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ENTIRE SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS OF FERMAT TYPE

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Abstract. In this paper, we extend some previous works by Liu et al. on the existence of transcendental entire solutions of differential-difference equations of Fermat type. In addition, we also present a precise description of the associated entire solutions.

1. Introduction and main results

Gross [3] proved that the functional equation of Fermat type

$$f(z)^n + g(z)^n = 1 \tag{1}$$

has no transcendental meromorphic solutions $f(z)$ and $g(z)$ when $n \geq 4$. Montel [12] showed that (1) has no transcendental entire solutions $f(z)$ and $g(z)$ when $n \geq 3$. Iyer [2] concluded that when $n = 2$, entire solutions of (1) have only the following forms

$$f(z) = \sin(h(z)), \quad g(z) = \cos(h(z))$$

except for interchangeable, where $h(z)$ is any entire function.

In 1970, Yang [15] investigated the following functional equation of Fermat type

$$a(z)f(z)^n + b(z)g(z)^m = 1, \tag{2}$$

where a, b are small functions with respect to f, that is, Nevanlinna's characteristic function $T(r, \alpha)$ of any $\alpha \in \{a, b\}$ satisfies $T(r, \alpha) = S(r, f)$ in which $S(r, f)$ denotes a real function of r with the property

$$S(r, f) = o(T(r, f)), \ r \to \infty$$

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possibly outside a set of values r of finite linear measure (see e.g., [7] and [17]), and where m, n are positive integers satisfying

$$\frac{1}{m}+\frac{1}{n}<1,$$

by proving that there are no non-constant entire solutions f and g satisfying (2). Yang's result shows that (2) has no non-constant entire solutions under the assumption $m>2$, $n>2$. However, when $m=n=2$, the problem is open.

In 2007, Tang and Liao [13] extended a study work of the open problem due to Yang and Li [16] through replacing g by $f^{(k)}$ to investigate entire solutions of the following equation

$$f(z)^2+\{P(z)f^{(k)}(z)\}^2=Q(z), \tag{3}$$

where P, Q are non-zero polynomials. In 2013, Liu and Yang [11] improved a researching result of the open problem in [9] through replacing $f^{(k)}(z)$ in (3) by $f(z+c)$, where c is a non-zero constant, by obtaining that if the following difference equation

$$f(z)^2+\{P(z)f(z+c)\}^2=Q(z) \tag{4}$$

admits a transcendental entire solution f of finite order, then $P(z)\equiv\pm1$ and Q reduces to a constant q, so that $f(z)=\sqrt{q}\sin(Az+B)$, where B is a constant, $A=\frac{(4k+1)\pi}{2c}$, in which k is an integer.

Recall that the *order* of f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

and the *hyper-order* of f is defined by

$$\rho_2(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

Note that the coefficients in the open problem are small functions of f, but where P,Q are assumed to be polynomials. A natural question is that what happens if P, Q in (3) or (4) are small functions of f?

Theorem 1.1. *Let P, Q be non-zero meromorphic functions. If the equation* (4) *admits a transcendental entire solution f with $\rho_2(f)<1$ such that P, Q are small functions of f, then we have $P^2(z)Q(z+c)=Q(z)$.*

Corollary 1.2. *Let P, Q be non-zero entire functions with $\rho(Q)<1$. If the equation* (4) *admits a transcendental entire solution f of $\rho_2(f)<1$ such that P, Q are small functions of f, then $P(z)=\pm1$ and Q reduces to a constant q, so that $f(z)=\frac{1}{2}(q_1\mathrm{e}^{az+b}+q_2\mathrm{e}^{-az-b})$, where a, b, q_1, q_2 are constants satisfying $ac=-\frac{\pi i}{2}+2k\pi i$ $(k\in\mathbb{Z})$, $q_1q_2=q$.*

It's easy to exhibit an example to show the existence of solutions in Corollary 1.2.

Example 1.3. Take $a = 1$ and $c = -\frac{\pi i}{2}$. Then $f(z) = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z})$ satisfies the equation (4).

Obviously, Corollary 1.2 is an extension and supplement of the result due to Liu and Yang [11], in which they proved that any transcendental entire solution f of finite order of the differential-difference equation

$$f(z+c)^2 + f^{(k)}(z)^2 = 1 \tag{5}$$

must be one of the following two cases:

(i) $f(z) = \mp \sin(Aiz + Bi)$, $A = \frac{k\pi i}{c}$, $A^k = \pm i$, where B is a constant and k is odd;

(ii) $f(z) = \pm \cos(Aiz + Bi)$, $A = \frac{(2k+1)\pi i}{2c}$, $A^k = \pm 1$, where B is a constant and k is even.

In this paper, we consider a slightly general form of the equation (5). More precisely, we get the following result.

Theorem 1.4. *Let $P(z)$, $Q(z)$ be non-zero polynomials and set*

$$L(f) = \sum_{j=0}^{k} b_j f^{(j)},$$

where k is a positive integer, and $b_0, b_1, \ldots, b_k (\neq 0)$ are constants. If the following differential-difference equation

$$f(z+c)^2 + \{P(z)L(f)(z)\}^2 = Q(z) \tag{6}$$

admits a transcendental entire solution f with $\rho_2(f) < 1$, then

$$f(z) = \frac{1}{2}\left(Q_1(z-c)\mathrm{e}^{az+b-ac} + Q_2(z-c)\mathrm{e}^{-az-b+ac}\right),$$

where $a\,(\neq 0)$, b are constants, Q_1 and Q_2 are factors of Q with $Q = Q_1Q_2$. Moreover, $P(z)$ can be determined by one of the following conditions:

(i) *$P(z)$ must be a constant if either Q_1 or Q_2 is a constant*;

(ii) *$P(z)$ must be a constant if either $l(a) \neq 0$ or $l(-a) \neq 0$, where $l(z) = \sum_{j=0}^{k} b_j z^j$*;

(iii) *$P(z)$ is a non-constant polynomial when $l(\pm a) = 0$ and if both Q_1 and Q_2 are non-constant polynomials. Further, if either $l'(a) \neq 0$ or $l'(-a) \neq 0$ holds, we have* $\deg P = 1$; *otherwise* $\deg P \geq 2$.

