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ON WEIGHTED BROWDER SPECTRUM

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Abstract. The main aim of the article is to introduce new generalizations of Fredholm and Browder classes of spectra when the underlying Hilbert space is not necessarily separable and study their properties. To achieve the goal the notions of α-Browder operators, α-B-Fredholm operators, α-B-Browder operators and α-Drazin invertibility have been introduced. The relation of these classes of operators with their corresponding weighted spectra has been investigated. An equivalence of α-Drazin invertible operators with α-Browder operators and α-B-Browder operators has also been established. The weighted Browder spectrum of the sum of two bounded linear operators has been characterised in the case when the Hilbert space (not necessarily separable) is a direct sum of its closed invariant subspaces.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators where $\mathcal{H}$ is a not necessarily separable complex Hilbert space of infinite dimension h where $\aleph_0 \leq h$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a Fredholm operator [1] if $n(T) < \infty$, $\beta(T) < \infty$ and $R(T)$, the range of T is closed, where $n(T)$ is the nullity of T and $\beta(T)$ is codimension of $R(T)$. $\Psi(\mathcal{H})$ shall denote the set of all Fredholm operators. For a cardinal number $\alpha \geq \aleph_0$, a subset $S \subset \mathcal{H}$ is said to be α-bounded [4, 7], if for each $\epsilon > 0$, there exists a set of points $\{r_i\}_{i \in I}$, $card(I) < \alpha$ and $r_i \in S$ with $S \subset \cup_{i \in I} B(r_i, \epsilon)$, where $B(r_i, \epsilon)$ is the open ball with radius ϵ and center r_i. A linear map $T \in \mathcal{L}(\mathcal{H})$ is said to be α-compact [4, 7] if the image $T(S)$ of each bounded subset $S \subset \mathcal{H}$ is α-bounded. We denote the closed two-sided ideal of all α-compact operators by $\mathcal{K}_\alpha(\mathcal{H})$. A subset $S \subset \mathcal{H}$ is called α-closed [4, 7] if there is a closed subspace $M \subset \mathcal{H}$ such that $M \subset S$ and $\dim(S \cap M^\perp) < \alpha$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called α-Fredholm (resp. upper semi α-Fredholm, lower semi α-Fredholm) operator [7], i.e., $T \in \Psi_\alpha(\mathcal{H})$ (resp. $\Psi_\alpha^+(\mathcal{H})$, $\Psi_\alpha^-(\mathcal{H})$), if $R(T)$ is α-closed and $\max\{n(T), n(T^*)\} < \alpha$ (resp. $n(T) < \alpha$, $n(T^*) < \alpha$),

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where T^* is the adjoint of T. Let $\phi_\alpha : \mathcal{L}(\mathcal{H}) \to \mathcal{C}_\alpha(\mathcal{H})$ denote the canonical homomorphism, where $\mathcal{C}_\alpha(\mathcal{H})$ denotes the quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}_\alpha(\mathcal{H})$. An Atkinson type of characterization for α-Fredholm operators was given by Edgar, Ernest and Lee in [7] as

$$T \in \Psi_\alpha(\mathcal{H}) \Leftrightarrow \phi_\alpha(T) \text{ is invertible in } \mathcal{C}_\alpha(\mathcal{H})$$

which implies

$$\sigma(\phi_\alpha(T)) = \sigma_\alpha(T), \text{ where } \sigma_\alpha(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_\alpha(\mathcal{H}\}$$

is the weighted spectrum with weight α. Since

$$\sigma_\alpha(T) \subset \sigma_e(T) = \sigma_{\aleph_0}(T) \subset \sigma(T)/\sigma_d(T),$$

where $\sigma_d(T)$ is the discrete spectrum of T, $\sigma(T)$ is the spectrum of T and $\sigma_e(T)$ is the essential spectrum of T. The non-discrete part of the spectrum is explored using a weighted spectrum. In 2018, Athmouni, Baloudi, Jeribi and Kacem in [4] discussed the relation between the weighted spectrum of the sum of two bounded linear operators and their weighted spectra. The present paper extends this study to the weighted Browder spectrum, α-B-Fredholm operator, α-B-Browder operator and α-Drazin invertible operator. The relation between these operators under some conditions along with their properties has also been discussed.

The paper has been organised in the following manner. Some preliminary definitions have been introduced in Section 2. The relation between the operators and corresponding spectra defined in Section 2 has been established in Section 3. Section 4, aims at the main result of the paper which is about the weighted Browder spectrum (with weight α) of the sum of two operators and property of operators defined on the direct sum of two arbitrary Hilbert spaces. The conclusion have been recorded in Section 5.

