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SEMICASCADES OF TORIC LOG DEL PEZZO SURFACES

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Abstract. A cascade of toric log del Pezzo surfaces of Picard number one was introduced as a language of classifying all such surfaces. In this paper, we introduce a generalized concept, a semicascade of toric log del Pezzo surfaces. As applications, we discuss Kähler–Einstein toric log del Pezzo surfaces and derive a bound on the Picard number in terms of the number of singular points, generalizing some results of Dais and Suyama.

1. Introduction

Classification problem of toric log del Pezzo surfaces was considered in [1, 2, 4, 7]. As a result, toric log del Pezzo surfaces are completely classified up to index 16 ([7]). Recently, [3] completely classifies all toric log del Pezzo surfaces with 1 singular point, and [11] provides the complete classification of all those surfaces with 2 or 3 singular points. In particular, for a singular toric log del Pezzo surface S of Picard number ρ with t singular points, if $t \leq 3$, then $\rho \leq t+2$ and the equality holds if and only if S is the blow up of the weighted projective plane $\mathbb{P}(1,1,n)$ at the two smooth torus-fixed points where $n \geq 2$. We shall generalize this observation.

Theorem 1.1. *Let S be a singular toric log del Pezzo surface of Picard number ρ with t singular points. Then $\rho \leq t+2$ and the equality holds if and only if S is the blow up of $\mathbb{P}(1,1,n)$ at the two smooth torus-fixed points where $n \geq 2$.*

On the other hand, recall that we always have $\rho \geq t-2$. We observe that singular Kähler–Einstein toric log del Pezzo surfaces attain the lower bound.

Theorem 1.2. *Let S be a singular Kähler–Einstein toric log del Pezzo surface of Picard number ρ with t singular points. Then we have $\rho = t-2$. Moreover, S admits a semicascade to one of $S_1(2,2)$, $S_2(2,3)$, or $S_2(2,4)$. See Notation 2.15 for the notation.*

Corollary 1.3. *Let S be a Kähler–Einstein toric log del Pezzo surface. Then the maximal cones of the corresponding fan are either all smooth or all singular.*

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The proofs of both theorems use a generalization of cascades of toric log del Pezzo surfaces of Picard number one, which was introduced in [6] as a language to describe the classification of all such surfaces. Unfortunately, this notion cannot be extended to those with higher Picard number. See Subsection 3.4 for more explanation. Nevertheless, we introduce the following generalization.

Definition 1.4. Let S be a toric log del Pezzo surface. We say that S *admits a semicascade* if there exists a diagram as follows:

$$\begin{array}{ccccccc} S' = S'_t & \xrightarrow{\phi_t} & S'_{t-1} & \xrightarrow{\phi_{t-1}} & \cdots & \xrightarrow{\phi_1} & S'_0 \\ \pi_t \downarrow & & \pi_{t-1} \downarrow & & & & \pi_0 \downarrow \\ S_t := S & & S_{t-1} & & \cdots & & S_0 \end{array}$$

where for each k

(1) ϕ_k is a (toric) blow-down.
(2) π_k is the minimal resolution,
(3) Either $\rho(S_k) = \rho(S_{k-1})$ or $\rho(S_k) = \rho(S_{k-1}) + 1$,
(4) S_0 is a basic surface (See Definition 3.2).

In this case, we also say that S *admits a semicascade to* S_0.

Note that, unlike the Picard number one case, we do not require that S_k is a log del Pezzo surface for each k.

Theorem 1.5. *Every singular toric log del Pezzo surface admits a semicascade.*

We introduce the notion of toric graphs to prove Theorem 1.5 by using a graph-theoretic argument. One can immediately read off the information of the $\mathbb{P}^1$-fibration structure on the corresponding smooth toric surface S'_k. See Section 2 for toric graphs.

Conversely, by inverting the semicascade process, one can obtain all toric log del Pezzo surfaces.

Theorem 1.6. *Except for* $\mathbb{P}(1,1,n)$*, the minimal resolution of every toric log del Pezzo surface is obtained from a basic toric surface by a semiinverting* (*See Definition 3.4 for semiinverting*).