We exhibit some examples to show the existence of solutions in Theorem 1.4.

Example 1.5. Take $a = 1$ and $c = \frac{\pi i}{2}$. Then $f(z) = \frac{i}{2}(-\mathrm{e}^z + \mathrm{e}^{-z})$ satisfies the equation

$$f(z+c)^2 + \left\{\frac{1}{2}f(z) + \frac{1}{2}f''(z)\right\}^2 = 1.$$

Example 1.6. Take $a = 1$ and $c = \frac{\pi i}{2}$. Then $f(z) = \frac{z\mathrm{e}^z + \mathrm{e}^{-z}}{2}$ satisfies the equation

$$f(z+c)^2 + \left\{(1-\frac{\pi i}{4})f'(z) + f''(z) - (1-\frac{\pi i}{4})f'''(z)\right\}^2 = z + \frac{\pi i}{2}.$$

Example 1.7. Take $a = 1$ and $c = \frac{\pi i}{2}$. Then $f(z) = \frac{z\mathrm{e}^z + z\mathrm{e}^{-z}}{2}$ satisfies the equation

$$f(z+c)^2 + \left\{-\frac{\pi i}{4}f'(z) + 2f''(z) + \frac{\pi i}{4}f'''(z) - f^{(4)}(z)\right\}^2 = (z + \frac{\pi i}{2})^2.$$

Example 1.8. Take $a = 1$ and $c = \frac{\pi i}{2}$. Then $f(z) = \frac{(z-\frac{\pi i}{2})\mathrm{e}^z + (z-\frac{\pi i}{2})\mathrm{e}^{-z}}{2}$ satisfies the equation

$$f(z+c)^2 + \left\{z(-\frac{1}{2}f'(z) + \frac{1}{2}f'''(z))\right\}^2 = z^2.$$

Example 1.9. Take $a = 1$ and $c = 2\pi i$. Then $f(z) = \frac{(z-2\pi i)^2\mathrm{e}^z + (z-2\pi i)^2\mathrm{e}^{-z}}{2}$ satisfies the equation

$$f(z+c)^2 + \left\{iz^2(\frac{1}{8}f'(z) - \frac{1}{4}f'''(z) + \frac{1}{8}f^{(5)}(z))\right\}^2 = z^4.$$

In 2018, Zhang [18] considered existence of transcendental entire solutions of the following equation

$$f(z)^2 + \{f(z+c) - f(z)\}^2 = \beta(z)^2, \tag{7}$$

where β is a small function of f, and raised a conjecture as follows:

Conjecture 1.10. *If f is a transcendental entire solution of finite order of (7) such that β is a small function of f, then $\beta \equiv 0$.*

In other words, these results or conjecture consider admissible solutions (see, e.g., [8]). In particular, Zhang [18] proved that the difference equation (7) admits no transcendental entire functions of finite order if β is a non-zero constant. Related to the conjecture above, we give the following theorem, which extends a result in [10].

Theorem 1.11. *If $a(z), b(z)$ are non-zero rational functions, then*

$$f(z)^2 + \{a(z)f(z) + b(z)f(z+c)\}^2 = \beta(z)^2 \tag{8}$$

has no any transcendental entire function f with $\rho_2(f) < 1$ such that β is a non-vanishing small function of f under one of the following conditions:

(i) *β is a non-constant periodic function of period c*;

(ii) *β is a non-constant entire function of finite order $\rho(\beta) = \varrho$.*

It is natural to ask whether Equation (8) has a transcendental entire function f with $\rho_2(f) < 1$ when β is a non-zero constant. We can get the following theorem.

Theorem 1.12. *Suppose that* $a(z), b(z)$ *are non-zero rational functions,* $\beta(\neq 0)$ *is a constant. If* (8) *has a transcendental entire solution* f *with* $\rho_2(f) < 1$, *then* $a(z)$, $b(z)$ *reduce to constant* a, b *respectively, and satisfy* $a^2+1=b^2$, *and* $f = \beta \sin(Az+B)$, *where* B *is a constant and* $\mathrm{e}^{iAc} = \frac{a-i}{-b}$.

Example 1.13. Take $A=1$, $B=0$, and $c = -i \ln \frac{-1-i}{\sqrt{2}}$. Then $f(z) = \beta \sin z$ satisfies the difference equation

$$f(z)^2 + \{f(z) + \sqrt{2} f(z+c)\}^2 = \beta^2.$$

Corollary 1.14. *Suppose that* $a(z), b(z)$ *are non-zero rational functions,* $\beta(\neq 0)$ *is a constant. Then the equation* (8) *has no transcendental meromorphic solution* $f(z)$ *satisfying* $\rho_2(f) < 1$ *under one of the following conditions*:
(i) $a(z), b(z)$ *are non-constant rational functions*;
(ii) $a(z), b(z)$ *are non-zero constants* a, b *and* $a^2+1 \neq b^2$.

2. Some lemmas

In order to prove the results above, we need the following lemmas.