2. Preliminaries

This section establishes some preliminary definitions which are required in the remaining sections. The important class of Fredholm operators was extended to the class of α-Fredholm operators on a non-separable Hilbert space $\mathcal{H}$ of uncountably infinite dimension by Edgar, Luft and Lee in [7] for an operator $T \in \mathcal{L}(\mathcal{H})$ whereas Berkani [5] extended this theory to B-Fredholm operators. Recall that T is semi B-Fredholm (upper semi B-Fredholm, lower semi B-Fredholm and B-Fredholm) operator, if for some integer $n \geq 0$, the range $R(T^n)$ is closed and the operator $T_{[n]} = T|_{R(T^n)}$ is a semi Fredholm (resp. upper semi Fredholm, lower semi Fredholm and Fredholm) operator. Motivated by these ideas, in the present article, the notions of upper semi-α-B-Fredholm and lower semi-α-B-Fredholm operators have been introduced. An operator T has been introduced which is upper semi α-B-Fredholm operator or lower semi α-B-Fredholm operator if $R(T^n)$ is α-closed and $T_{[n]}$ is respectively upper or lower semi α-Fredholm operator for some integer $n \geq 0$. The classes of upper semi α-B-Fredholm operator and lower semi α-B-Fredholm operator shall be denoted by $\Psi^{+}_{\alpha BF}(\mathcal{H})$ and $\Psi^{-}_{\alpha BF}(\mathcal{H})$, respectively. $\Psi_{\alpha BF}(\mathcal{H})$ shall denote the class of α-B-Fredholm operators where $\Psi_{\alpha BF}(\mathcal{H})= \Psi^{+}_{\alpha BF}(\mathcal{H}) \cap \Psi^{-}_{\alpha BF}(\mathcal{H})$.

For a Banach space X, let $T \in \mathcal{L}(X)$, the ascent $p = p(T)$ of T is the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$, where $N(T)$ denotes the null space of T. If such an integer does not exist, we put $p = \infty$. Similarly, the descent $q = q(T)$ of T is the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$. If no such integer exists, then we say that descent $q(T)$ is infinite, where $R(T)$ denotes the range space of T. An operator $T \in \mathcal{L}(X)$ is left Drazin invertible [1] if the ascent $p = p(T) < \infty$ and $R(T^{p+1})$ is closed, and $T \in \mathcal{L}(X)$ is right Drazin invertible if the descent $q = q(T) < \infty$ and $R(T^q)$ is closed. This gives impetus to the following idea where an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be α-left Drazin invertible if $p = p(T) < \infty$ and $R(T^{p+1})$ is α-closed and α-right Drazin invertible if $q = q(T) < \infty$ and $R(T^q)$ is α-closed, denoted by $\Psi_{\alpha lD}(\mathcal{H})$ and $\Psi_{\alpha rD}(\mathcal{H})$, respectively. T is α-Drazin invertible if and only if $T \in \mathcal{L}(\mathcal{H})$ is both α-left and α-right Drazin invertible. It is denoted by $\Psi_{\alpha D}(\mathcal{H})$.

Semi Browder operators are an important class of operators in Fredholm theory. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be Browder (upper semi Browder, lower semi Browder) operator [6] if T is Fredholm (resp. upper semi Fredholm, lower semi Fredholm) operator and $p(T) = q(T) < \infty$ (resp. $p(T) < \infty$, $q(T) < \infty$) where $p(T)$ and $q(T)$ denote the ascent and descent of T, respectively. Several variations on separable Hilbert spaces are available in the literature. The following generalization is motivated by these ideas.

$\Psi^{+}_{\alpha B}(\mathcal{H})$ and $\Psi^{-}_{\alpha B}(\mathcal{H})$ shall denote the classes of upper semi α-Browder and lower semi α-Browder operators where $\Psi^{+}_{\alpha B}(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) :$ if T is a upper semi α-Fredholm operator and $p(T) < \infty\}$, $\Psi^{-}_{\alpha B}(\mathcal{H}) =\{T \in \mathcal{L}(\mathcal{H}) :$ if T is a lower semi α-Fredholm operator and $q(T) < \infty\}$, and $\Psi_{\alpha B}(\mathcal{H})=\{ T \in \mathcal{L}(\mathcal{H}) :$ if T is an α-Fredholm operator and $p(T) = q(T) < \infty\}$ is the class of α-Browder operators. Indeed

$$\Psi_{\alpha B}(\mathcal{H}) = \Psi^{+}_{\alpha B}(\mathcal{H}) \cap \Psi^{-}_{\alpha B}(\mathcal{H}).$$

The following example shows that the classes of α-Browder operators and α-Drazin invertible operators are not void.

Example 2.1. Let $X = \oplus_{\alpha\in\Delta} l^2$, where delta is an uncountable index set. By definition each element of space X is $(x_\alpha)_{\alpha\in\Delta}$ such that $x_\alpha = 0$ for all but finitely many α. Define, $\|\bar{x}\| = (\Sigma_{\alpha\in\Delta}\|x_\alpha\|^2)^{1/2} < \infty$. Then space X with this norm is a complete norm space. Now, let $\{\bar{e}_\alpha\}_{\alpha\in\Delta}$ be a set in X, where $\bar{e}_\alpha = (e_{\alpha\beta})_{\beta\in\Delta}$ such that

$$e_{\alpha\beta} = \begin{cases} e, & \alpha = \beta, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $e \in l^2$ such that $\|e\| = 1$. Then $\{\bar{e}_\alpha\}_{\alpha\in\Delta}$ is an uncountable set in X such that $\|\bar{e}_\alpha\| = 1$ for all $\alpha \in \Delta$. Therefore X is a non-separable Hilbert space. Let

