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FURTHER RESULTS ON BIASES IN INTEGER PARTITIONS

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Abstract. Let $p_{a,b,m}(n)$ be the number of integer partitions of n with more parts congruent to a modulo m than parts congruent to b modulo m. We prove that $p_{a,b,m}(n) \geq p_{b,a,m}(n)$ whenever $1 \leq a < b \leq m$. We also propose some conjectures concerning series with nonnegative coefficients in their expansions.

1. Introduction

In analogy to *Chebyshev's bias* [3] concerning the excess of the number of primes of the form $4k+3$ over the number of primes of the form $4k+1$, B. Kim, E. Kim, and J. Lovejoy [5] introduced a phenomenon called *parity bias* for integer partitions.

Theorem 1.1 (B. Kim, E. Kim, and J. Lovejoy)**.** *Let $p_o(n)$ (resp. $p_e(n)$) denote the number of integer partitions of n with more odd parts than even parts (resp. with more even parts than odd parts). Then*

$$p_o(n) \geq p_e(n).$$

This phenomenon is called "parity bias" for integer partitions.

Recently, B. Kim and E. Kim [4] went on to investigate this phenomenon in a more general setting. Let us first adopt their notation.

Definition. We denote by $p_{a,b,m}(n)$ the number of partitions of n with more parts congruent to a modulo m than parts congruent to b modulo m.

Making use of the above notation, we have $p_o(n) = p_{1,2,2}(n)$ and $p_e(n) = p_{2,1,2}(n)$ and therefore arrive at the inequality $p_{1,2,2}(n) \geq p_{2,1,2}(n)$ from Theorem 1.1. Similar phenomena shown in [4] also include inequalities as follows.

Theorem 1.2 (B. Kim and E. Kim)**.** *Let $m \geq 2$ be an integer. Then*

$$p_{1,m,m}(n) \geq p_{m,1,m}(n),$$
$$p_{1,m-1,m}(n) \geq p_{m-1,1,m}(n).$$

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Our object here is to extend the above results for general $p_{a,b,m}(n)$.

Theorem 1.3. *Let $m \geq 2$ be an integer. For any two integers a and b with $1 \leq a < b \leq m$, we have*

$$p_{a,b,m}(n) \geq p_{b,a,m}(n). \tag{1}$$

We separate this theorem into two cases. First, we prove the case $(a, b) \neq (1, 2)$ using q-series manipulations. Then we provide an injective proof for $(a, b) = (1, 2)$.

2. Case $(a, b) \neq (1, 2)$

Let us first recall the notation of q-Pochhammer symbols: for $n \in \mathbb{N} \cup \{\infty\}$,

$$(A; q)_n := \prod_{k=0}^{n-1} (1 - Aq^k),$$
$$(A_1, A_2, \ldots, A_m; q)_n := (A_1; q)_n (A_2; q)_n \cdots (A_m; q)_n.$$

Next, given an integer partition λ, we denote by $|\lambda|$ the sum of parts in λ and by $\sharp_{a,m}(\lambda)$ the number of parts in λ that are congruent to a modulo m. Let $\mathscr{P}$ be the set of integer partitions.

Our starting point is the following trivial trivariate generating function:

$$\sum_{\lambda \in \mathscr{P}} x^{\sharp_{a,m}(\lambda)} y^{\sharp_{b,m}(\lambda)} q^{|\lambda|} = \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \frac{1}{(xq^a, yq^b; q^m)_\infty}, \tag{2}$$

provided that $1 \leq a, b \leq m$ and $a \neq b$.

We are then led to the following lemma.

Lemma 2.1. *Let $1 \leq a, b \leq m$ and $a \neq b$. We have*

$$\sum_{n \geq 0} p_{a,b,m}(n) q^n = \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{\substack{i,j \geq 0 \\ i > j}} \frac{q^{ai+bj}}{(q^m; q^m)_i (q^m; q^m)_j}. \tag{3}$$

Proof. Recall Euler's first identity [2, p. 19, (2.2.5)]:

$$\frac{1}{(z; q)_\infty} = \sum_{n \geq 0} \frac{z^n}{(q; q)_n}. \tag{4}$$

Setting $y = x^{-1}$ in (2) yields

$$\begin{aligned}
&\sum_{\lambda \in \mathscr{P}} x^{\sharp_{a,m}(\lambda) - \sharp_{b,m}(\lambda)} q^{|\lambda|} \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \frac{1}{(xq^a, x^{-1}q^b; q^m)_\infty} \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q; q)_\infty} \sum_{i \geq 0} \frac{x^i q^{ai}}{(q^m; q^m)_i} \sum_{j \geq 0} \frac{x^{-j} q^{bj}}{(q^m; q^m)_j} \quad \text{(by using (4) twice)}
\end{aligned}$$

$$= \frac{(q^a, q^b; q^m)_\infty}{(q;q)_\infty} \sum_{i,j\geq 0} \frac{x^{i-j} q^{ai+bj}}{(q^m;q^m)_i (q^m;q^m)_j}.$$

Noticing that $p_{a,b,m}(n)$ counts the number of partitions λ of n such that $\sharp_{a,m}(\lambda) > \sharp_{b,m}(\lambda)$, we must single out terms in the above with positive exponents in x and therefore terms with $i - j > 0$. The desired result immediately follows. □

Now, we are in a position to prove Theorem 1.3 for $(a, b) \neq (1, 2)$.