Lemma 2.1. *Let* $P(z)$ *be a non-zero entire function,* $Q(z)$ *be a non-constant entire function, and let* c *be a non-zero finite value. If* $Q(z+c)P(z) = Q(z)$, *then there exists a positive number* A *such that* $T(r,Q) \geq Ar$ *holds for sufficiently large* r.

Proof. It follows from $Q(z+c)P(z) = Q(z)$ that $Q(z)$ is transcendental. Otherwise, if $Q(z)$ is a polynomial, then $P(z)$ must be a polynomial. By comparing degrees and coefficients of the equation $Q(z+c)P(z) = Q(z)$, we find $P(z)=1$. Further, $Q(z+c) = Q(z)$ implies that Q is a constant. This is a contradiction. Next we distinguish two cases to prove the claim.

Case 1. $Q(z)$ has no zeros.

Then there exists a non-constant entire function $h(z)$ satisfying $Q(z) = \mathrm{e}^{h(z)}$, which means that there exists a positive number A such that $T(r,Q) \geq Ar$ holds for sufficiently large r.

Case 2. $Q(z)$ has at least one zero, say z_0.

Without loss of generality, we assume that $z_0 = 0$. Note that $Q(z+c)P(z) = Q(z)$ implies $Q(z)P(z-c) = Q(z-c)$. By induction, we find that $-jc$ are zeros of Q for positive integers j, so that the number $n(r, \frac{1}{Q})$ of zeros of Q in the disc $|z| \leq r$ satisfies $n(r, \frac{1}{Q}) \gtrapprox \frac{r}{|c|}$. Then there exists a positive number B such that $N(r, \frac{1}{Q}) \geq Br$ holds for sufficiently large r, and hence there exists a positive number A such that $T(r,Q) \geq N(r, \frac{1}{Q}) + O(1) \geq Ar$ holds for sufficiently large r. □

Lemma 2.2 (see, e.g., Lemma 5.1 in [17]). *If* f *is a non-constant periodic meromorphic function, then* $\rho(f) \geq 1$.

Lemma 2.3 (see, e.g., Theorem 1.45 in [17]). *If* h *is a non-constant entire function, then* $\rho_2(\mathrm{e}^h) = \rho(h)$.

Lemma 2.4 (see, e.g., Lemma 5.1 in [4])**.** *Let $a_j(z)$ be entire functions of finite order ρ and let $g_j(z)$ be entire functions such that $g_k(z) - g_j(z)\,(j \neq k)$ are transcendental entire functions or polynomials of degree greater than ρ. Then*

$$\sum_{j=1}^{n} a_j(z)\mathrm{e}^{g_j(z)} = a_0(z)$$

holds only when

$$a_0(z) = a_1(z) = \cdots = a_n(z) \equiv 0.$$

Lemma 2.5. *Let $b_j(z)$ be meromorphic functions of finite order ρ such that $b_j(z)$ has only finitely many poles for each j. Let $g_j(z)$ be entire functions such that $g_k(z) - g_j(z)\,(j \neq k)$ are transcendental entire functions or polynomials of degree greater than ρ. Then*

$$\sum_{j=1}^{n} b_j(z)\mathrm{e}^{g_j(z)} = b_0(z)$$

holds only when

$$b_0(z) = b_1(z) = \cdots = b_n(z) \equiv 0.$$

Proof. Suppose that $b_j(z)$ has a finite number of poles, say $z_{j1}, z_{j2}, \ldots, z_{jk_j}$ with multiplicity $m_{j1}, m_{j2}, \ldots, m_{jk_j}$, respectively, and set

$$p(z) = \prod_{j=0}^{n}\prod_{i=1}^{k_j}(z - z_{ji})^{m_{ji}}.$$

Applying Lemma 2.4 to the equation

$$\sum_{j=1}^{n} p(z)b_j(z)\mathrm{e}^{g_j(z)} = p(z)b_0(z),$$

we obtain

$$b_0(z) = b_1(z) = \cdots = b_n(z) \equiv 0. \qquad \square$$

Lemma 2.6 (see, e.g., [1])**.** *Let g be a transcendental meromorphic function of order less than 1, and let h be a positive constant. Then there exists an ε-set E such that as $\mathbb{C}\backslash E \ni z \to \infty$, one has*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \quad \frac{g(z+\eta)}{g(z)} \to 1$$

uniformly in η for $|\eta| \leq h$. Further, the ε-set E may be chosen so that for large z not in E, the function g has no zeros or poles in $|\zeta - z| \leq h$.

According to the works of Hayman (see, e.g., [6]), an ε-set E is defined to be any countable set of circles not containing the origin, and subtending angles at the origin whose sum s is finite, in which the number s is called the (angular) extent of the ε-set E. A basic fact remarked by Hayman [6] is that the set

S of r for which the circle $|z| = r$ meets the circles of an ε-set E has finite logarithmic measure.