As an illustration, we recover some results of [3] and [11]. See Theorem 4.1 and Theorem 4.2.

2. Toric graphs

For convenience, we use the following notation in this paper.

Notation 2.1. (1) The vertex-weighted graph $\overset{n_1}{\circ} - \overset{n_2}{\circ} - \cdots - \overset{n_k}{\circ}$ is denoted by $[n_1, n_2, \ldots, n_k]$.
(2) The vertex-weighted cycle with weights $n_1, n_2, \ldots, n_k$ in order is denoted by $[[n_1, n_2, \ldots, n_k]]$.

Remark 2.2. (1) The toric graph $[[1, 1, 1]]$ corresponds to the projective plane.
(2) The toric graph $[[-n, 0, n, 0]]$ corresponds to the Hirzebruch surface of degree n.

2.1. Toric graphs

We will use the following notion of a toric graph to describe a smooth projective toric surface.

Definition 2.3. (1) A *blowup* at an edge e of a vertex-weighted cycle G is a vertex-weighted cycle G' obtained by the following procedure.
   (a) Add a new vertex v of weight -1 and new edges between v and w_i for $i = 1, 2$ where w_1 and w_2 are the adjacent vertices of e.
   (b) Delete the edge e.
   (c) Decrease each of the weights of w_1 and w_2 by 1.
(2) A *blowdown* at a vertex v of a vertex-weighted cycle G is a vertex-weighted cycle G' obtained by the following procedure.
   (a) Add an edge between the adjacent vertices v_1 and v_2 of v.
   (b) Delete the vertex v and its adjacent edges.
   (c) Increase each of the weights of v_1 and v_2 by 1.
(3) A *toric graph* G is either $[[1, 1, 1]]$, $[[-n, 0, n, 0]]$ for some $n \geq 0$, or a vertex-weighted cycle obtained from them by a finite sequence of blowups.
   (a) G is said to be *of type* (O) if all the adjacent vertices of every (-1)-vertex have weight at least -2.
   (b) G is said to be *of type* (I) if it is not of type (O) and there is a (-1)-vertex one of whose adjacent vertices has the weight -2.
   (c) G is said to be *of type* (II) if it is neither of type (O) nor of type (I).
(4) We say that *there is a map* $G \to G'$ from a toric graph G to a toric graph G' if G can be obtained from G' by a finite sequence of blowups, or equivalently, G' can be obtained from G by a finite sequence of blowdowns.

Remark 2.4. The toric graph can be regarded as a 'minimal resolution' of a WVE^{2}C-graph introduced in [1].

Note that every toric graph is either of type (O), of type (I), or of type (II). The following lemma immediately follows from the standard theory of rational surfaces.

Lemma 2.5. *Let G be a toric graph. Then $N = 12 - 3n$ where N is the sum of all weights and n is the number of vertices.*

Definition 2.6. Let H be a subgraph of a toric graph G.
(1) H is said to be *contractable* if each of its vertices has weight at most -2.

(2) The *length* of H is defined to be the number of vertices of H.
(3) The *weight* of the vertex v of G is denoted by $wt(v)$. A vertex of weight n is said to be an *(n)-vertex*.
(4) A contractable subgraph H of G is said to be *maximally connected* if $wt(a) \geq -1$ and $wt(b) \geq -1$ where a and b are the adjacent vertices of H.
(5) G is said to be *singular* if it contains a contractible subgraph.
(6) We say that G *has t singular points* if G has t maximally connected contractible subgraphs.

The following easy observation is useful.

Lemma 2.7. (1) *Taking a blowup monotonically decreases each weight.*
(2) *Taking a blowup monotonically increases the number of maximally connected contractable subgraphs of a toric graph.*

2.2. Fibers of a toric graph

Definition 2.8. A vertex-weighted graph $H = [n_1, n_2, \ldots, n_k]$ is said to be a *fiber* of a toric graph G if it satisfies the following three conditions.
(1) H is a subgraph of G.
(2) Every vertex of H has a non-positive weight, i.e., $n_i \leq 0$ for each i.
(3) $((n_1, n_2, \ldots, n_k)) = 0$ where $((n_1, n_2, ..., n_k)) = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_k}}}$ denotes the Hirzebruch-Jung continued fraction.