$\dim(X)$ be h. Now define an operator $T_0 : l^2 \to l^2$ as

$$T_0(x) = T_0(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_2, \xi_3, \xi_4, \ldots, \xi_n, 0, 0, 0, \ldots) \text{ for all } x \in l^2,$$

where n is a fixed finite positive integer. Then,

$$T_0^n(x) = 0 \text{ for all } x \in l^2.$$

So, $p(T_0) = n = q(T_0) < \infty$, $n(T_0) = \aleph_0$ and $n(T_0^*) = \aleph_0$.

Let $T \in \mathcal{L}(X)$, defined as

$$T = \oplus_{\alpha\in\Delta} T_\alpha,$$

where

$$T_\alpha = \begin{cases} T_0, & \alpha = \alpha_0, \\ I, & \text{otherwise}, \end{cases} \tag{2.2}$$

and I is an identity map on l^2 space. Then, $N(T) \cong N(T_0)$ and $N(T^*) \cong N(T_0^*)$. Therefore, $n(T) = n(T_0) = \aleph_0$ and $n(T^*) = n(T_0^*) = \aleph_0$.

We can observe that, $R(T) \cong R(T_0) \oplus_{\Delta\setminus\{\alpha_0\}} l^2$, which is a closed subspace of X.

Now, $T^n = \oplus_{\alpha\in\Delta} T_\alpha^n$, where

$$T_\alpha^n = \begin{cases} 0, & \alpha = \alpha_0, \\ I, & \text{otherwise}. \end{cases} \tag{2.3}$$

Then, $N(T^n) = N(T^{n+1})$ and $R(T^n) = R(T^{n+1})$. Hence, for $\aleph_0 < \alpha \leq h$, T is an α-Browder and α-Drazin invertible operator.

Now we begin by exploring that onto operators are necessarily α-Fredholm and α-Browder operators.

Lemma 2.2. *For $T \in \mathcal{L}(\mathcal{H})$*

(1) *If T is onto, then T is a lower semi α-Fredholm operator.*
(2) *If T is onto, then T is a lower semi α-Browder operator.*
(3) *If there exists some integer $n \geq 0$ such that $R(T^n)$ is α-closed and $T_{[n]}$ is onto, then T is a lower semi α-B-Fredholm operator.*

Proof. (1) Since T is onto. T is a lower semi α-Fredholm operator.

(2) By ontoness of T and by (1), T is the lower semi α-Browder operator.

(3) Since $T_{[n]}$ is onto, then $T_{[n]}$ is the lower semi α-Fredholm operator. Therefore, T is a lower semi α-B-Fredholm operator. □

Carpintero et al. [6] have defined a bounded operator $T \in \mathcal{L}(\mathcal{H})$ to be B-Browder if for some integer $n \geq 0$, the range $R(T^n)$ is closed and $T_{[n]}$ is Browder. Motivated by this we have introduced the notion of the α-B-Browder operator where a bounded operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an α-B-Browder (upper semi α-B-Browder, lower semi α-B-Browder) operator if for some integer $n \geq 0$, the range $R(T^n)$ is α-closed and $T_{[n]}$ is an α-Browder (resp. upper semi α-Browder, lower semi α-Browder) operator. The corresponding classes of these operators are denoted by $\Psi_{\alpha BB}(\mathcal{H})$ (resp. $\Psi^{+}_{\alpha BB}(\mathcal{H}), \Psi^{-}_{\alpha BB}(\mathcal{H})$). We can easily observe that the operator defined in Example 2.1 is also an α-B-Browder operator.

Further we define an operator T to be α-bounded below if T is injective and $R(T)$ is α-closed and proceed to confirm that this property compels an operator to be upper semi α-Browder operator.

Proposition 2.3. *For a bounded linear operator $T \in \mathcal{L}(\mathcal{H})$,*

(1) *If T is α-bounded below, then T is the upper semi α-Browder operator.*
(2) *If there exists some integer $n \geq 0$ such that $R(T^n)$ is α-closed and $T_{[n]}$ is α-bounded below, then T is the upper semi α-B-Browder operator.*
(3) *If there exists some integer $n \geq 0$ such that $R(T^n)$ is α-closed and $T_{[n]}$ is onto, then T is the lower semi α-B-Browder operator.*

Proof. (1) Since T is α-bounded below, $n(T) = 0$, $p(T) = 0$ and $R(T)$ is α-closed. Hence, T is the upper semi α-Browder operator.

(2) Let $T_{[n]}$ be α-bounded below, then by (1), $T_{[n]}$ is a upper semi α-Browder operator. Hence T is the upper semi α-B-Browder operator.