3. Proof of Theorem 1.1

Suppose that (4) admits a transcendental entire solution f with $\rho_2(f) < 1$ such that P, Q are small functions of f. Set

$$G(z) = f^2(z), \quad H(z) = P^2(z)f^2(z+c). \tag{9}$$

Then (4) can be rewritten as

$$G(z) - Q(z) = -H(z) = -P^2(z)f^2(z+c), \tag{10}$$

which means $G(z-c) - Q(z-c) = -H(z-c)$. By (9) and (10), we have $H(z-c) = P^2(z-c)G(z)$ and

$$G(z) - R_c(z) = -\frac{f^2(z-c)}{P^2(z-c)}, \tag{11}$$

where R_c is a small function of f defined by

$$R_c(z) = \frac{Q(z-c)}{P^2(z-c)}.$$

Assume, to the contrary, that $P^2(z)Q(z+c) \not\equiv Q(z)$, that is, $R_c \neq Q$. By using the second main theorem for small functions (see, e.g., [14]), we get an inequality containing Nevanlinna's characteristic functions as follows:

$$2T(r,G) \le \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G-Q}\right) + \overline{N}\left(r, \frac{1}{G-R_c}\right) + S(r,G).$$

Note that $\overline{N}(r,G) = 0$ and

$$\overline{N}\left(r, \frac{1}{G}\right) \le \frac{1}{2}N\left(r, \frac{1}{G}\right) \le \frac{1}{2}T(r,G) + O(1),$$

$$\overline{N}\left(r, \frac{1}{G-Q}\right) \le \frac{1}{2}N\left(r, \frac{1}{G-Q}\right) \le \frac{1}{2}T(r,G) + S(r,G),$$

$$\overline{N}\left(r, \frac{1}{G-R_c}\right) \le \frac{1}{2}N\left(r, \frac{1}{G-R_c}\right) \le \frac{1}{2}T(r,G) + S(r,G).$$

Then we obtain

$$2T(r,G) \le \frac{3}{2}T(r,G) + S(r,G),$$

which is impossible. Therefore, we have

$$P^2(z)Q(z+c) = Q(z),$$

which completes the proof of Theorem 1.1.

4. Proof of Corollary 1.2

It follows from Theorem 1.1 that

$$P^2(z)Q(z+c) = Q(z). \tag{12}$$

If Q is not a constant, Lemma 2.1 yields $\rho(Q) \geq 1$, which contradicts the assumption $\rho(Q) < 1$. Thus $Q(z)$ reduces to a constant, say $Q(z) = q$, and hence $P^2(z) = 1$.

Furthermore, (4) gives

$$[f(z) + if(z+c)][f(z) - if(z+c)] = q, \tag{13}$$

which yields immediately

$$f(z) + if(z+c) = q_1 \mathrm{e}^{h(z)},\ f(z) - if(z+c) = q_2 \mathrm{e}^{-h(z)}, \tag{14}$$

where $h(z)$ is a non-constant entire function, and q_1, q_2 are constants with $q_1 q_2 = q$. It follows from (14) that

$$f(z) = \frac{q_1 \mathrm{e}^{h(z)} + q_2 \mathrm{e}^{-h(z)}}{2}, \quad f(z+c) = \frac{q_1 \mathrm{e}^{h(z)} - q_2 \mathrm{e}^{-h(z)}}{2i}. \tag{15}$$

Moreover, (15) implies

$$T(r, f) = 2T(r, \mathrm{e}^h) + O(1),$$

Lemma 2.3 yields $\rho(h) = \rho_2(f) < 1$.

Making use of (15) again, we obtain

$$iq_1 \mathrm{e}^{g_1(z)} + iq_2 \mathrm{e}^{g_2(z)} - q_1 \mathrm{e}^{g_3(z)} + q_2 = 0, \tag{16}$$

where

$$g_1(z) = h(z+c) + h(z),\ g_2(z) = h(z) - h(z+c),\ g_3(z) = 2h(z).$$

By applying Lemma 2.4 to (16), then either $-g_2(z) = g_1(z) - g_3(z) = h(z+c) - h(z)$ or $g_1(z) = g_3(z) - g_2(z) = h(z+c) + h(z)$ is a constant.

If $h(z+c) + h(z)$ is a constant, then $h(z)$ is not a non-constant polynomial. Otherwise, $0 = \deg[h(z+c) + h(z)] = \deg h(z) \geq 1$. This is a contradiction. Hence $h(z)$ is a transcendental entire function of order less than 1. We conclude that $h'(z+c) + h'(z) \equiv 0$, that is $\frac{h'(z+c)}{h'(z)} \equiv -1$. Since $\rho(h') = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h'(z+c)}{h'(z)} \to 1$$

as $\mathbb{C} \backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction.

Therefore, $h(z+c) - h(z)$ must be a constant. We know then that $h'(z+c) - h'(z) \equiv 0$. This implies that $h'(z)$ is a periodic function with period c. Since $\rho(h') = \rho(h) < 1$, it follows from Lemma 2.2 that $h' = a$, where a is a non-zero constant, so that $h(z) = az + b$, where b is a constant. Thus, we get $f(z) = \frac{q_1 \mathrm{e}^{az+b} + q_2 \mathrm{e}^{-az-b}}{2}$. And by $f(z+c) = f(z+c)$ in (15), we get $\mathrm{e}^{ac} = -i$, that is, $ac = -\frac{\pi i}{2} + 2k\pi i$ ($k \in \mathbb{Z}$). Corollary 1.2 follows.