Definition 2.9. Let $F = [n_1, n_2, \ldots, n_k]$ be a fiber of a toric graph G.
(1) F is said to be of *type* (O) if its length is at most two.
(2) F is said to be *of type* (I_0) if $F = [-2, -1, -2]$.
(3) F is said to be *of type* (I) if there is a map from F to a fiber of type (I_0).
(4) F is said to be of *type* (II_0) if F is of the form $[-1, -2, \ldots, -2, -1]$,
(5) F is said to be *of type* (II) if there is a map from F to a fiber of type (II_0).

Example 2.10. There are exactly two fibers $[0]$ and $[-1, -1]$ of type (O).

Remark 2.11. The fiber indeed corresponds to the fiber of a $\mathbb{P}^1$-fibration on the smooth toric surface corresponding to the toric graph. See Notation 2.15.

Lemma 2.12. (1) *Being a fiber is invariant under taking a blowup or a blowdown.*
(2) *Let G be a toric graph that is not* $[[1, 1, 1]]$. *Then there is a map $G \to G'$ to a dual graph G' of a Hirzebruch surface.*
(3) *If $H = [n_1, \ldots, n_k]$ is a fiber of a toric graph G, then there exists another fiber $H' = [m_1, \ldots, m_t]$ of G such that $G = [[s_1, n_1, \ldots, n_k, s_2,$*

$m_1, \dots, m_t]]$ for some s_1 and s_2. In this case, we also denote G by $[[s_1, H, s_2, H']]$.

(4) *If G is a toric graph that is neither $[[1,1,1]]$ nor $[[-n, 0, n, 0]]$ for some integer $n \geq 0$, then there exist at most two vertices of nonnegative weights.*

(5) *Let v be a vertex of a toric graph G that is not $[[1,1,1]]$. If $wt(v) > 0$, then there exist two fibers H_1 and H_2 of G such that $G = [[v, H_1, v', H_2]]$ with $wt(v') < 0$.*

Proof. These are well known facts from $\mathbb{P}^1$-fibrations on rational surfaces. See [5] and [9] for the details. (1) is an easy exercise on the finite negative continued fraction. (2) follows from the standard theory of rational surfaces or the toric Mori theory. (3) and (5) follow from (1) and (2). (4) follows from (1), (2) and the fact that taking a blowup decreases weights. □

2.3. Toric graphs of type (*O*)

Notation 2.13. (1) $G(\mathbb{P}^2) = [[1,1,1]]$.

(2) $G_n(0,0) = [[-n, 0, n, 0]]$.

(3) $G_n(0,1) = [-n, 0, n-1, -1, -1]]$.

(4) $G_n(1,1) = [-n, -1, -1, n-2, -1, -1]]$.

(5) $G_n(0,2) = [[-n, 0, n-1, -2, -1, -2]]$.

(6) $G_n(1,2) = [[-n, -1, -1, n-2, -2, -1, -2]]$.

(7) $G_n(2,2) = [[-n, -2, -1, -2, n-2, -2, -1, -2]]$.

(8) $G_2(2,3) = [[-2, -2, -1, -2, -1, -1, -2, -1]]$.

(9) $G_2(2,4) = [[-2, -2, -1, -2, -2, -1, -2, -2, -1]]$.

