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SASAKIAN 3-METRIC AS A $*$-CONFORMAL RICCI SOLITON REPRESENTS A BERGER SPHERE

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Abstract. In this article, the notion of $*$-conformal Ricci soliton is defined as a self similar solution of the $*$-conformal Ricci flow. A Sasakian 3-metric satisfying the $*$-conformal Ricci soliton is completely classified under certain conditions on the soliton vector field. We establish a relation with Fano manifolds and proves a homothety between the Sasakian 3-metric and the Berger Sphere. Also, the potential vector field V is a harmonic infinitesimal automorphism of the contact metric structure.

1. Introduction

Sasakian geometry is an odd dimensional analogue of the Kaehler geometry and perceived relevance in mathematical physics (see [4]). Due to this connection with physics, importance of Sasakian geometry increases to geometers and physicists. Ricci flow, $*$-Ricci flow, conformal Ricci flow and their different versions are topics of mathematical physics. Analogous to the notion of $*$-Ricci flow, the notion of $*$-conformal Ricci flow on an n-dimensional Riemannian manifold (M, g) can be defined as

$$\frac{\partial g}{\partial t} + 2(S^* + \frac{1}{n}g) = -pg,$$

where S^* is the $*$-Ricci tensor of g and p is a non-dynamical scalar field. We define the notion of $*$-conformal Ricci soliton as a self similar solution of the $*$-conformal Ricci flow as follows:

Definition. An almost contact metric manifold (M, g) of dimension 3 is said to admit $*$-conformal Ricci soliton (g, V, λ) if

$$\mathcal{L}_V g + 2S^* = [2\lambda - (p + \frac{2}{3})]g, \tag{1.1}$$

where λ is a constant, provided S^* is symmetric.

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If the vector field V is gradient of some smooth function f on M, then the above equation reduces to

$$\nabla^2 f + S^* = [\lambda - \frac{1}{2}(p + \frac{2}{3})]g, \tag{1.2}$$

where $\nabla^2 f$ is the Hessian of f. The $*$-conformal Ricci soliton is expanding, steady or shrinking according as λ is negative, zero or positive respectively.

Note that, the $*$-Ricci tensor is not symmetric in general. Hence, for a non-symmetric $*$-Ricci tensor of a manifold, the above notion is inconsistent. In a Sasakian 3-manifold, S^* is symmetric (given later) and hence, the above definition is well defined on Sasakian 3-manifolds. Now, it is worth considering a notion from mathematical physics on some geometrical space having connection with physics. In this article, we consider the notion of $*$-conformal Ricci soliton on Sasakian 3-manifolds and establish a relation with Fano manifolds and proves a homothety between the Sasakian 3-metric and the Berger sphere.

2. Sasakian 3-manifolds

An odd dimensional differentiable manifold M is said to be an almost contact metric manifold if it admits a structure (φ, ξ, η, g) satisfying

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta \circ \varphi = 0 \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \end{aligned} \tag{2.1}$$

for any vector fields X, Y on M, where φ is a $(1,1)$-tensor field, ξ is a unit vector field, η is a 1-form defined by $\eta(X) = g(X, \xi)$ and g is the Riemannian metric. Using (2.1), we can easily see that

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

An almost contact metric manifold with $d\eta = g(X, \varphi Y)$ is called a contact metric manifold. If the characteristic vector field ξ is Killing type, then a contact metric manifold is called a K-contact manifold and if the structure (φ, ξ, η, g) is normal, then a contact metric manifold is called Sasakian. Also, an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.2}$$

for any vector fields X, Y on M. A Sasakian manifold is K-contact but the converse holds only in dimension 3. It may not be true for higher dimension (see [8]). On a $(2n+1)$-dimensional Sasakian manifold, the following relations are well known:

$$\nabla_X \xi = -\varphi X,$$

$$(\nabla_X \eta)Y = g(X, \varphi Y), \tag{2.3}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.4}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

where R is the Riemann curvature tensor. Since a 3-dimensional Riemannian manifold is conformally flat, it's curvature tensor can be expressed as

$$\begin{aligned} R(X,Y)Z = {} & [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ & - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

where r is the scalar curvature defined by $r = S(e_i, e_i) = g(Qe_i, e_i)$ for any orthonormal basis $\{e_i\}$ of the tangent space at any point of M, S is the Ricci tensor. The scalar curvature r is not constant in general. Now, the Ricci tensor for a Sasakian 3-manifold can be obtained from here as

$$S(X,Y) = \frac{1}{2}[(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)]. \tag{2.6}$$

