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QUANTITATIVE WEIGHTED ESTIMATES FOR OSCILLATORY SINGULAR INTEGRALS WITH ROUGH KERNELS

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Abstract. In this paper, we obtain the quantitative weighted bounds of oscillatory singular integral with rough kernel. Moreover, the quantitative weighted bounds of maximally truncated oscillatory singular integral with rough kernel are also obtained.

1. Introduction

The following form of oscillatory singular integrals with standard Calderón-Zygmund kernel had been studied by F. Ricci and E. M. Stein in [15]:

$$T_P f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy, \tag{1}$$

where $\Omega \in C^1(\mathbb{S}^{n-1})$ is homogeneous of degree zero and has mean value zero on the unit sphere $\mathbb{S}^{n-1}$, and $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a real valued polynomial of two variables. As well-known, the operators of type (1) have arisen in the study of singular integrals on lower-dimensional varieties and the singular Radon transform, one can see [1, 4, 13, 18] and references therein. Ricci and Stein [15] proved that the operator T_P is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ provided that $\Omega \in C^1(\mathbb{S}^{n-1})$. Later on, the condition $\Omega \in C^1(\mathbb{S}^{n-1})$ was relaxed to $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$ by Lu and Zhang [12]. This paper will be devoted to discussing a class of oscillatory singular integrals with rough kernel, which is not necessary to be a standard Calderón-Zygmund kernel.

Before we give the main results, let's review some variants of the weight characteristic. For $1 < p < \infty$, we say that $w \in A_p$ if there exists a constant

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$C > 0$ such that

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} \le C, \tag{2}$$

where $p' = \frac{p}{p-1}$. The definition for the A_∞ constant of a weight w was introduced by N. Fujii [5] and J. M. Wilson [17]:

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(1_Q w)(x)dx.$$

Here, $w(Q) := \int_Q w(x)dx$, $1_Q w(x) = w(x)1_Q(x)$, where 1_Q is the characteristic function of Q, and the supremum above is taken over all cubes with edges parallel to the coordinate axes. When the supremum is finite, we will say that w belongs to the A_∞ class. In order to state the weighted estimates in this paper more effectively, we introduce the following variants of the weight characteristic:

$$\{w\}_{A_p} := [w]_{A_p}^{1/p}\max\{[w]_{A_\infty}^{1/p'}, [w^{1-p'}]_{A_\infty}^{1/p}\},$$
$$(w)_{A_p} := \max\{[w]_{A_\infty}, [w^{1-p'}]_{A_\infty}\}.$$

Recently, with the development of the key tool sparse domination (pointwise version originated in [9]), the quantitative weighted bounds for singular integrals with rough kernels have been studied intensively (see [3, 7, 11]). Among these quantitative weighted bounds for T_Ω which is defined by

$$T_\Omega f(x) = p.v.\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

we would like to highlight that Hytönen–Roncal–Tapiola [7] first proved that when $\Omega \in L^\infty(\mathbb{S}^{n-1})$,

$$\|T_\Omega\|_{L^p(w)\to L^p(w)} \le c_{n,p}\|\Omega\|_{L^\infty}\{w\}_{A_p}(w)_{A_p}. \tag{3}$$

Later, Conde-Alonso–Culiuc–Di Plinio–Ou [3] proved a sparse domination for the bilinear forms associated with T_Ω with $\Omega \in L^q(\mathbb{S}^{n-1})$ for $1 < q \le \infty$ satisfying the cancellation conditions, which leads to quantitative weighted bounds for T_Ω.

Moreover, Lu-Zhang [12] established a simple criterion on L^p boundedness of the rough oscillatory singular integral operator, and developed a general method for studying the L^p boundedness of oscillating singular integrals. In this paper, we will extend the criteria to the weighted background, and then use their method to analyze the weighted boundedness of oscillatory singular integral and maximally truncated oscillatory singular integral with rough kernel. Our first aim is to focus on the behavior of the quantitative weighted bound for a class of oscillatory singular integrals with rough kernel.

Theorem 1.1. *Suppose $P(x,y)$ is a real valued polynomial. Let Ω be homogeneous of degree zero, have mean value zero and $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then for $1<p<\infty$ and $w \in A_p$, the operator T_P satisfies*

$$\|T_P f\|_{L^p(w)} \le C\|\Omega\|_{L^\infty}\{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}, \tag{4}$$

where C depends only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.