5. Proof of Theorem 1.4

Suppose that f is a transcendental entire solution of (6) with $\rho_2(f) < 1$. Then we have

$$[f(z+c) + iP(z)L(f)(z)][f(z+c) - iP(z)L(f)(z)] = Q(z), \tag{17}$$

thus, both $f(z+c) + iP(z)L(f)(z)$ and $f(z+c) - iP(z)L(f)(z)$ have finitely many zeros, so that

$$f(z+c) + iP(z)L(f)(z) = Q_1(z)\mathrm{e}^{h(z)},$$
$$f(z+c) - iP(z)L(f)(z) = Q_2(z)\mathrm{e}^{-h(z)},$$

where Q_1, Q_2 are polynomials with $Q_1Q_2 = Q$ and h is a non-constant entire function. It follows that

$$f(z+c) = \frac{Q_1(z)\mathrm{e}^{h(z)} + Q_2(z)\mathrm{e}^{-h(z)}}{2}, \tag{18}$$

$$L(f)(z) = \frac{Q_1(z)\mathrm{e}^{h(z)} - Q_2(z)\mathrm{e}^{-h(z)}}{2iP(z)}. \tag{19}$$

Moreover, (18) shows that the function $f_c(z) = f(z+c)$ satisfies

$$T(r, f_c) = 2T(r, \mathrm{e}^h) + O(\log r).$$

Since $\rho_2(f) < 1$, we have

$$T(r, f) = T(r, f_c) + S(r, f),$$

see, e.g., [5], and hence

$$T(r, f) = 2T(r, \mathrm{e}^h) + S(r, f).$$

Thus Lemma 2.3 yields $\rho(h) = \rho_2(f) < 1$.

By differentiating (18), we have

$$f^{(j)}(z+c) = \frac{M_j(z)\mathrm{e}^{h(z)} + N_j(z)\mathrm{e}^{-h(z)}}{2}, \tag{20}$$

where

$$\begin{aligned} M_j &= Q_1^{(j)} + jQ_1^{(j-1)}h' + \cdots + jQ_1'[(h')^{j-1} + L_{j-2}(h')] \\ &\quad + Q_1[(h')^j + L_{j-1}(h')], \\ N_j &= Q_2^{(j)} + jQ_2^{(j-1)}(-h') + \cdots + jQ_2'[(-h')^{j-1} + R_{j-2}(-h')] \\ &\quad + Q_2[(-h')^j + R_{j-1}(-h')], \end{aligned}$$

in which L_{j-1}, L_{j-2}, R_{j-1}, R_{j-2} are polynomials of $h^{(k)}, \ldots, h'$ such that $\deg L_{j-1} \leq j$, $\deg R_{j-1} \leq j$, $\deg L_{j-2} \leq j-1$, $\deg R_{j-2} \leq j-1$. By (19) and (20), one can obtain

$$Q_1(z+c)\mathrm{e}^{h(z+c)} - Q_2(z+c)\mathrm{e}^{-h(z+c)} - iM(z)\mathrm{e}^{h(z)} = iN(z)\mathrm{e}^{-h(z)},$$

where

$$M(z) = P(z+c)\sum_{j=0}^{k} b_j M_j(z),\ N(z) = P(z+c)\sum_{j=0}^{k} b_j N_j(z),$$

or equivalently

$$Q_1(z+c)\mathrm{e}^{g_1(z)} - Q_2(z+c)\mathrm{e}^{g_2(z)} - iM(z)\mathrm{e}^{g_3(z)} = iN(z), \tag{21}$$

where

$$g_1(z) = h(z+c) + h(z),\ g_2(z) = h(z) - h(z+c),\ g_3(z) = 2h(z).$$

Moreover, it is easy to show that $\rho(M) < 1$ and $\rho(N) < 1$ since $\rho(h) < 1$. Next we distinguish four cases to discuss the equation (21).

Case 1. $M(z) \equiv 0$ and $N(z) \equiv 0$.

The equation (21) gives

$$Q_1(z+c)\mathrm{e}^{g_1(z)} = Q_2(z+c)\mathrm{e}^{g_2(z)},$$

that is

$$\mathrm{e}^{2h(z+c)} = \mathrm{e}^{g_1(z)-g_2(z)} = \frac{Q_2(z+c)}{Q_1(z+c)}.$$

That is a contradiction because $h(z)$ is a non-constant entire function, so that **Case 1** is ruled out.

Case 2. $M(z) \not\equiv 0$ and $N(z) \equiv 0$.

Now (21) turns into

$$Q_1(z+c)\mathrm{e}^{g_1(z)-g_3(z)} - Q_2(z+c)\mathrm{e}^{g_2(z)-g_3(z)} = iM(z). \tag{22}$$

By using Lemma 2.4, either $g_1(z) - g_3(z) = h(z+c) - h(z)$ or $g_3(z) - g_2(z) = h(z+c) + h(z)$ is a constant.

If $h(z+c) + h(z)$ is a constant, we can rule out the case that $h(z)$ is a non-constant polynomial because $0 = \deg[h(z+c) + h(z)] = \deg h(z) \geq 1$, which is a contradiction. Thus $h(z)$ is a transcendental entire function of order less than 1. We conclude that $h'(z+c) + h'(z) \equiv 0$, that is, $\frac{h'(z+c)}{h'(z)} \equiv -1$. Since $\rho(h') = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h'(z+c)}{h'(z)} \to 1$$

as $\mathbb{C}\backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction again.

If $h(z+c) - h(z)$ is a constant, say A, but $h(z+c) + h(z)$ is not a constant. Rewrite (22) into the following form

$$Q_2(z+c)\mathrm{e}^{-h(z+c)-h(z)} = Q_1(z+c)\mathrm{e}^{A} - iM(z).$$

By comparing the order of both sides, we get a contradiction again, so that **Case 2** is ruled out.

Case 3. $M(z) \equiv 0$ and $N(z) \not\equiv 0$.

Then (21) turns into

$$Q_1(z+c)e^{g_1(z)} - Q_2(z+c)e^{g_2(z)} = iN(z). \tag{23}$$

By Lemma 2.4, either $g_1(z) = h(z+c) + h(z)$ or $g_2(z) = h(z) - h(z+c)$ is a constant.