Proof of Theorem 1.3 for $(a,b) \neq (1,2)$. Recall that $1 \leq a < b \leq m$. The following is a simple consequence of Lemma 2.1:

$$\begin{aligned}
&\sum_{n\geq 0} \big(p_{a,b,m}(n) - p_{b,a,m}(n)\big) q^n \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q;q)_\infty} \sum_{\substack{i,j\geq 0 \\ i>j}} \left(\frac{q^{ai+bj}}{(q^m;q^m)_i (q^m;q^m)_j} - \frac{q^{bi+aj}}{(q^m;q^m)_i (q^m;q^m)_j} \right) \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q;q)_\infty} \sum_{\substack{i,j\geq 0 \\ i>j}} \frac{q^{ai+bj}(1 - q^{a(j-i)+b(i-j)})}{(q^m;q^m)_i (q^m;q^m)_j} \\
&= \frac{(q^a, q^b; q^m)_\infty}{(q;q)_\infty} \sum_{j\geq 0} \sum_{k\geq 1} \frac{q^{a(j+k)+bj}(1 - q^{(b-a)k})}{(q^m;q^m)_j (q^m;q^m)_{j+k}}.
\end{aligned}$$

We then consider two subcases.

Subcase I. $a \neq 1$. Noticing that $(b-a)k$ is always a positive integer, we may factor $1 - q^{(b-a)k}$ as $(1-q)(1 + q + q^2 + \cdots + q^{(b-a)k-1})$. Thus,

$$\begin{aligned}
&\sum_{n\geq 0} \big(p_{a,b,m}(n) - p_{b,a,m}(n)\big) q^n \\
&= \frac{(1-q)(q^a, q^b; q^m)_\infty}{(q;q)_\infty} \sum_{j\geq 0} \sum_{k\geq 1} \frac{q^{a(j+k)+bj}(1 + q + q^2 + \cdots + q^{(b-a)k-1})}{(q^m;q^m)_j (q^m;q^m)_{j+k}}.
\end{aligned}$$

Apparently, the Taylor expansion of the double series in the above has nonnegative coefficients. For the infinite product in the above, we have, as $2 \leq a < b \leq m$,

$$\frac{(1-q)(q^a, q^b; q^m)_\infty}{(q;q)_\infty} = \frac{(q^a, q^b; q^m)_\infty}{(q^2;q)_\infty},$$

which also has nonnegative coefficients in its series expansion. We therefore conclude that $p_{a,b,m}(n) \geq p_{b,a,m}(n)$ for $a \neq 1$.

Subcase II. $a = 1$ and $b \neq 2$. We have

$$\sum_{n\geq 0} \big(p_{1,b,m}(n) - p_{b,1,m}(n)\big) q^n = \frac{(q, q^b; q^m)_\infty}{(q;q)_\infty} \sum_{j\geq 0} \sum_{k\geq 1} \frac{q^{(j+k)+bj}(1 - q^{(b-1)k})}{(q^m;q^m)_j (q^m;q^m)_{j+k}}.$$

Notice that $b > a = 1$. This time we should factor $1 - q^{(b-1)k}$ as $(1 - q^{b-1})(1 + q^{b-1} + \cdots + q^{(b-1)(k-1)})$. Thus,

$$\begin{aligned}&\sum_{n\geq 0}\big(p_{1,b,m}(n) - p_{b,1,m}(n)\big)q^n\\ &= \frac{(1-q^{b-1})(q,q^b;q^m)_\infty}{(q;q)_\infty}\sum_{j\geq 0}\sum_{k\geq 1}\frac{q^{(j+k)+bj}(1+q^{b-1}+\cdots+q^{(b-1)(k-1)})}{(q^m;q^m)_j(q^m;q^m)_{j+k}}.\end{aligned}$$

Similarly, the double series in the above can be expanded as a nonnegative series in q. Also, as $b \neq 2$, we have $1 < b-1 < b \leq m$. This implies that the infinite product part in the above is also a nonnegative series in q. Therefore, $p_{1,b,m}(n) \geq p_{b,1,m}(n)$ for $b \neq 2$. □