In the last ten years, (3) was also extended to maximal singular integrals T_Ω^* which is defined by

$$T_\Omega^* f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|$$

in Di Plinio, Hytönen and Li [14] and Lerner [10] via sparse domination, which gives

$$\|T_\Omega^*\|_{L^2(w)\to L^2(w)} \le c_n\|\Omega\|_{L^\infty}[w]_{A_2}^2. \tag{5}$$

The second result of this paper is to focus on the quantitative weighted bound of maximal truncation of oscillatory singular integrals T_p^* with

$$T_P^* f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\right|.$$

Note that B. Krause and M. T. Lacey proved the quantitative weighted bounds for maximally truncated oscillatory singular integral with $\Omega \in C^1(\mathbb{S}^{n-1})$ in [8, Corollary 1.3] as follows:

$$\|T_P^*\|_{L^p(w)\to L^p(w)} \le C[w]_{A_p}^{1+\max\{\frac{1}{p-1},1\}}.$$

We shall get a similar result for the maximal truncation T_P^* with $\Omega \in L^\infty(\mathbb{S}^{n-1})$.

Theorem 1.2. *Suppose $P(x,y)$ is a real valued polynomial. Let Ω be homogeneous of degree zero, have mean value zero and $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then for $1<p<\infty$ and $w \in A_p$, the operator T_P^* satisfies*

$$\|T_P^* f\|_{L^p(w)} \le C\{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}, \tag{6}$$

where C depends only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$.

Remark 1.3. Note that $C^1(\mathbb{S}^{n-1}) \subset L^\infty(\mathbb{S}^{n-1})$, Theorem 1.2 can be regarded as an extension of [8, Corollary 1.3]. In [8], authors mainly studied the sparse form for oscillatory singular integral with Calderón-Zygmund kernel. However, the rough kernel studied in Theorems 1.1 and 1.2 does not belong to the above kernel, the sparse method for Calderón-Zygmund kernel fails. We used the existing conclusion that quantitative weighted bounds for singular integrals with rough kernels in [7] and the general method for studying the L^p boundedness of oscillating singular integrals in [12] to prove the main theorems in this paper.

This paper is organized as follows. In Section 2, we give some notation and lemmas, we will establish an quantitative weighted estimate for truncated oscillatory singular integrals with rough kernel. The proofs of our main theorems are given in Section 3.

Notation. Throughout the whole paper, $p' = p/(p-1)$ represents the conjugate index of $p \in [1, \infty)$; $X \lesssim Y$ stands for $X \leq CY$ for a constant $C > 0$ which is independent of the essential variables living on X & Y; and $X \approx Y$ denotes $X \lesssim Y \lesssim X$.

2. Some lemmas

First, we state several lemmas which play an important role in the proofs of Theorems 1.1 and 1.2. The following interpolation theorem with change of measures was presented by E. M. Stein and G. Weiss in [16, Theorem 2.11].

Lemma 2.1 ([16]). *Assume that $1 \leq p_0, p_1 \leq \infty$, that w_0 and w_1 are positive weights, and that T is a sublinear operator satisfying*

$$T : L^{p_i}(w_i) \to L^{p_i}(w_i),\ i = 0, 1,$$

with quasi-norms M_0 and M_1, respectively. Then

$$T : L^p(w) \to L^p(w),$$

with quasi-norm $M \leq M_0^{\lambda} M_1^{1-\lambda}$, where

$$\frac{1}{p} = \frac{\lambda}{p_0} + \frac{1-\lambda}{p_1},\ w = w_0^{p\lambda/p_0} w_1^{p(1-\lambda)/p_1}.$$

Lemma 2.2. *Suppose Ω is homogeneous of degree zero on $\mathbb{S}^{n-1}$, and $\Omega(x') \in L^{\infty}(\mathbb{S}^{n-1})$. If*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

satisfies

$$\|Tf\|_{L^p(w)} \lesssim \|T\|_{L^p(w)\to L^p(w)} \|f\|_{L^p(w)}$$

for $1 < p < +\infty$ and $w \in A_p$. Also $K(x, y)$ satisfies

$$|K(x, y)| \lesssim \frac{|\Omega(x-y)|}{|x-y|^n}.$$

Then the operators

$$T_{\varepsilon} f(x) = \int_{|x-y|<\varepsilon} K(x, y) f(y) dy$$

satisfy

$$\|T_{\varepsilon}\|_{L^p(w)\to L^p(w)} \lesssim C(\|T\|_{L^p(w)\to L^p(w)} + \|\Omega\|_{L^{\infty}} \{w\}_{A_p}),$$

where C is independent of T, ε and w.