If $h(z+c) + h(z)$ is a constant, then $h(z)$ is not a non-constant polynomial. Otherwise, $0 = \deg[h(z+c) + h(z)] = \deg h(z) \geq 1$, which is a contradiction. Hence $h(z)$ is a transcendental entire function of order less than 1. We conclude that $h'(z+c)+h'(z) \equiv 0$, that is, $\frac{h'(z+c)}{h'(z)} \equiv -1$. Since $\rho(h') = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h'(z+c)}{h'(z)} \to 1$$

as $\mathbb{C}\backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction.

If $h(z) - h(z+c)$ is a constant, say B, but $h(z+c) + h(z)$ is not a constant. Rewrite (23) into the following form

$$Q_1(z+c)e^{h(z+c)+h(z)} = Q_2(z+c)e^{B} + iN(z).$$

We also get a contradiction by comparing the order of both sides, so that **Case 3** is ruled out.

Case 4. $M(z) \not\equiv 0$ and $N(z) \not\equiv 0$.

Applying Lemma 2.4 to (21), either $-g_2(z) = g_1(z) - g_3(z) = h(z+c) - h(z)$ or $g_1(z) = g_3(z) - g_2(z) = h(z) + h(z+c)$ is a constant.

If $h(z+c) + h(z)$ is a constant, we easily see that $h(z)$ is not a non-constant polynomial. Otherwise, $0 = \deg[h(z+c) + h(z)] = \deg h(z) \geq 1$, which is a contradiction. Then $h(z)$ is a transcendental entire function of order less than 1. We conclude that $h'(z+c) + h'(z) \equiv 0$, that is, $\frac{h'(z+c)}{h'(z)} \equiv -1$. Since $\rho(h') = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h'(z+c)}{h'(z)} \to 1$$

as $\mathbb{C}\backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction.

Therefore, $h(z+c) - h(z)$ must be a constant, but $h(z+c) + h(z)$ is not a constant. Then we have $h'(z+c) - h'(z) \equiv 0$. This implies that $h'(z)$ is a periodic function with period c. Since $\rho(h') = \rho(h) < 1$, it follows from Lemma 2.2 that $h' = a$, where a is a non-zero constant, and hence $h(z) = az + b$, where b is a constant.

Thus, by the equation of (18), it yields the conclusion

$$f(z) = \frac{Q_1(z-c)e^{az+b-ac} + Q_2(z-c)e^{-az-b+ac}}{2}.$$

Moreover, the polynomial P can be determined as follows: Putting $h = az+b$ into (21), we get

$$Q_1(z+c)e^{2az+2b+ac} - Q_2(z+c)e^{-ac} - iM(z)e^{2az+2b} = iN(z),$$

which gives

$$\begin{cases} iM(z) = \mathrm{e}^{ac}Q_1(z+c), \\ iN(z) = -\mathrm{e}^{-ac}Q_2(z+c). \end{cases}$$

By using the expressions of M_j and N_j, the system above becomes

$$\tag{24} \begin{cases} iP(z+c)\sum_{j=0}^{k} b_j \left[a^j Q_1(z) + ja^{j-1}Q_1'(z) + \cdots + Q_1^{(j)}(z)\right] = \mathrm{e}^{ac}Q_1(z+c), \\ iP(z+c)\sum_{j=0}^{k} b_j \left[(-a)^j Q_2(z) + j(-a)^{j-1}Q_2'(z) + \cdots + Q_2^{(j)}(z)\right] = -\mathrm{e}^{-ac}Q_2(z+c). \end{cases}$$

Next, we distinguish three cases to determine $P(z)$.

Subcase 4.1. If either Q_1 or Q_2 is a constant, the equation (24) becomes either

$$iP(z+c)l(a) = \mathrm{e}^{ac}$$

if Q_1 is a constant, or

$$iP(z+c)l(-a) = -\mathrm{e}^{-ac}$$

if Q_2 is a constant, where $l(z) = \sum_{j=0}^{k} b_j z^j$, that is, P is a constant. For this case, we also have $l(\pm a) \neq 0$.

Subcase 4.2. If either $l(a) \neq 0$ or $l(-a) \neq 0$, say $l(a) \neq 0$, then we find that P is a constant by comparing the coefficients of the first equation in (24). For this case, we must have $l(-a) \neq 0$. Conversely, if $l(-a) \neq 0$, we can obtain similar conclusion by comparing the coefficients of the equation (24).

Subcase 4.3. When $l(\pm a) = 0$ and if both Q_1 and Q_2 are non-constant polynomials, the equation (24) becomes

$$\tag{25} \begin{cases} iP(z+c)\sum_{j=0}^{k} b_j \left[ja^{j-1}Q_1'(z) + \cdots + Q_1^{(j)}(z)\right] = \mathrm{e}^{ac}Q_1(z+c), \\ iP(z+c)\sum_{j=0}^{k} b_j \left[j(-a)^{j-1}Q_2'(z) + \cdots + Q_2^{(j)}(z)\right] = -\mathrm{e}^{-ac}Q_2(z+c). \end{cases}$$

Further, if either $l'(a) \neq 0$ or $l'(-a) \neq 0$, say $l'(a) \neq 0$, we find that P is linear by comparing the coefficients of the first equation in (25). For this case, we must have $l'(-a) \neq 0$. Conversely, if $l'(-a) \neq 0$, we can obtain similar conclusion by comparing the coefficients of the equation (25).