3. Case $(a, b) = (1, 2)$

When $(a,b) = (1,2)$, it looks like a q-theoretic proof is painfully difficult. Therefore, we consider this case in a combinatorial manner. First, for $d \in \mathbb{Z}$, we define

$$\mathscr{P}_d(n) = \mathscr{P}_d^{(m)}(n) := \big\{\lambda \in \mathscr{P} : |\lambda| = n \text{ and } \sharp_{1,m}(\lambda) - \sharp_{2,m}(\lambda) = d\big\}.$$

Then

$$p_{1,2,m}(n) = \sum_{d\geq 1} \operatorname{card} \mathscr{P}_d(n), \tag{5}$$

$$p_{2,1,m}(n) = \sum_{d\geq 1} \operatorname{card} \mathscr{P}_{-d}(n). \tag{6}$$

Our object is to show the following inequalities, from which our desired result $p_{1,2,m}(n) \geq p_{2,1,m}(n)$ follows as a direct consequence if we make use of the above two relations.

Theorem 3.1. *Let $m \geq 3$ be an integer. For $k \geq 0$,*

$$\operatorname{card} \mathscr{P}_{-(km+1)}(n) \leq \operatorname{card} \mathscr{P}_{km+2}(n), \tag{7}$$

$$\operatorname{card} \mathscr{P}_{-(km+2)}(n) \leq \operatorname{card} \mathscr{P}_{km+1}(n), \tag{8}$$

$$\operatorname{card} \mathscr{P}_{-(km+r)}(n) \leq \operatorname{card} \mathscr{P}_{km+r}(n), \tag{9}$$

where $3 \leq r \leq m$ in the third inequality.

Proof. We simply construct injections $\mathscr{P}_{-d}(n) \hookrightarrow \mathscr{P}_{d^*}(n)$ for $d = km + r > 0$ with $1 \leq r \leq m$ and

$$d^* = \begin{cases} km+2 & \text{if } r = 1,\\ km+1 & \text{if } r = 2,\\ km+r & \text{if } 3 \leq r \leq m.\end{cases}$$

Given any partition λ, we start with the following process.

Process (I). We replace any part in λ that is congruent to 1 modulo m, say $um+1$, by $um+2$ and replace any part in λ that is congruent to 2 modulo m, say $vm+2$, by $vm+1$. The resulting partition is called λ^*.

Now, if $\lambda \in \mathscr{P}_{-d}(n)$, then $\sharp_{1,m}(\lambda) - \sharp_{2,m}(\lambda) = -d$. Also, trivially,

$$|\lambda^*| = |\lambda| - d = n - d.$$

Thus, to arrive at a partition of size n, we need to append some additional parts that sum to d. We have three subcases.

Subcase I. $3 \le r \le m$. Recall that $d = km + r$. We append a part of size d to λ^* and call the new partition λ^{**}. Since $d \not\equiv 1, 2 \pmod{m}$, we have

$$\begin{aligned}\sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \sharp_{1,m}(\lambda^*) - \sharp_{2,m}(\lambda^*)\\ &= \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) \quad \text{(by Process (I))}\\ &= -(-d)\\ &= d^*.\end{aligned}$$

Thus, $\lambda^{**} \in \mathscr{P}_{d^*}(n)$.

Subcase II. $r = 1$. Recall that $d = km + 1$. We append a part of size 1 and a part of size km to λ^* and call the new partition λ^{**}. Notice that $km \equiv 0 \not\equiv 1, 2 \pmod{m}$ for $m \ge 3$. Thus,

$$\begin{aligned}\sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \big(1 + \sharp_{1,m}(\lambda^*)\big) - \sharp_{2,m}(\lambda^*)\\ &= 1 + \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) \quad \text{(by Process (I))}\\ &= 1 - (-d)\\ &= km + 2\\ &= d^*,\end{aligned}$$

which implies that $\lambda^{**} \in \mathscr{P}_{d^*}(n)$.

Subcase III. $r = 2$. Recall that $d = km + 2$. We append a part of size 2 and a part of size km to λ^* and call the new partition λ^{**}. We also have $km \equiv 0 \not\equiv 1, 2 \pmod{m}$ for $m \ge 3$. Thus,

$$\begin{aligned}\sharp_{1,m}(\lambda^{**}) - \sharp_{2,m}(\lambda^{**}) &= \sharp_{1,m}(\lambda^*) - \big(1 + \sharp_{2,m}(\lambda^*)\big)\\ &= -1 + \sharp_{2,m}(\lambda) - \sharp_{1,m}(\lambda) \quad \text{(by Process (I))}\\ &= -1 - (-d)\\ &= km + 1\\ &= d^*,\end{aligned}$$

and therefore, $\lambda^{**} \in \mathscr{P}_{d^*}(n)$.

