Bull. Korean Math. Soc. **59** (2022), No. 1, pp. 15–25
https://doi.org/10.4134/BKMS.b200730
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE EXISTENCE OF GRAHAM PARTITIONS WITH CONGRUENCE CONDITIONS

Byungchan Kim, Ji Young Kim, Chong Gyu Lee, Sang June Lee, and Poo-Sung Park

Dedicated to the memory of Professor Ronald L. Graham (1935–2020)

Abstract. In 1963, Graham introduced a problem to find integer partitions such that the reciprocal sum of their parts is 1. Inspired by Graham's work and classical partition identities, we show that there is an integer partition of a sufficiently large integer n such that the reciprocal sum of the parts is 1, while the parts satisfy certain congruence conditions.

1. Introduction

Integer partitions are one of the profound subjects and have been studied for many years [2]. We say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is an integer partition of a positive integer n if $\sum_{i=1}^{\ell} \lambda_i = n$, where a different order of λ_i's is considered the same partition. Here, we call λ_i a part of the partition λ. It was *Ronald L. Graham* [3] who first studied the partition with a specific condition on the reciprocal sum of parts. He showed that if $n \geq 78$, then n can be partitioned into distinct parts whose reciprocal sum is 1. We call a partition a *Graham partition* in the memory of R. Graham if the reciprocal sum of its parts is 1. For example, 78 has a Graham partition with distinct parts:

$$78 = 2 + 6 + 8 + 10 + 12 + 40 \quad \text{and} \quad 1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{40}.$$

Received August 27, 2020; Accepted October 29, 2021.
2020 *Mathematics Subject Classification.* Primary 11P81, 05A17.
Key words and phrases. Graham partition, sum of reciprocals, Rogers–Ramanujan identity.
Byungchan Kim and Poo-Sung Park were supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT (NRF-2019R1F1A1043415, NRF-2017R1A2B1010761). Ji Young Kim, Chong Gyu Lee and Sang June Lee were supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1I1A1A01053318, NRF-2016R1D1A1B01009208, NRF-2019R1F1A1058860).

Graham's work seems to be isolated for a few decades, but more recently, it draws attention in diverse ways. Alekseyev [1] showed that if $n \geq 8543$, then n can be partitioned into squares of distinct positive integers such that the reciprocal sum of the positive integers is 1. For example,

$$49 = 2^2 + 3^2 + 6^2 \quad \text{and} \quad 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

The four of the authors [4] showed that if $n \geq 1224$, then n can be partitioned into squares, not necessarily distinct, whose reciprocal sum of squares is 1. In other words, there is a Graham partition of n into squares for $n \geq 1224$. For example,

$$66 = 2^2 + 2^2 + 2^2 + 3^2 + 3^2 + 6^2 \quad \text{and} \quad 1 = \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{6^2}.$$

In this note, we consider Graham partitions into parts satisfying some congruence conditions. From the seminal work of Euler (see [2, Chapter 1]), it is now famous that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts. From Graham's work on partitions into distinct parts, it is natural to ask whether there is a Graham partition consisting of odd parts. In this direction, we obtain the following result.

Theorem 1.1. *We have that $n \equiv 1 \pmod 8$ and $n \neq 17$ if and only if there is a Graham partition of n into odd parts.*

While the number of partitions into distinct parts equals the number of partitions into odd parts, the number of Graham partitions into distinct parts may not be the same as the number of Graham partitions into odd parts.

Motivated by Theorem 1.1, we further consider Graham partitions into parts satisfying some congruence conditions. In this direction, we first consider *m-regular partitions* which do not have parts divisible by m. Let $\mathcal{R}_m(n)$ be the set of m-regular Graham partitions of n and let $r_m(n)$ be the size of $\mathcal{R}_m(n)$. Theorem 1.1 can be restated as $r_2(n) > 0$ if and only if $n \equiv 1 \pmod 8$ and $n \neq 17$. We obtain the following result for $m = 3$, 4, and 5.

