(e))\}$. It is easy to see that D' is another minimum one-two ve-dominating set of T containing $p(father(e))$. □

Claim 5. Suppose that $e \in B$. If $v \in F$, then there exists a minimum one-two ve-dominating set of T not containing v. If $v \in R$, then there exists a minimum one-two ve-dominating set of T containing v.

Proof. Let D be a minimum one-two ve-dominating set of T. Suppose that $e \in B$ and $v \in F$. If $v \in D$, then let $w \in N[u] \setminus \{v\}$ and $D' = (D \setminus \{v\}) \cup \{w\}$. Then D' is a minimum one-two ve-dominating set of T not containing v. Suppose that $e \in B$ and $v \in R$. By the definition of one-two ve-dominating set, there exists a minimum one-two ve-dominating set of T containing v. □

□

Acknowledgments. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2020R1I1A3A04036669).

References

[1] R. Boutrig, M. Chellali, T. W. Haynes, and S. T. Hedetniemi, *Vertex-edge domination in graphs*, Aequationes Math. **90** (2016), no. 2, 355–366. https://doi.org/10.1007/s00010-015-0354-2

[2] E. J. Cockayne, S. Goodman, and S. T. Hedetniemi, *A linear algorithm for the domination number of a tree*, Inform. Process. Lett. **4** (1975) 41–44.

[3] S. T. Hedetniemi, R. Laskar, and J. Pfaff, *A linear algorithm for finding a minimum dominating set in a cactus*, Discrete Appl. Math. **13** (1986), no. 2-3, 287–292. https://doi.org/10.1016/0166-218X(86)90089-2

[4] B. Krishnakumari, M. Chellali, and Y. B. Venkatakrishnan, *Double vertex-edge domination*, Discrete Math. Algorithms Appl. **9** (2017), no. 4, 1750045, 11 pp. https://doi.org/10.1142/S1793830917500458

[5] B. Krishnakumari, Y. B. Venkatakrishnan, and M. Krzywkowski, *Bounds on the vertex-edge domination number of a tree*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 5, 363–366. https://doi.org/10.1016/j.crma.2014.03.017

[6] J. Lewis, S. T. Hedetniemi, T. W. Haynes, and G. H. Fricke, *Vertex-edge domination*, Util. Math. **81** (2010), 193–213.

[7] J. W. Peters, *Theoretical and algorithmic results on domination and connectivity*, Ph.D. thesis, Clemson University, 1986.

[8] Y. Zhao, Z. Liao, and L. Miao, *On the algorithmic complexity of edge total domination*, Theoret. Comput. Sci. **557** (2014), 28–33. https://doi.org/10.1016/j.tcs.2014.08.005

Xue-Gang Chen
Department of Mathematics
North China Electric Power University
Beijing 102206, P. R. China
Email address: gxcxdm@163.com

Moo Young Sohn
Department of Mathematics
Changwon National University
Changwon 51140, Korea
Email address: mysohn@changwon.ac.kr