Otherwise, that is, $l(\pm a) = 0$ and $l'(\pm a) = 0$, the equation (25) becomes the following form

$$\tag{26} \begin{cases} iP(z+c)\sum_{j=0}^{k} b_j \left[A_j Q_1''(z) + \cdots + Q_1^{(j)}(z)\right] = \mathrm{e}^{ac}Q_1(z+c), \\ iP(z+c)\sum_{j=0}^{k} b_j \left[B_j Q_2''(z) + \cdots + Q_2^{(j)}(z)\right] = -\mathrm{e}^{-ac}Q_2(z+c), \end{cases}$$

where A_j, B_j are well-known constants, which obviously implies that $\deg P \geq 2$. For this case, we also have $\deg Q_1 \geq 2$ and $\deg Q_2 \geq 2$.

Therefore, Theorem 1.4 follows.

6. Proof of Theorem 1.11

Suppose, to the contrary, that f is a transcendental entire solution of (8) with $\rho_2(f) < 1$ such that β is a non-vanishing small function of f under one of the conditions (i) and (ii) of Theorem 1.11. Now we rewrite (8) into the following form

$$\left[\frac{f(z)}{\beta(z)}\right]^2 + \left[\frac{a(z)f(z) + b(z)f(z+c)}{\beta(z)}\right]^2 = 1,$$

which gives

$$f(z) = \beta(z)\sin h(z), \quad a(z)f(z) + b(z)f(z+c) = \beta(z)\cos h(z) \tag{27}$$

by Iyer's result [2], where h is an entire function. Obviously, h is non-constant. Moreover, by (27) and Lemma 2.3, we easily get $\rho(h) = \rho_2(f) < 1$. Elimating f from (27), we obtain

$$(a(z)-i)\beta(z)e^{g_1(z)} - (a(z)+i)\beta(z)e^{g_2(z)} + b(z)\beta(z+c)e^{g_3(z)} = b(z)\beta(z+c), \tag{28}$$

where

$$g_1(z) = ih(z) + ih(z+c),\ g_2(z) = ih(z+c) - ih(z),\ g_3(z) = 2ih(z+c).$$

Under the condition (i) of Theorem 1.11, that is, if β is a non-constant periodic function with period c, then we may rewrite (28) into the following form

$$(a(z) - i)e^{g_1(z)} - (a(z) + i)e^{g_2(z)} + b(z)e^{g_3(z)} = b(z). \tag{29}$$

Applying Lemma 2.5 to (29), we find that $g_2(z) = g_3(z) - g_1(z) = ih(z+c) - ih(z)$ or $g_1(z) = g_3(z) - g_2(z) = ih(z+c) + ih(z)$ is a constant.

If $ih(z+c)+ih(z)$ is a constant, then $h(z)$ is not a non-constant polynomial. Otherwise, $0 = \deg[h(z+c) + h(z)] = \deg h(z) \geq 1$, which is a contradiction. Hence $h(z)$ is a transcendental entire function of order less than 1. We conclude that $h'(z+c)+h'(z) \equiv 0$, that is, $\frac{h'(z+c)}{h'(z)} \equiv -1$. Since $\rho(h') = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h'(z+c)}{h'(z)} \to 1$$

as $\mathbb{C}\backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction.

Therefore, $ih(z+c) - ih(z)$ must be a constant. It follows that $h'(z+c) - h'(z) \equiv 0$, that is, $h'(z)$ is a periodic function with period c. Since $\rho(h') = \rho(h) < 1$, it follows from Lemma 2.2 that $h' = a$, where a is a non-zero constant, so that $h(z) = az + b$, where b is a constant. Putting $h(z) = az + b$ into (27), we deduce $f(z) = \beta(z)\sin(az+b)$, which tells us that $\rho(f) = 1$. However, since β is a non-constant periodic function with period c, it follows from Lemma 2.2

that $\rho(\beta) \geq 1$, which therefore implies that β is not a small function of f. This is a contradiction.

Under the condition (ii) of Theorem 1.11, that is, β is an non-constant non-vanishing entire function of finite order $\rho(\beta) = \varrho$, then we have $\beta(z) = \mathrm{e}^{p(z)}$, where $p(z)$ is a non-constant polynomial of degree ϱ. Now we can rewrite (28) into the following form

$$(a(z)-i)\mathrm{e}^{p(z)-p(z+c)+g_1(z)} - (a(z)+i)\mathrm{e}^{p(z)-p(z+c)+g_2(z)} + b(z)\mathrm{e}^{g_3(z)} = b(z). \tag{30}$$

Applying Lemma 2.5 to (30), we find that either

$$h_1 = p(z) - p(z+c) + g_1(z) = p(z) - p(z+c) + i[h(z) + h(z+c)]$$

or

$$h_2 = p(z) - p(z+c) + g_1(z) - g_3(z) = p(z) - p(z+c) + i[h(z) - h(z+c)]$$

or

$$h_3 = p(z) - p(z+c) + g_2(z) = p(z) - p(z+c) - i[h(z) - h(z+c)]$$

or

$$h_4 = p(z) - p(z+c) + g_2(z) - g_3(z) = p(z) - p(z+c) - i[h(z) + h(z+c)]$$

is a constant.

If h_1 is a constant, but $h(z)$ is a non-constant polynomial, then

$$h(z+c) + h(z) = -i\{p(z+c) - p(z) + h_1\}$$

is a polynomial with degree $s = \varrho - 1$. Note that $\beta(z) = \mathrm{e}^{p(z)}$ is a small function of f, that gives $\varrho = \deg p(z) < \deg h(z) = s$. This is a contradiction. When $h(z)$ is a transcendental entire function of order less than 1, we see that

$$h(z+c) + h(z) = -i\{p(z+c) - p(z) + h_1\}$$

is a polynomial with degree $s = \varrho - 1$, and hence $h^{(s+1)}(z+c) + h^{(s+1)}(z) \equiv 0$. Since $\rho(h^{(s+1)}) = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h^{(s+1)}(z+c)}{h^{(s+1)}(z)} \to 1$$

as $\mathbb{C}\backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction.