The $*$-Ricci tensor of a Sasakian 3-manifold is given by (see [3, 9])

$$S^*(X,Y) = \frac{1}{2}(r-4)[g(X,Y) - \eta(X)\eta(Y)]. \tag{2.7}$$

A contact metric manifold M is said to be η-Einstein if there exist two smooth functions α and β such that

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$$

for all vector fields X, Y on M.

A $(2n+1)$-dimensional η-Einstein Sasakian manifold such that $\alpha = -2$ and $\beta = 2n+2$ is known as null-Sasakian. Also, η-Einstein Sasakian manifold with $\alpha > -2$ is called positive-Sasakian. In this case, the transverse geometry of M is Fano, a compact manifold whose anticanonical line bundle is ample, that is, the first Chern class of the canonical line bundle is negative-definite. For more details, we refer the reader to go through [2]. From (2.6), we see that a Sasakian 3-manifold is η-Einstein. Thus a Sasakian 3-manifold is null-Sasakian if $r = -2$ and positive-Sasakian if $r > -2$. For $r > -2$, the transverse geometry of the Sasakian 3-manifold is Fano.

In Riemannian geometry, a Berger sphere is a standard 3-sphere with Riemannian metric from a one-parameter family, which can be obtained from the standard metric by shrinking along fibers of a Hopf fibration. The sphere S^3 and the Lie group $SU(2)$ can be identified. We consider the basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ such that

$$[X_1, X_2] = 2X_3,\ [X_2, X_3] = 2X_1,\ [X_3, X_1] = 2X_2.$$

The one-parameter family $\{g_\epsilon : \epsilon > 0\}$ of left-invariant Riemannian metrics on $S^3 = SU(2)$ given at the identity, with respect to the basis of left-invariant vector fields $X_1,\ X_2,\ X_3$ by

$$g_\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are called the Berger metrics on S^3. The Berger spheres are the simply connected complete Riemannian manifolds $S^3_\epsilon = (S^3, g_\epsilon)$, $\epsilon > 0$. For more details, we refer the reader to go through the references [1] and [5].

3. $*$-conformal Ricci soliton

In this section, we consider the notion of $*$-conformal Ricci soliton in the framework of Sasakian 3-manifolds. Observe that the $*$-Ricci tensor of a Sasakian 3-manifold is symmetric and hence the notion of $*$-conformal Ricci soliton is consistent in this setting.

Definition ([10])**.** A vector field V on an almost contact metric manifold M is said to be an infinitesimal contact transformation if $\mathcal{L}_V\eta = f\eta$ for some smooth function f on M. If $f = 0$, then V is said to be a strict infinitesimal contact transformation. If V leaves all the structure tensor fields ϕ, ξ, η and g invariant, then V is called an infinitesimal automorphism of the contact metric structure.

Theorem 3.1. *Let M be a Sasakian 3-manifold M admitting $*$-conformal Ricci soliton (g, V, λ). If one of the followings holds:*

(a) *V is an infinitesimal contact transformation,*
(b) *V is pointwise collinear with ξ,*
(c) *V is a gradient vector field,*

then

(1) *The manifold M is $*$-Ricci flat.*
(2) *The Sasakian 3-manifold M is positive Sasakian and the transverse geometry of M is Fano.*
(3) *The Sasakian 3-metric g is homothetic to a Berger sphere.*
(4) *The potential vector field V is a harmonic infinitesimal automorphism of the contact metric structure.*

Proof. (a) If V is an infinitesimal contact transformation, then we have

$$\mathcal{L}_V\eta = f\eta \tag{3.1}$$

for some smooth function f on M. Since $d\eta(X,Y) = g(X, \varphi Y)$, then

$$(\mathcal{L}_V d\eta)(X,Y) = (\mathcal{L}_V g)(X, \varphi Y) + g(X, (\mathcal{L}_V\varphi)Y). \tag{3.2}$$