Proposition 2.14. *A toric graph of type (O) is one of the following graphs: $G(\mathbb{P}^2)$, $G_n(0,0)$, $G_n(0,2)$, $G_n(2,2)$ for every n; $G_n(0,1)$, $G_n(1,1)$, $G_n(1,2)$ for $n = 0,1,2$; $G_2(2,3)$ and $G_2(2,4)$.*

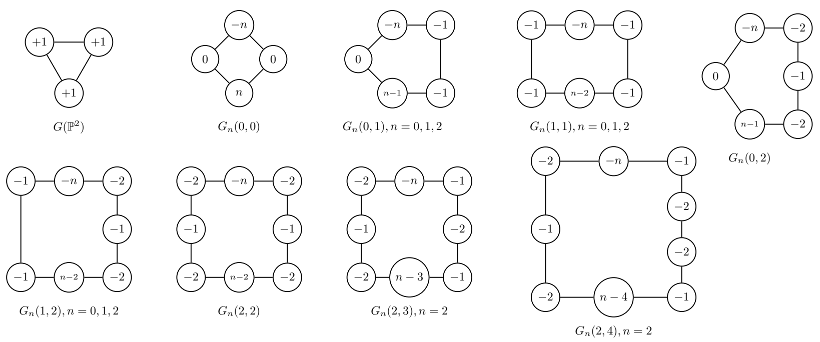


FIGURE 1. Toric graphs of type (O)

Proof. Let G be a toric graph of type (O). If there is no (-1)-vertex, G is either $G(\mathbb{P}^2)$ or $G_n(0,0)$ with $n \neq 1$. So we may assume that G has a (-1)-vertex. If every vertex of G has weight at least -1, then it is easy to see that G is either $G_1(0,0)$, $G_1(1,1)$ or $G_n(0,1)$ with $n = 0, 1$ by Lemma 2.7(1). Thus we may assume that there also exists a vertex v of G with $wt(v) \leq -2$.

We claim that except for $G_n(0,0)$ there exists a (-1)-vertex e adjacent to a maximally connected contractible subgraph of G. Assume the claim is not true. Then there exists an adjacent vertex e of v_1 with $wt(e) \neq -1$ where v_1 is an end vertex of a maximally connected contractible subgraph of G. Then $wt(e) \geq 0$. If $wt(e) > 0$, by Lemma 2.12(5), there exists a fiber F adjacent to e containing the vertex v_1. Since G is of type (O), we see that $F = [-2, -1, -2]$, which is a contradiction to the assumption. Thus $wt(e) = 0$. By Lemma 2.12(3), there exists a fiber containing the other adjacent vertex w of v_1. By a similar argument as above, we see that $G = G_n(0,0)$ with $n = -wt(v_1)$. This completes the proof of the claim.

Let e be a (-1)-vertex adjacent to a vertex v_1 of a maximally connected contractible subgraph of G. Let w be the other adjacent vertex of e. Since G is of type (O), $wt(w) \geq -2$. Assume first that $wt(w) = wt(v_1) = -2$. Since $[v_1, e, w]$ is a fiber of type (I), by Lemma 2.12(3), there exists another fiber F such that $G = [[s, v_1, e, w, s', F]]$ for some s and s'. It is easy to see that the fiber F is of type (O), (I_0) or (II_0) since G is of type (O). Then, by Lemma 2.5 and considering the effect of the symmetric shape of the graph with respect to s and s', G is one of the following 5 graphs: $G_n(2,3)$, $G_n(2,4)$, $G_n(1,2)$ with $n = 0, 1, 2$, $G_n(0,2)$ and $G_n(2,2)$ for every n.

Now we may assume that $wt(w) = -1$ and there is no (-1)-vertex such that both of its adjacent vertices have weight -2. In particular, there is no fiber of type (I). Since $[e, w]$ forms a complete fiber, by Lemma 2.12(3), there exists another fiber F such that $G = [[v_1, e, w, s, F]]$ for some s. Since G is of type (O), by assumption, we see that the fiber F is either $[0]$ or $[-1, -1]$. By Lemma 2.5, we see that $G = G_2(0,1)$ in the first case and $G = G_2(1,1)$ in the latter case.