(3) Since $T_{[n]}$ is onto, $q(T_{[n]}) = 0 < \infty$, $\beta(T_{[n]}) = 0 < \alpha$ and $R(T_{[n]}) = \mathcal{H}$. Therefore, T is the lower semi α-B-Browder operator. □

Corollary 2.4. *For a Hilbert space $\mathcal{H}$,*

(1) $\Psi^{+}_{\alpha BB}(\mathcal{H}) \subset \Psi^{+}_{\alpha BF}(\mathcal{H})$.
(2) $\Psi^{-}_{\alpha BB}(\mathcal{H}) \subset \Psi^{-}_{\alpha BF}(\mathcal{H})$.
(3) $\Psi_{\alpha BB}(\mathcal{H}) \subset \Psi_{\alpha BF}(\mathcal{H})$.

In this sequel now the following lemma characterises upper (lower) semi α-B-Browder operator when T is injective (resp. onto).

Lemma 2.5. *For $T \in \mathcal{L}(\mathcal{H})$,*

(1) *If T is injective, then T is a upper semi α-B-Browder operator if and only if T is a upper semi α-B-Fredholm operator.*
(2) *If T is onto, then T is a lower semi α-B-Browder operator if and only if T is a lower semi α-B-Fredholm operator.*

Proof. (1) Let T be a upper semi α-B-Fredholm operator and injective. Then $N(T_{[n]}) = \{0\}$ and $p(T_{[n]}) = 0 < \infty$. Hence T is the upper semi α-B-Browder operator.

Conversely, since T is the upper semi α-B-Browder operator. Therefore, T is a upper semi α-B-Fredholm operator.

(2) Similar to (1). □

Motivated by the works of Edger et al. [7] and Jeribi et al. [4] we define $\sigma_{U\alpha B}(T)$, $\sigma_{L\alpha B}(T)$, $\sigma_{\alpha B}(T)$ as;

$$\sigma_{U\alpha B}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi^{+}_{\alpha B}(\mathcal{H})\}.$$

$$\sigma_{L\alpha B}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi^{-}_{\alpha B}(\mathcal{H})\},$$

$$\sigma_{\alpha B}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_{\alpha B}(\mathcal{H})\} = \sigma_{U\alpha B}(T) \cup \sigma_{L\alpha B}(T),$$

and call these as upper semi weighted-Browder spectrum, lower semi weighted-Browder spectrum and weighted-Browder spectrum with weight α, respectively. Hence one can observe that for a bounded linear operator T defined on Hilbert space $\mathcal{H}$,

$$\sigma_{\alpha}(T) \subseteq \sigma_{\alpha B}(T).$$

Also, we define the upper semi weighted B-Fredholm spectrum and lower semi weighted B-Fredholm spectrum of a bounded linear operator T with weight α respectively as the set of all those complex numbers λ for which $T - \lambda I$ is not in $\Psi^{+}_{\alpha BF}(\mathcal{H})$ and $\Psi^{-}_{\alpha BF}(\mathcal{H})$ as;

$$\sigma_{U\alpha BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi^{+}_{\alpha BF}(\mathcal{H})\},$$
$$\sigma_{L\alpha BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi^{-}_{\alpha BF}(\mathcal{H})\}.$$

Hence, the weighted B-Fredholm spectrum is

$$\sigma_{\alpha BF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_{\alpha BF}(\mathcal{H})\} = \sigma_{U\alpha BF}(T) \cup \sigma_{L\alpha BF}(T).$$

We assign

$$\sigma_{U\alpha BB}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi^{+}_{\alpha BB}(\mathcal{H})\},$$
$$\sigma_{L\alpha BB}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi^{-}_{\alpha BB}(\mathcal{H})\},$$

and

$$\sigma_{\alpha BB}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_{\alpha BB}(\mathcal{H})\} = \sigma_{U\alpha BB}(T) \cup \sigma_{L\alpha BB}(T)$$

to the upper semi weighted B-Browder spectrum, the lower semi weighted B-Browder spectrum and the weighted B-Browder spectrum, respectively. Moreover $\sigma_{l\alpha D}(T)$, $\sigma_{r\alpha D}(T), \sigma_{\alpha D}(T)$ are left weighted Drazin invertible spectrum, right weighted Drazin invertible spectrum and weighted Drazin invertible spectrum, respectively, and defined as:

$$\sigma_{l\alpha D}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_{l\alpha D}(\mathcal{H})\},$$
$$\sigma_{r\alpha D}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_{r\alpha D}(\mathcal{H})\},$$

and

$$\sigma_{\alpha D}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Psi_{\alpha D}(\mathcal{H})\}.$$

3. On α-Drazin invertibility and α-Browderness

This section establishes the relation between the operators and their respective spectra defined in the previous section. Jeribi et al. in [4] have shown that, if T_1 and T_2 are α-Fredholm operators in $\mathcal{L}(\mathcal{H})$, then T_1T_2 is also an α-Fredholm operator. Hence, if we consider commuting operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ with finite ascent and descent it is very easy to observe that the ascent and descent of the product T_1T_2 are also finite and this leads to:

Proposition 3.1. *Let T_1 and T_2 be in $\mathcal{L}(\mathcal{H})$ such that $T_1T_2 = T_2T_1$. If $T_1 \in \Psi_{\alpha B}(\mathcal{H})$, $T_2 \in \Psi_{\alpha B}(\mathcal{H})$, then $T_1T_2 \in \Psi_{\alpha B}(\mathcal{H})$.*

The definitions of upper (lower) semi α-B-Browder operator and α-left(right) Drazin invertible operator aid in concluding the following lemmas.