Applying (1.1) in (3.2) yields

$$(\mathcal{L}_V d\eta)(X,Y) = -2S^*(X,\varphi Y) + [2\lambda - (p + \frac{2}{3})]g(X, \varphi Y) + g(X, (\mathcal{L}_V\varphi)Y). \tag{3.3}$$

With the help of (3.1), we obtain

$$\mathcal{L}_V d\eta = d\mathcal{L}_V\eta = df \wedge \eta + f d\eta, \tag{3.4}$$

which implies

$$(\mathcal{L}_V d\eta)(X,Y) = \frac{1}{2}[df(X)\eta(Y) - df(Y)\eta(X)] + fg(X, \varphi Y). \tag{3.5}$$

Equating (3.3) and (3.5), we get

$$g(X,(\mathcal{L}_V\varphi)Y) = \frac{1}{2}[(Xf)\eta(Y) - (Yf)\eta(X)] + [r - 4 + f - 2\lambda + (p + \frac{2}{3})]g(X,\varphi Y),$$

which implies

$$(\mathcal{L}_V\varphi)Y = \frac{1}{2}[\eta(Y)Df - (Yf)\xi] + [r - 4 + f - 2\lambda + (p + \frac{2}{3})]\varphi Y, \tag{3.6}$$

where D is the gradient operator. Substituting $Y = \xi$ in the above equation, we have

$$(\mathcal{L}_V\varphi)\xi = \frac{1}{2}[Df - (\xi f)\xi]. \tag{3.7}$$

Since $(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X)$, then tracing (1.1) yields

$$\operatorname{div} V = \frac{3}{2}[2\lambda - (p + \frac{2}{3})] - (r - 4), \tag{3.8}$$

where 'div' stands for divergence. Let Ω be the volume form of M, that is, $\Omega = \eta \wedge (d\eta)^n \neq 0$. Taking Lie derivative of this along the vector field V and applying the formula $\mathcal{L}_V\Omega = (\operatorname{div} V)\Omega$, (3.1) and (3.4), we obtain $(\operatorname{div} V)\Omega = (n+1)f\Omega$, which implies

$$\operatorname{div} V = (n+1)f. \tag{3.9}$$

Equating (3.8) and (3.9) yields

$$r = \frac{3}{2}[2\lambda - (p + \frac{2}{3})] + 4 - (n+1)f. \tag{3.10}$$

Now, from (1.1), we get

$$(\mathcal{L}_V g)(X,\xi) = [2\lambda - (p + \frac{2}{3})]\eta(X),$$

which gives

$$(\mathcal{L}_V\eta)X - g(X,\mathcal{L}_V\xi) = [2\lambda - (p + \frac{2}{3})]\eta(X). \tag{3.11}$$

First applying (3.1) in (3.11) and then substituting $X = \xi$ gives

$$\eta(\mathcal{L}_V\xi) = [f - 2\lambda + (p + \frac{2}{3})]. \tag{3.12}$$

Now, Putting $X = \xi$ in (3.11), we get

$$\eta(\mathcal{L}_V\xi) = -\frac{1}{2}[2\lambda - (p + \frac{2}{3})]. \tag{3.13}$$

Equating (3.12) and (3.13), we obtain $f = \frac{1}{2}[2\lambda - (p + \frac{2}{3})] =$ constant. Integrating (3.9) and applying 'divergence theorem' gives $f = 0$ and hence, $2\lambda - (p + \frac{2}{3}) = 0$. Equation (3.9) shows that V is harmonic. Therefore, Equation (3.1) gives $\mathcal{L}_V\eta = 0$ and (3.13) implies $\eta(\mathcal{L}_V\xi) = 0$. Equation (3.10) provides $r = 4$ and thus (2.7) gives $S^* = 0$ proving (1). Now, (3.6) gives $\mathcal{L}_V\varphi = 0$ and

(1.1) implies $\mathcal{L}_V g = 0$. Since $f = 0$, then Equation (3.7) gives $(\mathcal{L}_V\varphi)\xi = 0$, which implies $\varphi(\mathcal{L}_V\xi) = 0$. Operating φ on this and using $\eta(\mathcal{L}_V\xi) = 0$, we obtain $\mathcal{L}_V\xi = 0$. Therefore, V is harmonic and leaves all the structure tensor fields φ, ξ, η, g invariant. This proves (4). Since $r = 4 > -2$, then M is positive-Sasakian and the transverse geometry of M is Fano proving (3). The Tanaka-Webster curvature (see [6]) of a Sasakian 3-manifold is given by $W = \frac{1}{4}(r+2)$. Since $r = 4$, then $W = \frac{3}{2}$. Following the classification given by Guilfoyle [7] for $0 < W < 2$, we conclude that g is homothetic to a Berger sphere proving (4). This completes the proof of part (a).