Finally, we may assume that the other adjacent vertex of every (-1)-vertex adjacent to a maximally connected contractible subgraph of G have nonnegative weights. Let e be a (-1)-vertex adjacent to a maximally connected contractible subgraph G' of G and w be the other adjacent vertex of e. By assumption, $wt(w) \geq 0$. If $wt(w) > 0$, by Lemma 2.12(5), there exists two fibers H_1 and H_2 such that $G = [[w, H_1, s, H_2]]$ for some s where H_1 contains e. Since G is of type (O), H_1 is of type (II_0), so $wt(s) \geq 0$ by assumption, which is a contradiction by Lemma 2.12(5). If $wt(w) = 0$, by Lemma 2.12(3), there exists another fiber F such that $G = [[s, w, e, F]]$ for some s. By assumption we see that F is of type $[-1, -1]$ or of type (II_0). In the first case, we have $G = G_1(0,1)$, a contradiction since G has a vertex v with $wt(v) \leq 2$. In the latter case, e is adjacent to a (-1)-vertex that is adjacent to a maximally connected contractible subgraph of G, a contradiction. □

See Figure 2 for the maps between some toric graphs of type (O).

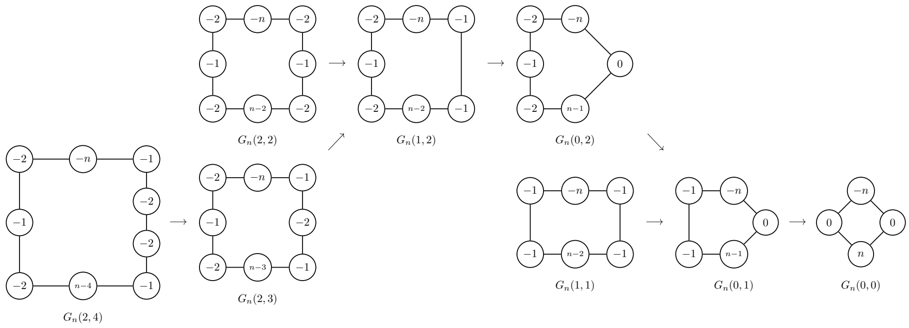


FIGURE 2. Maps between toric graphs of type (O)

2.4. Ray generators of the fan

In this section, we present explicit coordinates for the ray generators of the fans corresponding to toric graphs of type (O).

Notation 2.15. For each toric graph G in Notation 2.13, we denote by S' the corresponding smooth toric surface and by S the contraction of all rational curves with self-intersection number at most -2. We denote by P' the lattice polytope corresponding to S' and by P the lattice polytope corresponding to S.

See Table 1 for the explicit coordinates in $\mathbb{R}^2$. Note that $P_0(0,2) = P_1(0,2)$, $P_0(1,1) = P_2(1,1)$, $P_0(1,2) = P_2(1,2)$, and $P_0(2,2) = P_2(2,2)$.

TABLE 1. Ray generators of the fans of type (O)

$P(\mathbb{P}^2)$	$\{(0,1),(-1,-1),(1,0)\}$
$P_n(0,0)$	$\{(0,1),(-1,0),(n,-1),(1,0)\}$
$P_n(0,1)$	$\{(0,1),(-1,1),(-1,0),(n,-1),(1,0)\}$
$P'_n(1,1), n \geq 1$	$\{(0,1),(-1,1),(-1,0),(n-1,-1),(n,-1),(1,0)\}$
$P'_n(0,2), n \geq 1$	$\{(1,1),(-n,-n+1),(-1,-1),(0,-1),(1,-1),(1,0)\}$
$P'_n(1,2), n \geq 1$	$\{(1,1),(-n+1,-n+2),(-n,-n+1),(-1,-1),(0,-1),(1,-1),(1,0)\}$
$P'_1(2,2)$	$\{(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1),(1,0)\}$
$P'_n(2,2), n \geq 2$	$\{(-1,-1),(1,0),(3,1),(2,1),(1,1),(n-2,n-1),(2n-5,2n-3),(n-3,n-2)\}$
$P'_2(2,3)$	$\{(1,1),(0,1),(-1,0),(-2,-1),(-1,-1),(0,-1),(1,-1),(1,0)\}$
$P'_2(2,4)$	$\{(1,2),(0,1),(-1,0),(-2,1),(-1,-1),(0,-1),(1,-1),(1,0),(1,1)\}$

See Figure 3 for the drawings of some Fano polygons of type (O). See [8] for more about Fano polytopes.