Lemma 3.2. *For $T \in \mathcal{L}(\mathcal{H})$,*

(1) *If $T-\lambda I$ is injective, then $\lambda \in \sigma_{U\alpha BF}(T)$ if and only if $\lambda \in \sigma_{U\alpha BB}(T)$.*
(2) *If $T-\lambda I$ is onto, then $\lambda \in \sigma_{L\alpha BF}(T)$ if and only if $\lambda \in \sigma_{L\alpha BB}(T)$.*

Lemma 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$.*

(1) *If T is α-bounded below, then T is α-left Drazin invertible.*
(2) *If T is onto, then T is α-right Drazin invertible.*

Proof. (1) Let T be α-bounded below. Then $N(T) = \{0\}$, $p(T) < \infty$ and $R(T)$ is α-closed. Hence T is α-left Drazin invertible.

(2) Let T be onto. Then $q = q(T) = 0 < \infty$ and $R(T^q) = \mathcal{H}$. Hence, T is α-right Drazin invertible. □

The surjectivity spectrum [6] of an operator T is defined by

$$\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not onto}\}.$$

Further, these lemmas provide the subsets of the surjectivity spectrum as follows:

Theorem 3.4. *For an operator $T \in \mathcal{L}(\mathcal{H})$,*

(1) $\sigma_{L\alpha F}(T) \subseteq \sigma_s(T)$.
(2) $\sigma_{L\alpha B}(T) \subseteq \sigma_s(T)$.
(3) $\sigma_{\alpha rD}(T) \subseteq \sigma_s(T)$.

The following equivalences are observed when T is injective (onto).

Lemma 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be injective. The following are equivalent*:

(1) *T is α-left Drazin invertible.*
(2) *T is the upper semi α-Browder operator.*

Proof. (1) $\Rightarrow$ (2) In this case $n(T) = 0 < \alpha$, $p = p(T) = 0 < \infty$ and α-closedness of $R(T^{p+1}) = R(T)$ lead to (2).

(2) $\Rightarrow$ (1) Since T is an injective and upper semi α-Browder operator, $p = p(T) = 0 < \infty$ and $R(T^{p+1})$ is α-closed conclude to (1). □

Lemma 3.6. *For an onto operator $T \in \mathcal{L}(\mathcal{H})$ the following dual result holds*:

(1) *T is α-right Drazin invertible.*
(2) *T is a lower semi α-Browder operator.*

Proof. (1) $\Rightarrow$ (2) T is onto and α-right Drazin invertible. Then $n(T^*) = 0 < \infty$, $q = q(T) = 0 < \alpha$ and $R(T^q)$ α-closed. Hence T is a lower semi α-Browder operator.

(2) $\Rightarrow$ (1) Let T be onto and a lower semi α-Browder operator. Then $q = 0 = q(T) < \infty$ and $R(T^q) = \mathcal{H}$. Therefore, T is α-right Drazin invertible. □

The above sequel leads to the following corollary:

Corollary 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$.*

(1) *If T is injective and α-left Drazin invertible, then T is a upper semi α-B-Browder operator.*
(2) *If T is onto and α-right Drazin invertible, then T is a lower semi α-B-Browder operator.*

Proof. (1) Let T be injective and T is α-left Drazin invertible. Then T is a upper semi α-Browder operator. Hence, for $n = 0$, T is a upper semi α-B-Browder operator.

(2) Let T be onto and α-right Drazin invertible. Then T is a lower semi α-Browder operator. Hence, for $n = 0$, T is a lower semi α-B-Browder operator. □

By [4, Proposition 2.6] and [8, Theorem 2.2] we have the following lemma:

Lemma 3.8. *If $T \in \mathcal{L}(\mathcal{H})$ and $K \in K_\alpha(\mathcal{H})$ such that K commutes with T and $\dim(R(K^n)) < \infty$ for some integer $n \geq 1$. Then, if $T \in \Psi_{\alpha B}(\mathcal{H})$, $T + K \in \Psi_{\alpha B}(\mathcal{H})$.*

From Lemma 3.8, we can conclude the following:

Corollary 3.9. *If $T \in \mathcal{L}(\mathcal{H})$ and $K \in K_\alpha(\mathcal{H})$ such that K commutes with T and $\dim(R(K^n)) < \infty$ for some integer $n \geq 1$. Then,*

$$\sigma_{\alpha B}(T) = \sigma_{\alpha B}(T + K).$$

By Lemma 3.8 and [2], [3, Lemma (2.1)], we conclude:

Corollary 3.10. *Let T be an α-Browder operator. Then there is an $\epsilon > 0$ such that for any S in $\mathcal{L}(\mathcal{H})$ commuting with T, satisfying $\|S\| < \epsilon$ and $\dim R(S^n) < \infty$ for some integer $n \geq 1$, $T + S$ is also the α-Browder operator and $i(T + S) = i(T)$.*