If h_2 is a constant, then

$$h(z+c) - h(z) = i\{p(z+c) - p(z) + h_2\}$$

is a polynomial with degree $s = \varrho - 1$, so that $h^{(s+1)}(z+c) - h^{(s+1)}(z) \equiv 0$. This implies that $h^{(s+1)}(z)$ is a periodic function with period c. Since $\rho(h^{(s+1)}) = \rho(h) < 1$, it follows from Lemma 2.2 that $h^{(s+1)}$ is a constant, that is, h is a polynomial with $\deg h \leq s+1$. Note that β is a small function of f and $f(z) = \beta(z) \sin h(z)$. These results therefore deduce $\varrho < s+1$. This is a contradiction.

If h_3 is a constant, then

$$h(z+c) - h(z) = -i\{p(z+c) - p(z) + h_3\}$$

is a polynomial with degree $s = \varrho-1$, so that $h^{(s+1)}(z+c)-h^{(s+1)}(z) \equiv 0$. This implies that $h^{(s+1)}(z)$ is a periodic function with period c. Since $\rho(h^{(s+1)}) = \rho(h) < 1$, it follows from Lemma 2.2 that $h^{(s+1)}$ is a constant, that is, h is a polynomial with $\deg h \leq s+1$. Note that β is a small function of f and $f(z) = \beta(z) \sin h(z)$. These results deduce $\varrho < s+1$. This is a contradiction.

If h_4 is a constant, but $h(z)$ is a non-constant polynomial, then

$$h(z+c) + h(z) = i\{p(z+c) - p(z) + h_4\}$$

is a polynomial with degree $s = \varrho-1$. Note that $\beta(z) = \mathrm{e}^{p(z)}$ is a small function of f. It gives $\varrho = \deg p(z) < \deg h(z) = s$. This is a contradiction. When $h(z)$ is a transcendental entire function of order less than 1, we see that

$$h(z+c) + h(z) = i\{p(z+c) - p(z) + h_4\}$$

is a polynomial with degree $s = \varrho - 1$, and hence $h^{(s+1)}(z+c) + h^{(s+1)}(z) \equiv 0$. Since $\rho(h^{(s+1)}) = \rho(h) < 1$, Lemma 2.6 yields

$$-1 \equiv \frac{h^{(s+1)}(z+c)}{h^{(s+1)}(z)} \to 1$$

as $\mathbb{C}\backslash E \ni z \to \infty$, where E is an ε-set. This is a contradiction, and Theorem 1.11 follows.

7. Proof of Theorem 1.12

Similar to the case β is a non-constant periodic function in Theorem 1.11, we can also get (29). Thus Lemma 2.5 yields that h' is a non-zero constant, say A, so that $h(z) = Az + B$, where B is a constant. Then (27) gives

$$f(z) = \beta \sin(Az + B). \tag{31}$$

Putting $h = Az + B$ into (29), we can obtain

$$\begin{cases} (a(z) - i)\mathrm{e}^{-iAc} = -b(z), \\ -(a(z) + i)\mathrm{e}^{iAc} = b(z), \end{cases} \tag{32}$$

which implies

$$a(z)^2 + 1 = b(z)^2. \tag{33}$$

Now we rewrite equation (8) into the following form

$$f^2(z) + a^2(z)f^2(z) + 2a(z)b(z)f(z)f(z+c) + b^2(z)f^2(z+c) = \beta^2.$$

By using (33), the above equation can be converted into

$$b^2(z)f^2(z) + 2a(z)b(z)f(z)f(z+c) + b^2(z)f^2(z+c) = \beta^2. \tag{34}$$

Further, together with (31), we have

$$\begin{aligned} &[b^2(z)\mathrm{e}^{2ib} + 2a(z)b(z)\mathrm{e}^{2ib+iac} + b^2(z)\mathrm{e}^{2ib+2iac}]\mathrm{e}^{2iaz} \\ &+ [b^2(z)\mathrm{e}^{-2ib} + 2a(z)b(z)\mathrm{e}^{-2ib-iac} + b^2(z)\mathrm{e}^{-2ib-2iac}]\mathrm{e}^{-2iaz} \\ = {}& 4b^2(z) - 2a(z)b(z)[\mathrm{e}^{iac} + \mathrm{e}^{-iac}] - 4. \end{aligned} \tag{35}$$

Applying Lemma 2.5 to equation (35), we see

$$\begin{cases} b^2(z)e^{2ib} + 2a(z)b(z)e^{2ib+iac} + b^2(z)e^{2ib+2iac} \equiv 0, \\ b^2(z)e^{-2ib} + 2a(z)b(z)e^{-2ib-iac} + b^2(z)e^{-2ib-2iac} \equiv 0, \\ 4b^2(z) + 2a(z)b(z)[e^{iac} + e^{-iac}] \equiv 4. \end{cases} \tag{36}$$

The first equation of (36) yields

$$2a(z)b(z) = -b^2(z)[e^{iac} + e^{-iac}].$$

Combining this with the third equation in (36), we see

$$b^2(z)[4 - (e^{iac} + e^{-iac})^2] = 4,$$

which implies that $b(z)$ is a constant b, and thus $a(z)$ reduce to a constant a. It follows from (33) that $a^2 + 1 = b^2$. The first equation in (32) implies that $e^{iAc} = \frac{a-i}{-b}$. Thus, Theorem 1.12 follows.

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