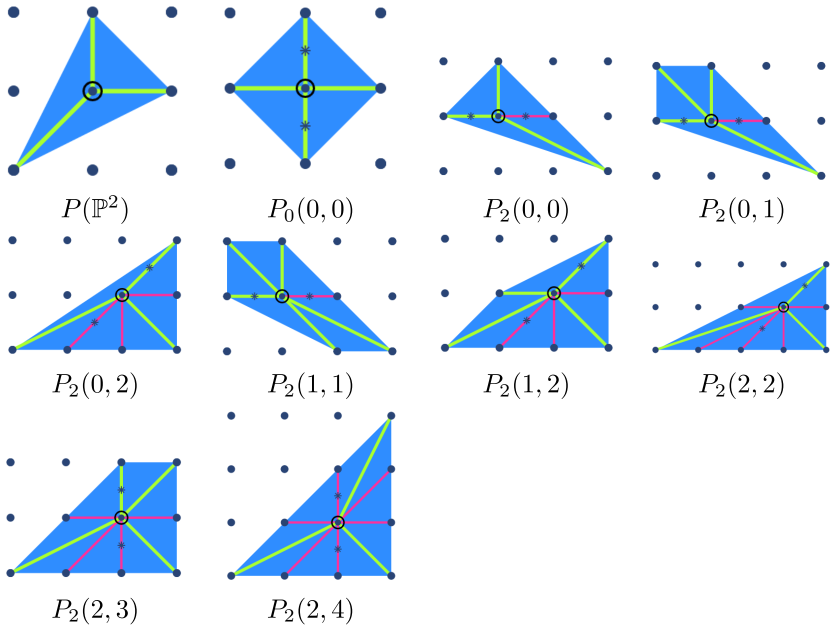


FIGURE 3. Some Fano polygons of type (O)

3. Cascades of toric log del Pezzo surfaces

Note that the set of all toric graphs has a one-to-one correspondence with the set of all smooth projective toric surfaces up to natural isomorphisms. In this correspondence, a toric graph with n vertices corresponds to a smooth projective toric surface of Picard number $n-2$.

Example 3.1. Note that $S' = S$ for smooth toric del Pezzo surfaces. The cascades for five smooth toric del Pezzo surfaces can be written as follows:

$$S_1(1,1) \to S_1(0,1) \to S_1(0,0) = \mathbb{F}_1 \to S(\mathbb{P}^2) = \mathbb{P}^2, S_0(0,0) = \mathbb{F}_0.$$

Note that for a toric graph G of type (O) the corresponding toric surface S is a toric log del Pezzo surface since the morphism $S' \to S$ is a minimal resolution. For example, $S_2(2,4)$ is the cubic surface with three cusps, i.e., the toric log del Pezzo surface of Picard number one with 3 singular points of type A_2.

3.1. Semicascades of toric log del Pezzo surfaces

Definition 3.2. The toric log del Pezzo surface is said to be *of type* (O) if the corresponding toric graph is of type (O). A singular toric log del Pezzo surface of type (O) is called *a basic toric log del Pezzo surface* (or *a basic surface* in short) if it is not isomorphic to $S_n(0,0)$.

Definition 3.3. Let S' be the minimal resolution of a toric log del Pezzo surface. A point of S' is said to be *invertible* if it is the intersection point of two negative rational curves, not both of them being (-1)-curves.

Definition 3.4. Let S be a toric log del Pezzo surface.

(1) We say that S *admits a one-step semicascade* if there exists a diagram as follows:

$$\begin{array}{ccc} S' & \xrightarrow{\phi} & \bar{S}' \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \bar{\pi}}\big\downarrow \\ S & & \bar{S} \end{array}$$

where
  (a) ϕ is a blow-down of a (-1)-curve,
  (b) π and $\bar{\pi}$ are minimal resolutions, and
  (c) $\bar{S}$ is a projective toric surface with $\rho(S) = \rho(\bar{S})$ or $\rho(S) = \rho(\bar{S})+1$.