Theorem 1.2. *The following hold.*

(a) $r_3(n) > 0$ *for* $n \equiv 1 \pmod 3$ *and* $n \neq 7, 13, 19$, *and* $r_3(n) = 0$ *for all the other* n.
(b) $r_4(n) > 0$ *for* $n \geq 48$.
(c) $r_5(n) > 0$ *for* $n \geq 26$.

Instead of proving Theorem 1.2 directly, we consider the subset $\overline{\mathcal{R}}_m(n) \subset \mathcal{R}_m(n)$ which consists of m-regular Graham partitions having parts congruent to $i \pmod m$ for each $i \in \{1, 2, \ldots, m-1\}$. We denote $\overline{r}_m(n)$ to be the size of $\overline{\mathcal{R}}_m(n)$. Note that $r_2(n) = \overline{r}_2(n)$. We also obtain the following result for $m = 3$, 4, and 5.

Theorem 1.3. *The following hold.*

(a) $\overline{r}_3(n) > 0$ *for* $n \equiv 1 \pmod 3$ *and* $n \neq 1, 4, 7, 13, 16, 19, 25, 34$, *and* $\overline{r}_3(n) = 0$ *for all the other* n.
(b) $\overline{r}_4(n) > 0$ *for* $n \geq 67$.
(c) $\overline{r}_5(n) > 0$ *for* $n \geq 69$.

As an analog of Theorems 1.2 and 1.3, for a fixed positive integer m, we investigate a Graham partition of n such that there is a part congruent to $i \pmod m$ for each $i \in \{0, 1, 2, \ldots, m-1\}$. We let $\mathcal{F}_m(n)$ be the set of such partitions of n and $f_m(n) = |\mathcal{F}_m(n)|$. Then we obtain the following result for $f_m(n)$ for $m = 2$, 3, 4, and 5.

Theorem 1.4. *The following hold.*

(a) $f_2(n) > 0$ *for* $n \geq 34$.
(b) $f_3(n) > 0$ *for* $n \geq 42$.
(c) $f_4(n) > 0$ *for* $n \geq 59$.
(d) $f_5(n) > 0$ *for* $n \geq 82$.

We also study another subset of $\mathcal{R}_5(n)$ inspired by the Rogers–Ramanujan identities. The product-side of the first Rogers–Ramanujan identity

$$1 + \sum_{n\geq 1} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^k)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and the second Rogers–Ramanujan identity

$$1 + \sum_{n\geq 1} \frac{q^{n^2+n}}{\prod_{k=1}^{n}(1-q^k)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

are generating functions for the number of partitions of n into parts congruent to 1 or 4 (mod 5) and for the number of partitions of n into parts congruent to 2 or 3 (mod 5), respectively.

Inspired from this fact, we define $\mathcal{R}_5^+(n)$ be the set of Graham partitions of n into parts congruent to $\pm 1 \pmod 5$, that is, quadratic residues modulo 5. Similarly, let $\mathcal{R}_5^-(n)$ be the set of Graham partitions of n into parts congruent to $\pm 2 \pmod 5$, that is, quadratic non-residues modulo 5. Let $r_5^+(n) = |\mathcal{R}_5^+(n)|$ and $r_5^-(n) = |\mathcal{R}_5^-(n)|$. We obtain the following result.

Theorem 1.5. *The following hold.*

(a) $r_5^+(n) > 0$ *for* $n \geq 101$ *and* $n \equiv 1 \pmod 5$.
(b) $r_5^-(n) > 0$ *for* $n \geq 124$ *and* $n \equiv 4 \pmod 5$.

It is well-known that the sum-side of the first (resp. second) Rogers–Ramanujan identity is a generating function for the number of partitions of n into distinct parts (resp. distinct parts at least 2) with gaps at least 2, meaning that the difference between any two parts is at least 2 (see [2, Chapter 7]). As Graham partitions of $n > 1$ do not contain the part 1, the condition that each part is at least 2 has no role in the sense of Graham partitions unless $n = 1$.

Let $\mathcal{G}_d(n)$ be the set of Graham partitions of n into distinct parts with gaps at least d and let $g_d(n) = |\mathcal{G}_d(n)|$. Note that Graham's result corresponds to the fact that $g_1(n) > 0$ for $n \geq 78$. We obtain the following result on $g_2(n)$.