(b) If V is pointwise collinear with ξ, then there is a non-zero smooth function b on M such that $V = b\xi$. Now,

$$(\mathcal{L}_V g)(X,Y) = (\mathcal{L}_{b\xi} g)(X,Y) = (Xb)\eta(Y) + (Yb)\eta(X). \tag{3.14}$$

Using (3.14) in (1.1), we have

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2S^*(X,Y) = [2\lambda - (p + \frac{2}{3})]g(X,Y). \tag{3.15}$$

Substituting $X = Y = \xi$ in the preceding equation yields

$$2(\xi b) = [2\lambda - (p + \frac{2}{3})]. \tag{3.16}$$

Let $\{e_i\}$ be any orthonormal frame on M. Substituting $X = Y = e_i$ in (3.15) and then summing over i, we get

$$2(\xi b) = 3[2\lambda - (p + \frac{2}{3})] - 2(r - 4). \tag{3.17}$$

Equating (3.16) and (3.17), we obtain

$$[2\lambda - (p + \frac{2}{3})] = r - 4. \tag{3.18}$$

Now, differentiating (1.1) covariantly, we obtain

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = -2(\nabla_Z S^*)(X,Y). \tag{3.19}$$

The well known commutation formula (see [11])

$$\begin{aligned}&(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y,Z)\\ = &-g((\mathcal{L}_V\nabla)(X,Y),Z) - g((\mathcal{L}_V\nabla)(X,Z),Y)\end{aligned}$$

leads to

$$\begin{aligned}g((\mathcal{L}_V\nabla)(X,Y),Z) = &\frac{1}{2}(\nabla_X \mathcal{L}_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \mathcal{L}_V g)(X,Z)\\ &- \frac{1}{2}(\nabla_Z \mathcal{L}_V g)(X,Y).\end{aligned}$$

Using (3.19) in the preceding equation yields

$$g((\mathcal{L}_V\nabla)(X,Y),Z) = (\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(X,Z). \tag{3.20}$$

Now, differentiating (2.7) covariantly along any vector field Z and applying (2.3), we obtain

$$(\nabla_Z S^*)(X,Y) = \frac{1}{2}(Zr)[g(X,Y) - \eta(X)\eta(Y)] - \frac{1}{2}(r-4)[\eta(X)g(\varphi Y, Z) + \eta(Y)g(\varphi X, Z)]. \tag{3.21}$$

Applying (3.21) in (3.20), we get

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X,Y), Z) = & \frac{1}{2}(Zr)[g(X,Y) - \eta(X)\eta(Y)] \\ & - \frac{1}{2}(Xr)[g(Y,Z) - \eta(Y)\eta(Z)] \\ & - \frac{1}{2}(Yr)[g(X,Z) - \eta(X)\eta(Z)] \\ & - (r-4)[\eta(X)g(\varphi Y, Z) + \eta(Y)g(\varphi X, Z)], \end{aligned}$$

which implies

$$\begin{aligned} (\mathcal{L}_V \nabla)(X,Y) = & \frac{1}{2}[g(X,Y) - \eta(X)\eta(Y)]Dr \\ & - \frac{1}{2}(Xr)[Y - \eta(Y)\xi] \\ & - \frac{1}{2}(Yr)[X - \eta(X)\xi] \\ & - (r-4)[\eta(X)\varphi Y + \eta(Y)\varphi X]. \end{aligned} \tag{3.22}$$

Substituting $Y = \xi$ in the foregoing equation and noting $\xi r = 0$ (as ξ is Killing), we get

$$(\mathcal{L}_V \nabla)(X,\xi) = -(r-4)\varphi X. \tag{3.23}$$

Now,

$$(\nabla_Y \mathcal{L}_V \nabla)(X,\xi) = \nabla_Y (\mathcal{L}_V \nabla)(X,\xi) - (\mathcal{L}_V \nabla)(\nabla_Y X, \xi) - (\mathcal{L}_V \nabla)(X, \nabla_Y \xi).$$