(2) A one-step semicascade is said to be *of type* (I) if $\rho(S) = \rho(\bar{S})$.

(3) A one-step semicascade is said to be *of type* (II) if $\rho(S) = \rho(\bar{S}) + 1$.

(4) A one-step semicascade is said to be a *one-step cascade* if $\bar{S}$ is a log del Pezzo surface.

(5) We say that a toric log del Pezzo surface $\bar{S}$ *admits a one-step semiinverting* if there exists a diagram in (1) exists where (c) is replaced by (c') S is a projective toric surface with $\rho(S) = \rho(\bar{S})$ or $\rho(S) = \rho(\bar{S})+1$.

(6) We say that a toric log del Pezzo surface $\bar{S}$ *admits a one-step inverting* if it admits a one-step semiinverting such that S is a toric log del Pezzo surface.

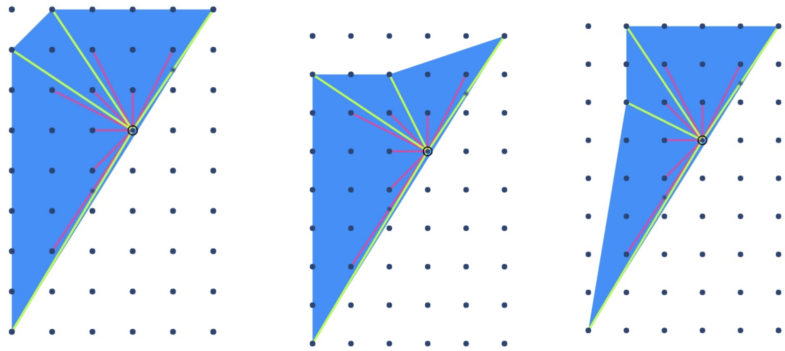
Proposition 3.5. *The number of singular points is monotonely increasing under a one-step semicascade.*

3.2. Existence of a semicascade (=Proof of Theorem 1.5)

Let S be a toric log del Pezzo surface and S' be its minimal resolution. Let G be the corresponding toric graph of S'. Then G is either of type (O), of type (I), or of type (II). Note that a toric graph of type (I) (or (II), resp.) induces a one-step semicascade of type (I) (or (II), resp.) to the corresponding toric log del Pezzo surface. Now Proposition 2.14 completes the proof.

3.3. Semiinverting (=Proof of Theorem 1.6)

As we already saw in the proof of Theorem 1.5, a toric graph of type (I) (or (II), resp.) induces a one-step semicascade of type (I) (or (II), resp.). In this process, the blowing-up locus of each one-step semiinverting is exactly one of the invertible points. Note that $\mathbb{P}(1,1,n)$ for some integer $n \geq 0$ is not a basic surface but the corresponding toric graph $G_n(0,0)$ is of type (O) with no invertible points. This completes the proof.



P P_1 P_2

FIGURE 4.

3.4. An example

Consider the toric log del Pezzo surface S corresponding to P in Figure 4. By Theorem 1.5, S admits a one-step semicascade. However, one can see that S does not admit a one-step cascade since any one-step semicascade results in a nonconvex lattice polygon, hence is not Fano, as P_1 and P_2 in Figure 4 show. Note that S has 3 singularities of type $[2,4,2],[5],[2,3,3,2]$ and $\rho(S)=2$.

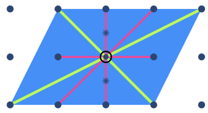
4. Applications

4.1. Bounds on the Picard number: Proof of Theorem 1.1

Let s be the number of smooth torus-fixed points. Then it is easy to see that $t+s=\rho+2$, so $\rho-t=s-2$. Since S is singular, by Theorem 1.6, it is enough to count the number of maximal dimensional smooth cones of the fan corresponding to basic surfaces. Now, by looking at Figure 1, it is easy to see that $s\leq 4$ and $s=4$ if and only if $S=S_n(1,1)$ with $n\geq 2$ since S is singular. This completes the proof.