Theorem 1.6. *We have $g_2(n) > 0$ for $n \geq 108$.*

The key idea of proofs is that, in order to show the existence of a desired Graham partition for sufficiently large integers, it suffices to check a bounded range of integers using an induction. Since the number of partitions of n grows exponentially, one might expect that it is likely that there is a partition of n with prescribed properties. However, at the same time, it is hard to check whether there is such a partition or not by the exhaustive search. Moreover, there are relatively very few Graham partitions of n. With an additional condition, the desired partition could be rare. For example, among $20,506,255$ partitions of 82, there are 59 Graham partitions of 82. Among them, there are only 3 partitions in the set $\mathcal{F}_5(82)$.

The main obstacle of the proof is that we cannot list all the partitions for large integers, so we need computational tricks to settle this issue down. We illustrate one of the computational tricks to verify Theorem 1.5 and give the lists of desired partitions for other results without computational details.

The rest of the paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 and this will be a guide for the proof of the other results. In Section 3.1, we state two lemmas. The first lemma enables us to search the target partitions more effectively, and the other lemma is for a necessary condition to have desired partitions with congruence conditions. In Section 3.2, we give brief proofs of other theorems. We conclude the paper with some remarks.

2. Proof of Theorem 1.1

We first prove that if there is a Graham partition of n into odd parts, then $n \equiv 1 \pmod 8$ and $n \neq 17$. Suppose that there exist odd positive integers $\lambda_1, \ldots, \lambda_\ell$ such that

$$n = \sum_{i=1}^{\ell} \lambda_i \qquad \text{and} \qquad 1 = \sum_{i=1}^{\ell} \frac{1}{\lambda_i}.$$

Then we find that

$$\begin{aligned} n &= (8a_1+1) + \cdots + (8a_i+1) + (8b_1-1) + \cdots + (8b_j-1) \\ &\quad + (8c_1+3) + \cdots + (8c_k+3) + (8d_1-3) + \cdots + (8d_r-3) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} 1 &= \frac{1}{8a_1+1} + \cdots + \frac{1}{8a_i+1} + \frac{1}{8b_1-1} + \cdots + \frac{1}{8b_j-1} \\ &\quad + \frac{1}{8c_1+3} + \cdots + \frac{1}{8c_k+3} + \frac{1}{8d_1-3} + \cdots + \frac{1}{8d_r-3} \end{aligned}$$

with $i, j, k, r \geq 0$.

Multiplying the product of all denominators to both sides of the second equality, we obtain

$$\begin{aligned} 1^i(-1)^j3^k(-3)^r \ &\equiv i \cdot 1^{i-1}(-1)^j3^k(-3)^r + j \cdot 1^i(-1)^{j-1}3^k(-3)^r \\ &\quad + k \cdot 1^i(-1)^j3^{k-1}(-3)^r + r \cdot 1^i(-1)^j3^k(-3)^{r-1} \pmod 8. \end{aligned}$$

Then, by dividing both sides by $1^{i-1}(-1)^{j-1}3^{k-1}(-3)^{r-1}$, we have

$$1(-1)3(-3) \ \equiv \ i(-1)3(-3) + j \cdot 1 \cdot 3(-3) + k \cdot 1(-1)(-3) + r \cdot 1(-1)3 \pmod 8$$

and thus,

$$1 \ \equiv \ i - j + 3k - 3r \pmod 8. \tag{2.2}$$

The equality (2.1) implies that

$$n \equiv i \cdot 1 + j(-1) + k \cdot 3 + r(-3) \equiv 1 \pmod 8$$

by the modular equality (2.2). It can be easily checked that 17 cannot be represented as the desired partition.