4. Weighted Browder spectrum for the sum of two bounded linear operators

We begin the following sequel with $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space of infinite dimension h and is assumed to be a direct sum of its T-invariant closed subspaces. With these assumptions the following discussion leads to the necessary and sufficient conditions for an operator to be an upper semi α-Browder operator, a lower semi α-Browder operator and finally we prove that the non-zero spectral values in the weighted Browder spectrum of the sum of operators T and S are the part of non-zero spectral values of the union of weighted Browder spectra of T and S respectively. We know [4], for $T \in \mathcal{L}(\mathcal{H})$ where $\mathcal{H}$ is a direct sum of closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are T-invariant, if $T_1 = T|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 = T|_{\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_2$, then T is an α-Fredholm operator if and only if T_1 and T_2 are α-Fredholm operators. This motivates the following dual results:

Lemma 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a direct sum of closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are T-invariant. If $T_1 = T|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 = T|_{\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_2$, then T is an upper semi α-Browder operator if and only if T_1 and T_2 are upper semi α-Browder operators.*

Proof. The operator T has the following matrix form with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}.$$

Suppose T is a upper semi α-Browder operator. Then T is a upper semi α-Fredholm operator and $m = p(T) < \infty$ for some integer $m \geq 0$. Using [4, Lemma (3.1)], we have T_1 and T_2 are upper semi α-Fredholm operators.

We show that: $p(T_1) < \infty$ and $p(T_2) < \infty$.

Consider, $T = T_1 \oplus T_2$. Therefore, $T^c = T_1^c \oplus T_2^c$ for all integers $c \geq 0$.

Let $x \in N(T^c) \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$. Then $x = y_1 + y_2$, where $y_1 \in \mathcal{H}_1$ and $y_2 \in \mathcal{H}_2$ and $T^c x = 0$. So, $(T_1^c + T_2^c)(y_1 + y_2) = 0$. This implies $T_1^c(y_1) = -T_2^c(y_2) \in R(T_1^c) \cap R(T_2^c) = \{0\}$.

So, $y_1 \in N(T_1^c)$ and $y_2 \in N(T_2^c)$ and $y_1 + y_2 \in N(T_1^c) + N(T_2^c)$. Thus $x \in N(T_1^c) + N(T_2^c)$. Hence

$$N(T^c) = N(T_1^c) + N(T_2^c). \tag{4.1}$$

Since $m = p(T) < \infty$, $N(T^m) = N(T^{m+1})$. By equation (4.1),

$$N(T^{m+1}) = N(T_1^m) + N(T_2^m),$$

$$N(T_1^{m+1}) + N(T_2^{m+1}) = N(T_1^m) + N(T_2^m),$$

$$N(T_1^{m+1}) = N(T_1^m) \text{ and } N(T_2^m) = N(T_2^{m+1}).$$

Thus $p(T_1) \leq m < \infty$ and $p(T_2) \leq m < \infty$. Therefore, T_1 and T_2 are upper semi α-Browder operators.

Conversely, suppose T_1 and T_2 are upper semi α-Browder operators. Then T_1 and T_2 are upper semi α-Fredholm operators and $p(T_1) < \infty$ and $p(T_2) < \infty$. Since T_1 and T_2 are upper semi α-Fredholm operators and $T = T_1 \oplus T_2$, by [4, Lemma (3.1)], T is also a upper semi α-Fredholm operator. Let $p(T_1) = c_1$ and $p(T_2) = c_2$, where c_1 and c_2 are some non-negative integers.

Then, $N(T_1^{c_1}) = N(T_1^{c_1+1})$ and $N(T_2^{c_2}) = N(T_2^{c_2+1})$. Now consider $c = c_1.c_2$. Thus $N(T^c) = N(T^{c+1})$. Therefore, $p(T) \leq c < \infty$. Hence T is a upper semi α-Browder operator. $\square$

Analogously we have the following:

Lemma 4.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{H}$ be a direct sum of closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are T-invariant. If $T_1 = T|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 = T|_{\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_2$, then T is a lower semi α-Browder operator if and only if T_1 and T_2 are lower semi α-Browder operators.*

Proof. The operator T has the following matrix form with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}.$$

Suppose T is a lower semi α-Browder operator. Then T is a lower semi α-Fredholm operator and $m = q(T) < \infty$ for some integer $m \geq 0$. Using [4, Lemma 3.1], we have T_1 and T_2 are lower semi α-Fredholm operators.

Now we show that: $q(T_1) < \infty$ and $q(T_2) < \infty$.

Consider, $T = T_1 \oplus T_2$. Therefore, $T^c = T_1^c \oplus T_2^c$ for all integers $c \geq 0$.