4.2. Kähler–Einstein toric log del Pezzo surfaces: Proof of Theorem 1.2

By Theorem 1.6, it is enough to consider the basic surfaces and its semi-invertings. Recall that a toric log del Pezzo surface admits a Kähler–Einstein metric if and only if the moment polygon has the origin as its barycenter by [10]. A Fano polygon is said to be *Kähler–Einstein* if the corresponding toric log del Pezzo surface admits a Kähler–Einstein metric.



Let T be the toric log del Pezzo surface whose corresponding Fano polygon has vertices $(2,1)$, $(-1,-1)$, $(-2,-1)$, and $(1,-1)$ as in the above figure. Note that T can be obtained from $S_2(2,3)$ by two one-step invertings. Then we can easily see that $S_2(2,3)$, $S_1(2,2)$ and T are Kähler–Einstein. By looking at their Fano polygons, it is easy to find infinitely many Kähler–Einstein Fano polygons by taking suitable semiinvertings of $S_2(2,3)$, $S_1(2,2)$, $S_2(2,4)$, or T.

For the other basic surfaces, by using the coordinates of the corresponding Fano polygons from Table 1, we see that none of them or their semiinvertings are Kähler–Einstein since the y-coordinate of the dual polygon is always negative. This completes the proof.

4.3. Toric log del Pezzo surfaces with at most two singular points

We recover the classification results in [3] and [11].

Theorem 4.1 ([3])**.** *Let S be a toric log del Pezzo surface with one singular point. Then S is one of the following*: (1) $S_n(0,0)$, (2) $S_n(0,1)$, (3) $S_n(1,1)$, *for some integer $n \geq 2$. In particular, the singularity type of S is always $\frac{1}{n}(1,1)$.*

Proof. By Theorem 1.6 and Proposition 3.5, we only need to consider basic surfaces with 1 singular point, i.e., $S_n(0,0)$, $S_n(0,1)$, $S_n(1,1)$ for $n \geq 2$. Note that any one-step semiinverting results in a toric surface with at least two singular points. This completes the proof. □

Theorem 4.2 (cf. [11])**.** *Let S be a toric log del Pezzo surface with two singular points. Then S admits a semicascade, only consisting of finitely many semicascades of type (I), to one of the following*: (1) $G_n(0,1)$ *with* $n \geq 2$, (2) $G_n(1,1)$ *with* $n \geq 2$, (3) $G_n(0,2)$ *with* $n \geq 1$, (4) $G_n(1,2)$ *with* $n \geq 1$. *Moreover, the toric graph of S is one of the following*: $[[-n-1,0,n-1,F_2]]$, $[[-n-1,-1,-1,n-2,F_2]]$, $[[-n,0,n-1,F_1]]$, $[[-n,-1,-1,n-2,F_1]]$ *where F_1 is a fiber of type (I) and F_2 is a fiber of type (II).*

Proof. By Proposition 3.5, we see that S is obtained by semiinverting from $S_n(0,1)$, $S_n(1,1)$, $S_n(0,2)$, $S_n(1,2)$. Consider the first two cases. If $n=1$, it has no singular points and no invertible points. Assume that $n \geq 2$. Taking a one-step semiinverting at any invertible point produces a toric log del Pezzo surface with 2 singular points, and the minimal resolution has a $\mathbb{P}^1$-fibration with a fiber of type (II). Consider the remaining two cases. In this case, S already has two singular points. Note that a semiinverting at some invertible point leads to a toric surface with three singular points. For instance, $S_1(0,2)$ has 3 invertible points but one of them induces a semiinverting that produces a toric surface with three singular points. Avoiding such invertible points, the semiinverting preserves the fiber of type (I). This completes the proof. □

Remark 4.3. One can further classify toric log del Pezzo surfaces with more singular points by a similar but more detailed analysis.

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