Next, we prove the opposite direction, that is, if $n \equiv 1 \pmod 8$ and $n \neq 17$, then there is a Graham partition of n into odd parts. Suppose that

$$n = \lambda_1 + \cdots + \lambda_\ell \qquad \text{and} \qquad 1 = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_\ell}$$

with odd numbers $\lambda_1, \ldots, \lambda_\ell$. Then, letting $n = 8k + 1$, we have partitions of $3n + 6$, $3n + 38$, and $3n + 70$ as follows.

$$\begin{cases} 3n + 6 = 3\lambda_1 + \cdots + 3\lambda_\ell + 3 + 3 = 8(3k + 1) + 1, \\ \qquad 1 = \dfrac{1}{3\lambda_1} + \cdots + \dfrac{1}{3\lambda_\ell} + \dfrac{1}{3} + \dfrac{1}{3}. \end{cases}$$

$$\begin{cases} 3n + 38 = 3\lambda_1 + \cdots + 3\lambda_\ell + 3 + 5 + 15 + 15 = 8(3(k + 1) + 2) + 1, \\ \qquad 1 = \dfrac{1}{3\lambda_1} + \cdots + \dfrac{1}{3\lambda_\ell} + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{15} + \dfrac{1}{15}. \end{cases}$$

$$\begin{cases} 3n + 70 = 3\lambda_1 + \cdots + 3\lambda_\ell + 5 + 5 + 15 + 15 + 15 + 15 = 8 \cdot 3(k + 3) + 1, \\ \qquad 1 = \dfrac{1}{3\lambda_1} + \cdots + \dfrac{1}{3\lambda_\ell} + \dfrac{1}{5} + \dfrac{1}{5} + \dfrac{1}{15} + \dfrac{1}{15} + \dfrac{1}{15} + \dfrac{1}{15}. \end{cases}$$

So, if n is a sufficiently large integer congruent to $1 \pmod 8$, then a desired partition of n can be obtained from a partition of some smaller integer n'. More precisely, we have

a partition of $n = 8(3s) + 1$ from $n' = 8(s - 3) + 1$;
a partition of $n = 8(3s + 1) + 1$ from $n' = 8s + 1$;
a partition of $n = 8(3s + 2) + 1$ from $n' = 8(s - 1) + 1$.

There are three cases $k \equiv 0, 1, 2 \pmod 3$.

Case 1: $k = 3s$. For a desired partition of n, we cannot use a partition of a small integer if $s - 3 < 0$, that is, $s = 0, 1, 2$. It happens when $k = 0, 3, 6$.

Case 2: $k = 3s + 1$. For a desired partition of n, we can always use a partition of a small integer.

Case 3: $k = 3s + 2$. For a desired partition of n, we cannot use a partition of a small integer if $s - 1 < 0$, that is, $s = 0$. It happens when $k = 2$.

Therefore, it remains to check a desired partition of

$$n = 8 \cdot 0 + 1,\ 8 \cdot 2 + 1,\ 8 \cdot 3 + 1,\ 8 \cdot 6 + 1.$$

We have that

$$\begin{aligned} &8 \cdot 0 + 1 = 1, \\ &8 \cdot 2 + 1 = 17 : \text{N/A}, \\ &8 \cdot 3 + 1 = 25 = 5 + 5 + 5 + 5 + 5, \\ &8 \cdot 6 + 1 = 49 = 7 + 7 + 7 + 7 + 7 + 7 + 7. \end{aligned}$$

Since $17 = 8 \cdot 2 + 1$ does not have a desired partition, we only have to check a desired partition of $n = 8k + 1$ related to $n' = 8 \cdot 2 + 1$. All such n's are

$$n = 8 \cdot 7 + 1,\ 8 \cdot 11 + 1,\ 8 \cdot 15 + 1.$$

We observe that

$$\begin{aligned} 8 \cdot 7 + 1 &= 57 = 3 + 9 + 9 + 9 + 9 + 9 + 9, \\ 8 \cdot 11 + 1 &= 89 = 3 + 5 + 9 + 9 + 9 + 9 + 45, \\ 8 \cdot 15 + 1 &= 121 = 11 + 11 + \cdots + 11. \end{aligned}$$

Since all such n's have desired partitions, it completes our proof.

3. Proofs of other theorems

3.1. Two basic lemmas

To restrict the number of parts and the size of the largest part during the search, we prove the following lemma.

Lemma 3.1. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ be a Graham partition of n. Then, the following hold*:

(1) $\ell \leq \sqrt{n}$.
(2) $\lambda_1 \geq \sqrt{n}$.