Using (3.22) and (3.23) in the foregoing equation, we obtain

$$\begin{aligned} (\nabla_Y \mathcal{L}_V \nabla)(X,\xi) = & -(Yr)\varphi X - (r-4)(\nabla_Y \varphi)X + \frac{1}{2}g(X,\varphi Y)Dr \\ & - \frac{1}{2}(Xr)\varphi Y - \frac{1}{2}g(\varphi Y, Dr)[X - \eta(X)\xi] \\ & + (r-4)\eta(X)[Y - \eta(Y)\xi]. \end{aligned} \tag{3.24}$$

Due to Yano [11], we have

$$(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z).$$

Using (3.24) in the above formula and applying (2.2), we obtain

$$\begin{aligned} (\mathcal{L}_V R)(X,\xi)\xi & = (\nabla_X \mathcal{L}_V \nabla)(\xi,\xi) - (\nabla_\xi \mathcal{L}_V \nabla)(X,\xi) \\ & = 2(r-4)[X - \eta(X)\xi]. \end{aligned} \tag{3.25}$$

From (1.1), we have

$$(\mathcal{L}_V g)(X,\xi) = [2\lambda - (p+\frac{2}{3})]\eta(X),$$

which leads to

$$(\mathcal{L}_V \eta)X - g(X,\mathcal{L}_V\xi) = [2\lambda - (p+\frac{2}{3})]\eta(X). \tag{3.26}$$

Putting $X=\xi$ in the preceding equation, we get

$$\eta(\mathcal{L}_V\xi) = -\frac{1}{2}[2\lambda - (p+\frac{2}{3})]. \tag{3.27}$$

Now, with the help of (2.4), (2.5), (3.26) and (3.27), we obtain

$$(\mathcal{L}_V R)(X,\xi)\xi = [2\lambda - (p+\frac{2}{3})][X-\eta(X)\xi]. \tag{3.28}$$

Comparing (3.25) and (3.28), we infer that

$$[2\lambda - (p+\frac{2}{3})] = 2(r-4). \tag{3.29}$$

Therefore, (3.18) and (3.29) together implies $r=4$. Then by the same argument as in part (a), the results (1), (2) and (3) holds. Now, from (3.18), we get $[2\lambda - (p+\frac{2}{3})]=0$ and hence, from (3.16), we get $(\xi b)=0$. Equation (3.15) reduces to $(Xb)\eta(Y)+(Yb)\eta(X)=0$. Substituting $Y=\xi$ here, we get $(Xb)=0$ for all vector fields X, which implies b is constant. Then (3.14) gives $\mathcal{L}_V g=0$. Since $V=b\xi$ and b is constant, then it is easy to see that $\mathcal{L}_V\xi=0$. Now, (3.26) gives $\mathcal{L}_V\eta=0$. It can be easily calculated that $\mathcal{L}_V\varphi=0$. Therefore, V leaves all the structure tensor fields invariant. Further, Since $V=b\xi$, then $\nabla_X V = b\nabla_X\xi = -b\varphi X$, which implies $\operatorname{div} V=0$ and therefore V is harmonic. This proves (4) and the proof of part (b) is complete.

(c) Let V be gradient of some smooth function f on M, that is, $V=Df$. Then Equation (1.2) can be exhibited as

$$\nabla_X Df = [\lambda - \frac{1}{2}(p+\frac{2}{3})]X - Q^*X. \tag{3.30}$$

It is known that

$$R(X,Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df.$$

Applying (3.30) in the above formula, we can easily obtain

$$R(X,Y)Df = (\nabla_Y Q^*)X - (\nabla_X Q^*)Y. \tag{3.31}$$

From (3.21), we can write

$$\begin{aligned}(\nabla_X Q^*)Y &= \frac{1}{2}(Xr)[Y-\eta(Y)\xi] \\ &\quad - \frac{1}{2}(r-4)[g(\varphi Y,X)\xi - \eta(Y)\varphi X].\end{aligned} \tag{3.32}$$