Lastly, it is straightforward to verify that the map $\lambda \mapsto \lambda^{**}$ is injective. $\square$

Proof of Theorem 1.3 for $(a,b) = (1,2)$. For $m = 2$, see Theorem 1.1 due to B. Kim, E. Kim and Lovejoy. For $m \ge 3$, we have

$$p_{2,1,m}(n) = \sum_{d \ge 1} \operatorname{card} \mathscr{P}_{-d}(n) \quad \text{(by (6))}$$

$$
\begin{aligned}
&= \sum_{k\geq 0} \operatorname{card} \mathscr{P}_{-(km+1)}(n) + \sum_{k\geq 0} \operatorname{card} \mathscr{P}_{-(km+2)}(n) \\
&\quad + \sum_{3\leq r\leq m} \sum_{k\geq 0} \operatorname{card} \mathscr{P}_{-(km+r)}(n) \\
&\leq \sum_{k\geq 0} \operatorname{card} \mathscr{P}_{km+2}(n) + \sum_{k\geq 0} \operatorname{card} \mathscr{P}_{km+1}(n) \\
&\quad + \sum_{3\leq r\leq m} \sum_{k\geq 0} \operatorname{card} \mathscr{P}_{km+r}(n) \quad \text{(by Theorem 3.1)} \\
&= \sum_{d\geq 1} \operatorname{card} \mathscr{P}_{d}(n) \\
&= p_{1,2,m}(n). \quad \text{(by (5))}
\end{aligned}
$$

This is exactly what we need. □

4. Closing remarks

Following Section 2, the case $(a, b) = (1, 2)$ of Theorem 1.3 is equivalent to the nonnegativity of

$$
\frac{(q, q^2; q^m)_\infty}{(q; q)_\infty} \sum_{j\geq 0} \sum_{k\geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j (q^m; q^m)_{j+k}}, \tag{10}
$$

that is, its series expansion has nonnegative coefficients. Although we do not find a q-theoretic proof of this fact, our numerical calculations indicate the following conjecture.

Conjecture 4.1. *For $m \geq 2$, the double series*

$$
\sum_{j\geq 0} \sum_{k\geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j (q^m; q^m)_{j+k}} \tag{11}
$$

has nonnegative coefficients in its expansion.

Notice that

$$
\sum_{j\geq 0} \sum_{k\geq 1} \frac{q^{3j+k}(1-q^k)}{(q^m; q^m)_j (q^m; q^m)_{j+k}} = \sum_{j\geq 0} \frac{q^{3j}}{(q^m; q^m)_j (q^m; q^m)_j} \sum_{k\geq 0} \frac{q^k(1-q^k)}{(q^{(j+1)m}; q^m)_k}.
$$

Regarding the inner series, we also have a more surprising conjecture.

Conjecture 4.2. *For $m, s \geq 1$,*

$$
\sum_{k\geq 0} \frac{q^k(1-q^k)}{(q^s; q^m)_k} \tag{12}
$$

has nonnegative coefficients in its expansion.

Here the case $s = m$ is to some extent easier.

Proof of Conjecture 4.2 for $s = m$. We have

$$\begin{aligned}\sum_{k\geq 0}\frac{q^k(1-q^k)}{(q^m;q^m)_k} &= \sum_{k\geq 0}\frac{q^k}{(q^m;q^m)_k}-\sum_{k\geq 0}\frac{q^{2k}}{(q^m;q^m)_k}\\ &= \frac{1}{(q;q^m)_\infty}-\frac{1}{(q^2;q^m)_\infty} \quad \text{(by (4))}\\ &= \sum_{n\geq 0}\rho_{1,m}(n)q^n-\sum_{n\geq 0}\rho_{2,m}(n)q^n,\end{aligned}$$

where for $i = 1$ or 2, we denote by $\rho_{i,m}(n)$ the number of partitions of n with parts of the form $km + i$ with $k \geq 0$.

Now we recall a result due to Andrews [1, Theorem 3]:

Let $S = \{a_i\}_{i\geq 1}$ and $T = \{b_i\}_{i\geq 1}$ be two strictly increasing sequences of positive integers such that $b_1 = 1$ and $a_i \geq b_i$ for all i. Then for any $n \geq 0$,

$$\rho_T(n) \geq \rho_S(n),$$

where $\rho_S(n)$ (resp. $\rho_T(n)$) denotes the number of partitions of n into parts taken from S (resp. T).

By the above theorem, we immediately have $\rho_{1,m}(n) \geq \rho_{2,m}(n)$ for all n. Thus, (12) is a nonnegative series in q when $s = m$. □

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