Proof. From the Cauchy–Schwarz inequality

$$\left(\sum_{i=1}^{\ell} \lambda_i \right) \left(\sum_{i=1}^{\ell} \frac{1}{\lambda_i} \right) \geq \ell^2,$$

we have $\ell \leq \sqrt{n}$. Moreover, $\ell \lambda_1 \geq n$ implies that $\lambda_1 \geq n/\ell \geq \sqrt{n}$. $\square$

The above lemma reduces the number of candidate partitions. However, when n gets larger, it quickly becomes infeasible to check all the candidates. In practice, we decompose the set of partitions into several pieces to decrease the computation time.

As we have seen Theorem 1.1, the congruence conditions on the parts prevent the existence of Graham partitions for certain arithmetic progressions. The following lemma shows that this is a general phenomenon in Graham partitions with congruence conditions.

Lemma 3.2. *Let p be a prime and let n be a positive integer which has a Graham partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Suppose that*

$$\lambda_i \equiv \pm\zeta \pmod{p}. \tag{3.1}$$

Then

$$n \equiv \zeta^2 \pmod{p}.$$

Proof. Since $\zeta = 0$ case is trivial, we assume that $\zeta \neq 0$. Let n be a positive integer which has a p-regular Graham partition satisfying (3.1);

$$n = \sum_{i=1}^{k}(pa_i + \zeta) + \sum_{j=1}^{r}(pb_j - \zeta) \quad \text{and} \quad \sum_{i=1}^{k} \frac{1}{pa_i + \zeta} + \sum_{j=1}^{r} \frac{1}{pb_j - \zeta} = 1. \tag{3.2}$$

The first equality provides

$$n \equiv (k - r)\zeta \pmod{p}. \tag{3.3}$$

Furthermore, multiplying $\prod(pa_i + \zeta) \cdot \prod(pb_j - \zeta)$ on both sides of the second equality of (3.2), we obtain

$$k(-1)^r\zeta^{k-1+r} + r(-1)^{r-1}\zeta^{k+r-1} \equiv (-1)^r\zeta^{k+r} \pmod{p},$$

and hence

$$\zeta \equiv k - r \pmod{p}. \tag{3.4}$$

Thus, (3.3) and (3.4) give

$$n \equiv \zeta^2 \pmod{p}.$$

This completes the proof. □

3.2. Proofs of Theorems 1.2–1.6

We use mathematical induction to prove all the theorems in this section. Thanks to the induction, we can conclude the claimed results once we find Graham partitions of n for some range of integers n. We use Python to find such partitions and the list of partitions to verify theorems can be found at

`https://github.com/math-bkim/Graham_ptn`

for readers.

We first give a proof of Theorem 1.3. Theorem 1.2 follows from Theorem 1.3 by checking finite cases between the claimed lower bounds in Theorem 1.2 and in Theorem 1.3.

Proof of Theorem 1.3. Lemma 3.2 immediately implies that $\overline{r}_3(n) = 0$ if $n \not\equiv 1 \pmod 3$. We now assume that $n \equiv 1 \pmod 3$. We note that if $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \overline{\mathcal{R}}_3(n)$, then

$$\begin{aligned} &(10, 5, 5, 2\lambda_1, \dots, 2\lambda_\ell) \in \overline{\mathcal{R}}_3(2n + 20) \quad \text{and} \quad 2n + 20 \equiv 4 \pmod 6, \\ &(20, 5, 4, 2\lambda_1, \dots, 2\lambda_\ell) \in \overline{\mathcal{R}}_3(2n + 29) \quad \text{and} \quad 2n + 29 \equiv 1 \pmod 6. \end{aligned} \tag{3.5}$$

Therefore, to prove $\overline{r}_3(n) > 0$, it suffices to show that $\overline{r}_3(n) > 0$ from $n = 37$ to $2 \times 37 + 29 - 3 = 100$ with $n \equiv 1 \pmod 3$. This is because we have Graham partitions of n from $n = 2 \times 37 + 29 = 103$ to $n = 2 \times 100 + 20 = 220$ by (3.5), and thus there are Graham partitions of n from $n = 2 \times 97 + 29 = 223$ and $2 \times 220 + 20 = 460$ again by (3.5), and so on. For $n \le 34$, we list $\overline{\mathcal{R}}_3(n)$ by exhaustive searches, which completes the proof of (a).