Applying (3.32) in (3.31), we obtain

$$R(X,Y)Df = \frac{1}{2}(Yr)[X - \eta(X)\xi] - \frac{1}{2}(Xr)[Y - \eta(Y)\xi] + \frac{1}{2}(r-4)[2g(X,\varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y].$$

Substituting $X = \xi$ in the preceding equation and using $\xi r = 0$ (as ξ is Killing), we obtain

$$R(\xi,Y)Df = \frac{1}{2}(r-4)\varphi Y.$$

Taking inner product of the preceding equation with X, we have

$$g(R(\xi,Y)Df, X) = \frac{1}{2}(r-4)g(X,\varphi Y). \tag{3.33}$$

Since $g(R(\xi,Y)Df, X) = -g(R(\xi,Y)X, Df)$, then with the help of (2.5), we obtain

$$g(R(\xi,Y)Df, X) = -g(X,Y)(\xi f) + \eta(X)(Yf). \tag{3.34}$$

Equating (3.33) and (3.34) and then antisymmetrizing yields

$$(r-4)g(X,\varphi Y) = \eta(X)(Yf) - \eta(Y)(Xf).$$

Replacing X by ξ in the foregoing equation, we get

$$(Yf) - \eta(Y)(\xi f) = 0,$$

which implies $V = Df = (\xi f)\xi$, that is, V is pointwise collinear with ξ. Rest of the proof follows from part (b). This completes the proof. □

Remark 3.2. We have obtained $2\lambda - (p + \frac{2}{3}) = 0$, that is, $\lambda = \frac{1}{2}(p + \frac{2}{3})$. Hence, the ∗-conformal Ricci soliton is expanding, steady or shrinking according as $\frac{1}{2}(p + \frac{2}{3})$ is negative, zero or positive respectively.

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References

[1] Y. Aryanejad, *Some geometrical properties of Berger spheres*, Caspian J. Math. Sci. https://doi.org/10.22080/CJMS.2021.3055

[2] C. P. Boyer, K. Galicki, and P. Matzeu, *On eta-Einstein Sasakian geometry*, Comm. Math. Phys. **262** (2006), no. 1, 177–208. https://doi.org/10.1007/s00220-005-1459-6

[3] D. Dey and P. Majhi, ∗*-Ricci solitons and* ∗*-gradient Ricci solitons on 3-dimensional trans-Sasakian manifolds*, Commun. Korean Math. Soc. **35** (2020), no. 2, 625–637. https://doi.org/10.4134/CKMS.c190121

[4] J. Figueroa-O'Farrill and A. Santi, *Sasakian manifolds and M-theory*, Classical Quantum Gravity **33** (2016), no. 9, 095004, 18 pp. https://doi.org/10.1088/0264-9381/33/9/095004

[5] P. M. Gadea and J. A. Oubiña, *Homogeneous Riemannian structures on Berger 3-spheres*, Proc. Edinb. Math. Soc. (2) **48** (2005), no. 2, 375–387. https://doi.org/10.1017/S0013091504000422
[6] B. S. Guifoyle, *Einstein metrics adapted to a contact structure on 3-manifolds*, Preprint. arXiv:math/0012027v1 [math.DG] (2000).
[7] B. S. Guifoyle, *The local moduli of Sasakian* 3*-manifolds*, Int. J. Math. Math. Sci. **32** (2002), no. 2, 117–127. https://doi.org/10.1155/S0161171202006774
[8] J.-B. Jun, I. B. Kim, and U. K. Kim, *On* 3*-dimensional almost contact metric manifolds*, Kyungpook Math. J. **34** (1994), no. 2, 293–301.
[9] P. Majhi, U. C. De, and Y. J. Suh, $*$*-Ricci solitons and Sasakian 3-manifolds*, Publ. Math. Debrecen **93** (2018), no. 1-2, 241–252. https://doi.org/10.5486/pmd.2018.8245
[10] S. Tanno, *Note on infinitesimal transformations over contact manifolds*, Tohoku Math. J. (2) **14** (1962), 416–430. https://doi.org/10.2748/tmj/1178244078
[11] K. Yano, *Integral formulas in Riemannian geometry*, Pure and Applied Mathematics, No. 1, Marcel Dekker, Inc., New York, 1970.

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