Proof. We split f into three parts $f(y) = f_1(y) + f_2(y) + f_3(y)$ for $h \in \mathbb{R}^n$. Here

$$\begin{aligned} f_1(y) &= f(y)\chi_{\{|y-h|<\varepsilon/2\}}(y), \\ f_2(y) &= f(y)\chi_{\{\varepsilon/2\le|y-h|<5\varepsilon/4\}}(y), \\ f_3(y) &= f(y)\chi_{\{|y-h|\ge5\varepsilon/4\}}(y). \end{aligned}$$

When $|x-h|<\varepsilon/4$, it is easy to see $T_\varepsilon f_1(x) = Tf_1(x)$. Then we have

$$\begin{aligned} \int_{|x-h|<\varepsilon/4} |T_\varepsilon f_1(x)|^p w(x)dx &\lesssim \int_{\mathbb{R}^n} |Tf_1(x)|^p w(x)dx \\ &\lesssim \|T\|^p_{L^p(w)\to L^p(w)} \int_{|y-h|<\varepsilon/2} |f(y)|^p w(y)dy. \end{aligned} \tag{7}$$

If $|x-h|<\varepsilon/4$, $\varepsilon/2 \le |y-h| < 5\varepsilon/4$, then $\varepsilon/4 < |x-y| < 3\varepsilon/2$. So we have

$$|T_\varepsilon f_2(x)| \le \int_{\varepsilon/4<|y|\le\varepsilon} \frac{|\Omega(y')|}{|y|^n} |f_2(x-y)|dy \lesssim \|\Omega\|_{L^\infty} Mf_2(x).$$

Then by the sharp weighted boundedness of the Hardy–Littlewood maximal operator M (see Hytönen–Pérez [6, Corollary 1.10], the original version was due to Buckley [2]),

$$\|Mf\|_{L^p(w)} \le c_n \cdot p' \cdot [w]^{\frac{1}{p}}_{A_p} [w^{1-p'}]^{\frac{1}{p}}_{A_\infty} \|f\|_{L^p(w)}, \quad 1<p<\infty, \tag{8}$$

it follows that

$$\begin{aligned} &\left(\int_{|x-h|<\varepsilon/4} |T_\varepsilon f_2(x)|^p w(x)dx \right)^{1/p} \\ &\lesssim \|\Omega\|_{L^\infty} \left(\int_{|x-h|<\varepsilon/4} |Mf_2(x)|^p w(x)dx \right)^{1/p} \\ &\lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p} \left(\int_{\mathbb{R}^n} |f_2(x)|^p w(x)\,dx \right)^{1/p} \\ &\simeq \|\Omega\|_{L^\infty} \{w\}_{A_p} \left(\int_{|y-h|<5\varepsilon/4} |f(y)|^p w(y)dy \right)^{1/p}. \end{aligned} \tag{9}$$

If $|x-h|<\varepsilon/4, |y-h|\ge 5/4$, then $|x-y|>\varepsilon$. So we have

$$T_\varepsilon f_3(x) = 0. \tag{10}$$

From (7), (9) and (10) it follows that the estimate

$$\begin{aligned} \int_{|x-h|<\varepsilon/4} |T_\varepsilon f(x)|^p w(x)dx \le\ & C(\|T\|_{L^p(w)\to L^p(w)} + \|\Omega\|_{L^\infty}\{w\}_{A_p})^p \\ &\times \int_{|y-h|<5\varepsilon/4} |f(y)|^p w(y)dy \end{aligned}$$

holds uniformly in $h \in \mathbb{R}^n$. The above estimates imply

$$\|T_\varepsilon f\|_{L^p(w)} \leq C(\|T\|_{L^p(w)\to L^p(w)} + \|\Omega\|_{L^\infty}\{w\}_{A_p})\|f\|_{L^p(w)}. \qquad \square$$

3. Proofs of the main theorems

3.1. Proof of Theorem 1.1

We shall carry out the argument by a double induction on the degrees in x and y of the polynomial P as follows. We assume the theorem is known for all polynomials which are sums of monomials degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y. Our inductive step will be to add to this all the monomials which have degree k in x and degree l in y.