For the proof of (b), we remark that if $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \overline{\mathcal{R}}_4(n)$, then

$$\begin{aligned} (6, 6, 3, 3\lambda_1, \dots, 3\lambda_\ell) &\in \overline{\mathcal{R}}_4(3n + 15) \quad \text{and} \quad 3n + 15 \equiv 0 \pmod 3, \\ (10, 6, 5, 5, 3\lambda_1, \dots, 3\lambda_\ell) &\in \overline{\mathcal{R}}_4(3n + 26) \quad \text{and} \quad 3n + 26 \equiv 2 \pmod 3, \\ (15, 10, 6, 3, 3\lambda_1, \dots, 3\lambda_\ell) &\in \overline{\mathcal{R}}_4(3n + 34) \quad \text{and} \quad 3n + 34 \equiv 1 \pmod 3. \end{aligned}$$

Thus, we find that it suffices to show that $\overline{\mathcal{R}}_4(n)$ is non-empty from $n = 67$ to $n = 3 \times 67 + 34 - 3 = 255$.

Finally, for the proof of (c), we notice that if $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \overline{\mathcal{R}}_5(n)$, then

$$\begin{aligned} (6, 6, 6, 2\lambda_1, \dots, 3\lambda_\ell) &\in \overline{\mathcal{R}}_5(2n + 18) \quad \text{and} \quad 2n + 18 \equiv 0 \pmod 2, \\ (6, 4, 3, 2\lambda_1, \dots, 3\lambda_\ell) &\in \overline{\mathcal{R}}_5(2n + 13) \quad \text{and} \quad 2n + 13 \equiv 1 \pmod 2. \end{aligned}$$

Therefore, we can conclude the proof if there is a partition in the set $\overline{\mathcal{R}}_5(n)$ from $n = 69$ to $n = 2 \times 69 + 18 - 2 = 154$. □

Now, we prove Theorem 1.4.

Proof of Theorem 1.4. Suppose that $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{F}_m(n)$. When $m = 2$, 3, or 5, we find that

$$\begin{aligned} (6, 3, 2\lambda_1, \dots, 2\lambda_\ell) &\in \mathcal{F}_m(2n + 9) \quad \text{and} \quad 2n + 9 \equiv 1 \pmod 2, \\ (10, 5, 5, 2\lambda_1, \dots, 2\lambda_\ell) &\in \mathcal{F}_m(2n + 20) \quad \text{and} \quad 2n + 20 \equiv 0 \pmod 2. \end{aligned}$$

Therefore, to prove $f_m(n) > 0$, we need to check from $n = 34$ to $n = 2 \times 34 + 20 - 2 = 86$ for $m = 2$, from $n = 42$ to $n = 2 \times 42 + 20 - 2 = 102$ for $m = 3$, and from $n = 82$ to $n = 2 \times 82 + 20 - 2 = 182$ for $m = 5$.

If $m = 4$, then

$$\begin{aligned} (3, 3, 3\lambda_1, \dots, 3\lambda_\ell) &\in \mathcal{F}_4(3n + 6) \quad \text{and} \quad 3n + 6 \equiv 0 \pmod 3, \\ (12, 4, 3, 3\lambda_1, \dots, 3\lambda_\ell) &\in \mathcal{F}_4(3n + 19) \quad \text{and} \quad 3n + 19 \equiv 1 \pmod 3, \\ (6, 2, 3\lambda_1, \dots, 3\lambda_\ell) &\in \mathcal{F}_4(3n + 8) \quad \text{and} \quad 3n + 8 \equiv 2 \pmod 3. \end{aligned}$$

To complete the proof, we need to verify that there are such Graham partitions from $n = 59$ to $n = 3 \times 59 + 19 - 3 = 193$. □

Finally, we prove results motivated from the Rogers–Ramanujan identities.