Let $y \in R(T^c)$. Then there exists $x \in \mathcal{H}$ such that $y = T^c(x)$. So, $y = T_1^c(x) + T_2^c(x) \in R(T_1^c) + R(T_2^c)$. Therefore,

$$R(T^c) \subseteq R(T_1^c) + R(T_2^c). \tag{4.2}$$

Now let $y \in R(T_1^c) + R(T_2^c)$ then $y = z_1 + z_2$, where $z_1 \in R(T_1^c)$ and $z_2 \in R(T_2^c)$. So, $y = T_1^c(x_1) + T_2^c(x_2)$ for some $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$. Then $y = T^c(x)$ for some $x \in \mathcal{H}$ and

$$R(T_1^c) + R(T_2^c) \subseteq R(T^c). \tag{4.3}$$

Hence

$$R(T_1^c) + R(T_2^c) = R(T^c). \tag{4.4}$$

Since $q(T) = m < \infty$, we have $R(T^m) = R(T^{m+1})$.

So, by equation (4.4)

$$R(T_1^m) + R(T_2^m) = R(T^{m+1}),$$
$$R(T_1^m) + R(T_2^m) = R(T_1^{m+1}) + R(T_2^{m+1}),$$
$$R(T_1^{m+1}) = R(T_1^m) \text{ and } R(T_2^c) = R(T_2^{c+1}).$$

Therefore, $q(T_1) \leq m < \infty$ and $q(T_2) \leq m < \infty$. Hence T_1 and T_2 are lower semi α-Browder operators.

Conversely, let T_1 and T_2 be lower semi α-Browder operators. Then T_1 and T_2 are lower semi α-Fredholm operators and $q(T_1) < \infty$ and $q(T_2) < \infty$. Since $T = T_1 \oplus T_2$ and T_1 and T_2 are lower semi α-Fredholm operators, by [4, Lemma 3.1], T is also a lower semi α-Fredholm operator.

Let $q(T_1) = c_1$ and $q(T_2) = c_2$, where c_1 and c_2 are non-negative integers. Then $R(T_1^{c_1}) = R(T_1^{c_1+1})$ and $R(T_2^{c_2}) = R(T_2^{c_2+1})$.

Now, consider $c = c_1 \cdot c_2$. Then $R(T_1^c) = R(T_1^{c+1})$ and $R(T_2^c) = R(T_2^{c+1})$. Therefore, $R(T^c) = R(T^{c+1})$ and $q(T) \leq c < \infty$. Hence T is a lower semi α-Browder operator. □

As a final conclusion we have the following characterisation:

Theorem 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{H}$ be a direct sum of closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ which are T-invariant. If $T_1 = T|_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 = T|_{\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_2$, then T is an α-Browder operator if and only if T_1 and T_2 are α-Browder operators.*

For each $0 \neq \beta \in \mathbb{C}$, Lemma 4.4 and Lemma 4.5 imply the invertibilty of the sum of the two bounded linear operators and also the fact that the non-zero spectral values in the weighted Browder spectrum of the sum are part of the non-zero spectral values of the union of weighted Browder spectra of T and S in Theorem 4.6 in the following discussion.

Lemma 4.4. *For bounded linear operators T and S in $\mathcal{L}(\mathcal{H})$, if for each $0 \neq \beta \in \mathbb{C}$, there exists some integer $k \geq 0$ such that $(T+S-\beta I)_{[k]}$ is injective and $TS \in \mathcal{K}_\alpha(\mathcal{H})$, then*

$$\sigma_{U\alpha B}(T+S)\backslash\{0\} \subseteq [\sigma_{U\alpha B}(T) \cup \sigma_{U\alpha B}(S)]\backslash\{0\}.$$

Proof. Let $0 \neq \lambda \notin [\sigma_{U\alpha B}(T) \cup \sigma_{U\alpha B}(S)] \setminus \{0\}$. Then $T-\lambda I \in \Psi^+_{\alpha B}(\mathcal{H})$ and $S-\lambda I \in \Psi^+_{\alpha B}(\mathcal{H})$. Therefore, $T-\lambda I \in \Psi^+_\alpha(\mathcal{H})$ and $S-\lambda I \in \Psi^+_\alpha(\mathcal{H})$ and $p(T-\lambda I) < \infty, p(S-\lambda I) < \infty$. So, $(T-\lambda I)(S-\lambda I) \in \Psi^+_\alpha(\mathcal{H})$. Since $TS \in \mathcal{K}_\alpha(\mathcal{H})$ and $(T-\lambda I)(S-\lambda I) = TS - \lambda(T+S-\lambda I)$. Therefore, $(T+S-\lambda I) \in \Psi^+_\alpha(\mathcal{H})$.