Specifically, if the theorem for the polynomials which are sums of monomials degree equal to k in x times monomials of degree equal to l in y is true, then from the assumption for all polynomials which are sums of monomials degree equal to k in x times monomials of degree less than l in y, by double induction we can get the theorem for the polynomials which are sums of monomials degree equal to k in x times monomials of any degree in y. Combining with the above result, the assumption for all polynomials which are sums of monomials degree less than k in x times monomials of any degree in y and the known theorem for the polynomials which are sums of monomials degree equal to 0 in y times monomials of any degree in x (see [7] and [11]), by double induction we have the theorem for all polynomials which are sums of monomials of any degree in x times monomials of any degree in y is finished. Thus, in order to prove Theorem 1.1, we only need to prove the theorem for the polynomials which are sums of monomials degree equal to k in x times monomials of degree equal to l in y.

For general $P(x, y)$, we may write

$$P(x,y) = \sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta + R_0(x,y),$$

where $R_0(x, y)$ satisfies the above induction assumption.

Without loss of generality, we may assume $\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| > 0$.

Case 1. $\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| = 1$.

We write

$$\begin{aligned} T_P f(x) &= \int_{|x-y|\leq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy + \int_{|x-y|>1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \\ &=: T_0 f(x) + T_\infty f(x). \end{aligned}$$

Take $h \in \mathbb{R}^n$, and write

$$P(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta}(x-h)^\alpha (y-h)^\beta + R(x,y,h),$$

where the polynomial $R(x,y,h)$ satisfies the induction assumption, and the coefficients of $R(x,y,h)$ depend on h. Namely,

$$\widetilde{T}f(x) = p.v. \int_{\mathbb{R}^n} e^{iR(x,y,h)} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$$

satisfies

$$\|\widetilde{T}f\|_{L^p(w)} \lesssim \{w\}_{A_p}(w)_{A_p} \|f\|_{L^p(w)}. \tag{11}$$

We have

$$\begin{aligned}
|T_0 f(x)| \le & \left| \int_{|x-y|\le 1} \exp\big\{i[R(x,y,h) + \sum_{|\alpha|=k} a_{\alpha\beta}(y-h)^{\alpha+\beta}]\big\} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right| \\
& + \left| \int_{|x-y|\le 1} \big\{ \exp(iP(x,y) - \exp(i[R(x,y,h) \right. \\
& \left. + \sum_{|\alpha|=k} a_{\alpha\beta}(y-h)^{\alpha+\beta}])\big\} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \right| \\
& =: |T_{01}f(x)| + |T_{02}f(x)|.
\end{aligned}$$

For T_{01}. Denote by,

$$\begin{aligned}
T_h f(x) &:= p.v. \int_{\mathbb{R}^n} \exp\big\{i \sum_{|\alpha|=k} a_{\alpha\beta}(y-h)^{\alpha+\beta}]\big\} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \\
&= p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} g_h(y)dy,
\end{aligned}$$

where $g_h(y) = \exp\big\{i \sum_{|\alpha|=k} a_{\alpha\beta}(y-h)^{\alpha+\beta}]\big\} f(y)$. Then by (3), we get

$$\begin{aligned}
\|T_h f\|_{L^p(w)} &\lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p}(w)_{A_p} \|g_h\|_{L^p(w)} \\
&\lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p}(w)_{A_p} \|f\|_{L^p(w)}.
\end{aligned} \tag{12}$$

From the induction assumption (11), (12) and Lemma 2.2 we obtain that

$$\|T_{01}f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p}(w)_{A_p} \|f\|_{L^p(w)}. \tag{13}$$