Proof of Theorem 1.5. We first note that the set $\mathcal{R}_5^+(n)$ could be non-empty only if $n \equiv 1 \pmod 5$ by Lemma 3.2 with $p = 5$ and $\zeta = 1$. We now note that if $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{R}_5^+(n)$, then

$$\begin{aligned}
(4,4,4,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^+(4n+12) \quad \text{and} \quad 4n+12 \equiv 16 \pmod{20},\\
(6,6,6,4,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^+(4n+22) \quad \text{and} \quad 4n+22 \equiv 6 \pmod{20},\\
(9,9,9,6,4,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^+(4n+37) \quad \text{and} \quad 4n+37 \equiv 1 \pmod{20},\\
(21,21,14,9,9,9,4,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^+(4n+87) \quad \text{and} \quad 4n+87 \equiv 11 \pmod{20}.
\end{aligned}$$

Therefore, it suffices to check that $\mathcal{R}_5^+(n)$ is not empty from 101 to $486 = 4 \times 101 + 87 - 5$.

For the second claim, we note that the set $\mathcal{R}_5^-(n)$ is empty if $n \not\equiv 4 \pmod 5$ by Lemma 3.2 with $p = 5$ and $\zeta = 2$. We now assume that $n \equiv 4 \pmod 5$ and observe that if $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{R}_5^-(n)$, then

$$\begin{aligned}
(8,8,2,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^-(4n+18) \quad \text{and} \quad 4n+18 \equiv 14 \pmod{20},\\
(12,12,8,8,3,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^-(4n+43) \quad \text{and} \quad 4n+43 \equiv 19 \pmod{20},\\
(8,8,8,8,8,8,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^-(4n+48) \quad \text{and} \quad 4n+48 \equiv 4 \pmod{20},\\
(18,18,18,8,8,3,4\lambda_1,\ldots,4\lambda_\ell) &\in \mathcal{R}_5^-(4n+73) \quad \text{and} \quad 4n+73 \equiv 9 \pmod{20}.
\end{aligned}$$

Thus, checking that $r_5^-(n) > 0$ from $n = 124$ to $n = 4 \times 124 + 73 - 5 = 564$ is enough. Here we give a pseudo-code to find a partition $\lambda \in \mathcal{R}_5^-(5n-1)$ for $n \geq 80$.

- Compute the sets
  $R_1 = \mathcal{P}(3n+1, S(4, \lfloor (5n-1)/10 \rfloor))$ and $R_2 = \mathcal{P}(2n-2, S(0,5))$,
  where $\mathcal{P}(n, S)$ is the set of partitions of n into parts in S whose reciprocal sum is less than 1, and $S(a,b) = \{5i+2, 5i+3 : a \leq i < b\}$.
- While listing elements of the set R_1 (R_2, resp.), we partition the set into 10 subsets $R_{1,j}$ (resp. $R_{2,j}$), where $j = \lfloor 10 \sum_{i=1}^{\ell} \frac{1}{\lambda_i} \rfloor$, where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is the partition in the set R_1 (R_2, resp.).
- for j=0 to 9 do
  for $(\pi, \lambda) \in R_{1,9-j} \times R_{2,j}$ do
  if $\sum(1/\pi_i) + \sum(1/\lambda_i) = 1$
  then we find a desired partition $(\pi_1, \ldots \pi_k, \lambda_1, \ldots, \lambda_r)$.
  stop the search.

We note that we find a partition in $\mathcal{R}_5^-(404)$ in the set $R_{1,9} \times R_{2,0}$ and $|R_{1,9}| = 385$ and $|R_{2,0}| = 1109$ for this case, while there are $214,001,655,327$ partitions of 404 into parts congruent to 2 or 3 $\pmod 5$. □

Proof of Theorem 1.6. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathcal{G}_2(n)$. If λ does not contain 3, 5, 6, 8, or 13 as a part, then we observe that

$$\begin{aligned}(2, 10, 15, 3\lambda_1, \ldots, 3\lambda_\ell) &\in \mathcal{G}_2(3n + 27) \quad \text{and} \quad 3n + 27 \equiv 0 \pmod 3,\\ (2, 9, 18, 3\lambda_1, \ldots, 3\lambda_\ell) &\in \mathcal{G}_2(3n + 29) \quad \text{and} \quad 3n + 29 \equiv 2 \pmod 3,\\ (2, 10, 24, 40, 3\lambda_1, \ldots, 3\lambda_\ell) &\in \mathcal{G}_2(3n + 76) \quad \text{and} \quad 3n + 76 \equiv 1 \pmod 3.\end{aligned}$$