Now we show that: $N((T+S-\lambda I)^c) = N((T+S-\lambda I)^{c+1})$ for some integer $c \geq 0$. Clearly,

$$N((T+S-\lambda I)^k) \subseteq N((T+S-\lambda I)^{k+1}) \text{ for all integer } k \geq 0. \tag{4.5}$$

We know that, for each $0 \neq \lambda \in \mathbb{C}$ there exists some integer $k \geq 0$ such that $(T+S-\lambda I)_{[k]}$ is injective. Let $x \in N((T+S-\lambda I)^{k+1})$. Then $(T+S-\lambda I)^k(x) \in N(T+S-\lambda I)$. So,

$$(T+S-\lambda I)^k(x) \in N(T+S-\lambda I) \cap R((T+S-\lambda I)^k) = N((T+S-\lambda I)_{[k]}) = \{0\}$$

and $(T+S-\lambda I)^k(x) = 0$. Therefore, $x \in N((T+S-\lambda I)^k)(x)$ and

$$N((T+S-\lambda I)^{k+1}) \subseteq N((T+S-\lambda I)^k). \tag{4.6}$$

By equations (4.5) and (4.6)

$$N((T+S-\lambda I)^{k+1}) = N((T+S-\lambda I)^k).$$

Therefore $p(T+S-\lambda I) \leq k < \infty$ and $\lambda \notin \sigma_{U\alpha B}(T+S)$.
Hence

$$\sigma_{U\alpha B}(T+S) \setminus \{0\} \subset [\sigma_{U\alpha B}(T) \cup \sigma_{U\alpha B}(S)] \setminus \{0\}. \qquad \square$$

Lemma 4.5. *For bounded linear operators T and S in $\mathcal{L}(\mathcal{H})$, if for each $0 \neq \beta \in \mathbb{C}$, there exists some integer $k \geq 0$ such that $(T+S-\beta I)_{[k]}$ is onto and $TS \in \mathcal{K}_\alpha(\mathcal{H})$, then*

$$\sigma_{L\alpha B}(T+S)\backslash\{0\} \subseteq [\sigma_{L\alpha B}(T) \cup \sigma_{L\alpha B}(S)]\backslash\{0\}.$$

Proof. Let $0 \neq \lambda \notin [\sigma_{L\alpha B}(T) \cup \sigma_{L\alpha B}(S)] \setminus \{0\}$. Then $T-\lambda I \in \Psi^-_{\alpha B}(\mathcal{H})$ and $S-\lambda I \in \Psi^-_{\alpha B}(\mathcal{H})$. Therefore, $T-\lambda I \in \Psi^-_\alpha(\mathcal{H})$, $S-\lambda I \in \Psi^-_\alpha(\mathcal{H})$ and $q(T-\lambda I) < \infty, q(S-\lambda I) < \infty$. So, $(T-\lambda I)(S-\lambda I) \in \Psi^-_\alpha(\mathcal{H})$. Since $TS \in \mathcal{K}_\alpha(\mathcal{H})$ and $(T-\lambda I)(S-\lambda I) = TS-\lambda(T+S-\lambda I)$. Thus $(T+S-\lambda I) \in \Psi^-_\alpha(\mathcal{H})$.

Since $(T+S-\lambda I)_{[k]}$ is onto, so $R((T+S-\lambda I)_{[k]}) = R((T+S-\lambda I)^k)$. Then

$$R((T+S-\lambda I)^{k+1}) = R((T+S-\lambda I)^k).$$

Thus $q(T+S-\lambda I) \leq k < \infty$. Therefore, $\lambda \notin \sigma_{L\alpha B}(T+S)$. Hence

$$\sigma_{L\alpha B}(T+S) \setminus \{0\} \subseteq [\sigma_{L\alpha B}(T) \cup \sigma_{L\alpha B}(S)] \setminus \{0\}.$$ □

Theorem 4.6. *For bounded linear operators T and S in $\mathcal{L}(\mathcal{H})$, if for each $0 \neq \beta \in \mathbb{C}$, there exists some integer $k \geq 0$ such that $(T+S-\beta I)_{[k]}$ is invertible and $TS \in \mathcal{K}_\alpha(\mathcal{H})$, then*

$$\sigma_{\alpha B}(T+S)\setminus\{0\} \subseteq [\sigma_{\alpha B}(T) \cup \sigma_{\alpha B}(S)]\setminus\{0\}.$$

5. Conclusion

In the articles [1, 5.6], authors studied Drazin invertibility, semi-B-Fredholm operators, and B-Browder spectra on the Complex Banach space. Motivated by their works, in the present paper, we introduced the notions of α-Browder operator, α B-Fredholm operator, α-B-Browder operator, and α-Drazin invertible operator and their corresponding spectra. The discussion on the relation between these operators and their corresponding spectra have lead to many interesting conclusions. We have obtained that, whenever T is onto, T is lower semi α-Fredholm operator, lower semi α-Browder operator, and α-right Drazin invertible operator. Also in the case of ontoness of T, equivalence has been found between lower semi α-B-Fredholm operators, lower semi α-B-Browder operators and also between α-right Drazin invertible operators and lower semi α-Browder operators. When T is α-right Drazin invertible operator with ontoness then T is lower semi α-B-Browder operator. The similar results holds whenever T is injective or α-bounded below. The example constructed in the discussion proves that these classes are not void. With the help of the deduced results and some previous results we develop the relation between α-Browder operators and their invariant subspaces. We characterize the weighted Browder spectrum of the sum of two bounded linear operators defined over the arbitrary Hilbert space.

We finish with a question of whether the reverse containment of the conclusion of Theorem 4.6 holds.

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