For T_{02}, when $|x-h| < 1/4, |x-y| < 1$, we have

$$\begin{aligned}
& \big| \exp\{iP(x,y)\} - \exp\big\{i\big[R(x,y,h) + \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta}(y-h)^{\alpha+\beta}\big]\big\}\big| \\
\lesssim & \sum_{\substack{|\alpha|=k \\ |\beta|=l}} \big|a_{\alpha\beta}\big||x-y| \lesssim |x-y|.
\end{aligned}$$

Thus

$$\begin{aligned}|T_{02}f(x)| &\lesssim \int_{|x-y|\le 1} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)|dy\\ &= \sum_{k\le 0}\int_{2^{k-1}<|x-y|\le 2^k} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|f(y)\chi_{B(h,5/4)}|dy\\ &\lesssim \|\Omega\|_{L^\infty}\sum_{k\le 0} 2^k Mf\chi_{B(h,5/4)}(x)\\ &\lesssim \|\Omega\|_{L^\infty} Mf\chi_{B(h,5/4)}(x).\end{aligned}$$

Then combining with (8), we get

$$\int_{|x-h|<1/4} |T_{02}f(x)|^p w(x)dx \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}\int_{|y-h|<5/4}|f(y)|^p w(y)dy.$$

Thus

$$\|T_{02}f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}\|f\|_{L^p(w)}. \tag{14}$$

Combining (13) and (14), we get

$$\|T_0 f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}. \tag{15}$$

We write

$$T_\infty f(x) = \sum_{j=1}^{+\infty}\int_{2^{j-1}<|x-y|\le 2^j} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy = \sum_{j=1}^{+\infty} T_j f(x).$$

We take $\delta \in (0,1]$, such that $\delta < \min\{k/l, 2k/((k+l)q')\}$. From [12, (2.7)], we get

$$\|T_j f\|_{L^p} \lesssim \|\Omega\|_{L^\infty}2^{-j\delta/2}\|f\|_{L^p}. \tag{16}$$

On the other hand,

$$|T_j f(x)| \lesssim \|\Omega\|_{L^\infty} Mf(x). \tag{17}$$

Thus by (8)

$$\|T_j f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}\|f\|_{L^p(w)}. \tag{18}$$

Further, let c_n be a small positive constant and denote $\varepsilon := \frac{1}{2}c_n/(w)_{A_p}$, $\{w^{1+\varepsilon}\}_{A_p} \le c_n\{w\}_{A_p}^{(1+\varepsilon)}$(see [7, Corollary 3.18]). By (18), we have

$$\begin{aligned}\|T_j f\|_{L^p(w^{1+\varepsilon})} &\lesssim \|\Omega\|_{L^\infty}\{w^{1+\varepsilon}\}_{A_p}\|f\|_{L^p(w^{1+\varepsilon})}\\ &\lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}^{(1+\varepsilon)}\|f\|_{L^p(w^{1+\varepsilon})}.\end{aligned} \tag{19}$$

We apply Lemma 2.1 to $T = T_j$ with $p_0 = p_1 = p$, $w_0 = w^0 = 1$ and $w_1 = w^{1+\varepsilon}$, so that by $\lambda = \varepsilon/(1+\varepsilon)$, (16) and (19), we have for some θ, $\gamma > 0$ such that

$$\|T_j\|_{L^p(w)\to L^p(w)} \lesssim \|\Omega\|_{L^\infty}2^{-\theta j\varepsilon/(1+\varepsilon)}\{w\}_{A_p} \lesssim \|\Omega\|_{L_\infty}2^{-\gamma j/(w)_{A_p}}\{w\}_{A_p}.$$

Then use

$$\sum_{j=1}^{\infty} 2^{-\gamma j/(w)_{A_p}} \lesssim \Big(\sum_{j:j\leq (w)_{A_p}} + \sum_{j:j\geq (w)_{A_p}} \Big) 2^{-\gamma j/(w)_{A_p}} \lesssim (w)_{A_p}.$$

This gives that

$$\|T_\infty f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p} (w)_{A_p} \|f\|_{L^p(w)}. \tag{20}$$

Combining (15) and (20), we obtain

$$\|T_P f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty} \{w\}_{A_p} (w)_{A_p} \|f\|_{L^p(w)}. \tag{21}$$