We find a partition in $\mathcal{G}_2(n)$ without the part 3, 5, 6, 8, or 13 from $n = 163$ to $n = 3 \times 163 + 76 - 3 = 562$. This proves that $g_2(n) > 0$ for $n \geq 163$. To complete the proof, we also find a partition $\lambda \in \mathcal{G}_2(n)$ from $n = 108$ to $n = 162$. □

Remark. One might try to use that for $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{G}_2(n)$,

$$(2, 2\lambda_1, \ldots, 2\lambda_\ell) \in \mathcal{G}_2(2n + 2) \quad \text{and} \quad (3, 6, 2\lambda_1, \ldots, 2\lambda_\ell) \in \mathcal{G}_2(2n + 9)$$

if the smallest part of λ is greater than or equal to 4. However, it seems that there are few partitions in $\mathcal{G}_2(n)$ with the smallest part at least 4 for $n \leq 200$.

4. Concluding remark

To prove the positivity, one may think of a generating function. For example,

$$\begin{aligned}\sum_{n\geq 1} r_5^+(n) q^n &= [z^1] \prod_{n=0}^{\infty} \frac{1}{(1 - z^{1/(5n+1)} q^{5n+1})(1 - z^{1/(5n+4)} q^{5n+4})}\\ &= q + q^{16} + q^{26} + q^{36} + q^{41} + q^{51} + 3q^{66} + 5q^{76} + q^{81} + \cdots,\end{aligned}$$

where $[z^k]f(z, q)$ denotes the coefficient of z^k for the power series $f(z, q) \in \mathbb{Z}[[z, q]]$. While computation cost for expanding the above generating function looks too expensive to verify theorems, an algebraic/analytic proof of the positivity would be interesting. We also conjecture that for any odd prime p, there is a Graham partition of $n \equiv 1 \pmod p$ into parts congruent to $\pm 1 \pmod p$ for sufficiently large integers n.

We expect a generalization of Theorem 1.5. Since ± 1 are quadratic residues and ± 2 are quadratic non-residues modulo 5, we may consider a Graham partition of n into quadratic residues or quadratic non-resides modulo p for a prime p larger than 5. We believe that there is such a Graham partition for sufficiently large n without necessary conditions.

One may also consider Graham partitions corresponding to other Rogers–Ramanujan type identities like Schur's partition identity or Bressoud's generalized Rogers–Ramanujan identities.

Acknowledgements. The authors are indebted to the referee for comments and corrections, which improved the presentation of this paper.

References

[1] M. A. Alekseyev, *On partitions into squares of distinct integers whose reciprocals sum to 1*, in The mathematics of various entertaining subjects. Vol. 3, 213–221, Princeton Univ. Press, Princeton, NJ, 2019.

[2] G. E. Andrews, *The theory of partitions*, reprint of the 1976 original, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998.

[3] R. L. Graham, *A theorem on partitions*, J. Austral. Math. Soc. **3** (1963), 435–441.

[4] B. Kim, J. Y. Kim, C. G. Lee, and P. Park, *On the partitions into squares whose reciprocal sum is one*, Publ. Math. Debrecen **95** (2019), no. 1-2, 243–247.

Byungchan Kim
School of Liberal Arts
Seoul National University of Science and Technology
Seoul 01811, Korea
Email address: bkim4@seoultech.ac.kr

Ji Young Kim
Department of Mathematical Sciences
Seoul National University
Seoul 08826, Korea
Email address: jykim98@snu.ac.kr

Chong Gyu Lee
Department of Mathematics
Soongsil University
Seoul 06978, Korea
Email address: cglee@ssu.ac.kr

Sang June Lee
Department of Mathematics
Kyung Hee University
Seoul 02447, Korea
Email address: sjlee242@khu.ac.kr

Poo-Sung Park
Department of Mathematics Education
Kyungnam University
Changwon-si 51767, Korea
Email address: pspark@kyungnam.ac.kr