Case 2. $\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| \neq 1$

Denote $A = \big(\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}|\big)^{1/(k+l)}$. We can write $p(x,y)$ as follows:

$$p(x,y) = \sum_{\substack{|\alpha|=k\\|\beta|=l}} \frac{a_{\alpha\beta}}{A^{k+l}} (Ax)^\alpha (Ay)^\beta + R_0(\frac{Ax}{A}, \frac{Ay}{A}) =: Q(Ax, Ay).$$

Thus

$$\begin{aligned} T_P f(x) &= \int e^{iQ(Ax,Ay)} K(x-y) f(y) dy \\ &= \int e^{iQ(Ax,y)} K(Ax-y) f(\frac{y}{A}) dy, \end{aligned}$$

where $K(x) = \frac{\Omega(x')}{|x|^n}$. From the result in Case 1, we obtain

$$\|T_P f\|_{L^p(w)} \leq C \|\Omega\|_{L^\infty} \{w\}_{A_p} (w)_{A_p} \|f\|_{L^p(w)},$$

where C depends only on the total degree of $P(x,y)$, but not on the coefficients of $P(x,y)$. So Theorem 1.1 holds for any polynomial $P(x,y)$ by induction principle.

3.2. Proof of Theorem 1.2

We shall carry out the argument by a double induction on the degrees in x and y of the polynomial P as in the proof of Theorem 1.1. As in the proof of Theorem 1.1, we write

$$P(x,y) = \sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta + R(x,y).$$

Since our conclusion is clearly invariant under dilation, we may assume that

$$\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| = 1.$$

If $k = 0$, we know that the conclusion holds from the result in [14] and [10].

$$T_\Omega^* f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \Big|$$

satisfies

$$\|T_\Omega^* f\|_{L^p(w)} \lesssim \{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}. \tag{22}$$

For general $P(x,y)$, we have

$$\begin{aligned}
T_P^* f(x) &\le \sup_{0<\varepsilon<1}\Big|\int_{|x-y|>\varepsilon} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&\quad + \sup_{\varepsilon\ge 1}\Big|\int_{|x-y|>\varepsilon} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&\le \sup_{0<\varepsilon<1}\Big|\int_{\varepsilon<|x-y|<1} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&\quad + \Big|\int_{|x-y|\ge 1} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&\quad + \sup_{\varepsilon\ge 1}\Big|\int_{|x-y|>\varepsilon} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&= T_{*,0}f(x) + T_\infty f(x) + T_{*,\infty}f(x).
\end{aligned}$$

By (20),

$$\|T_\infty f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}.$$

Now, it suffices to estimate $T_{*,0}$ and $T_{*,\infty}$.

By the method similar to proving (15) we can easily prove that

$$\|T_{*,0} f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}.$$

Notice that for fixed $\epsilon > 0$, we can always find a unique $J \in Z^+$ such that $2^{J-1} \le \epsilon < 2^J$. Thus

$$\begin{aligned}
T_{*,\infty}f(x) &\le \sup_{J\in\mathbb{Z}+}\int_{2^{J-1}\le|y|<2^J}\frac{|\Omega(y')|}{|y|^n}|f(x-y)|dy \\
&\quad + \sup_{J\in\mathbb{Z}^+}\sum_{j=J+1}\Big|\int_{2^{j-1}\le|x-y|<2^j} e^{ip(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&\le \sup_{J\in\mathbb{Z}^+}\int_{2^{J-1}\le|y|<2^J}\frac{|\Omega(y')|}{|y|^n}|f(x-y)|dy \\
&\quad + \sum_{j=1}^{+\infty}\Big|\int_{2^{j-1}\le|x-y|<2^j} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big| \\
&\le Mf(x) + \sum_{j=1}^{+\infty}\Big|\int_{2^{j-1}\le|x-y|<2^j} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big|.
\end{aligned}$$

From (8) and the method similar to proving (20), we get

$$\|T_{*,\infty} f\|_{L^p(w)} \lesssim \|\Omega\|_{L^\infty}\{w\}_{A_p}(w)_{A_p}\|f\|_{L^p(w)}.$$

So we have finished the proof of the theorem.

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