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GROTHENDIECK GROUP FOR SEQUENCES

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Abstract. For any category with a distinguished collection of sequences, such as n-exangulated category, category of N-complexes and category of precomplexes, we consider its Grothendieck group and similar results of Bergh-Thaule for n-angulated categories [1] are proven. A classification result of dense complete subcategories is given and we give a formal definition of K-groups for these categories following Grayson's algebraic approach of K-theory for exact categories [4].

1. Introduction

Algebraic K-theory, developed by Quillen, Waldhausen (among others), are homotopy groups of a space (or spectra) to a given category. In particular, the K_0 group of K-theory, i.e., the Grothendieck group, is one of the most fundamental and studied among all K-groups. For example, a famous classification theorem of Thomason states that there is a one-to-one correspondence between dense triangulated subcategories of a triangulated category and subgroups of its Grothendieck group. Later, Matsui gave an analogous result for exact categories with a (co)generator [9] and a long list of similar classification results via the Grothendieck group were proved to hold for various categories appeared in higher homological algebra.

Higher homological algebra and higher dimensional Auslander-Reiten theory attract lots of attention in recent years, partly because of the fundamental role the theory of n-cluster tilting subcategories of exact and triangulated categories plays. The theory of $(n+2)$-angulated categories [3] and n-abelian/exact categories [8] were introduced and the above classification result is generalized to $(n+2)$-angulated categories in [1].

Moreover, a simultaneous generalization of exact categories and triangulated categories called extriangulated categories is introduced by Nakaoka-Palu [10] so many similar results can be unified and extended to the extriangulated setting. Further, the higher version of extriangulated categories, the n-exangulated categories, was defined by Herschend-Liu-Nakaoka [7] as a

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unification of $(n+2)$-angulated and n-abelian/exact categories. The dense classification theorem via Grothendieck group was extended to n-exangulated categories by Haugland in [6].

The aforementioned categories all comes with a distinguished collection of sequences form by morphisms in the underlying categories, for example, conflations in exact categories, distinguished triangles in triangulated categories, conflations in extriangulated categories, $n-\Sigma$ sequences in n-angulated categories, etc. and the classification results are in fact formal consequences of the following common properties of the distinguished collections: let $\mathcal{C}$ be a category with a zero object, together with a collection of length $(n+1)$ sequences

$$\mathcal{F} = \{X^0 \to X^1 \to \cdots \to X^{n+1} : X^i \text{ objects of } \mathcal{C}\}$$

satisfying the following properties:

(1) $0 \to 0 \to \cdots \to 0 \in \mathcal{F}$;
(2) if $A \cong B \in \mathcal{C}$, then $0 \to \cdots \to A \xrightarrow{\cong} B \to 0 \to \cdots \to 0$ belongs to $\mathcal{F}$, with $A \xrightarrow{\cong} B$ placed anywhere in the sequence;
(3) the direct sum of any two sequences in $\mathcal{F}$ also belongs to $\mathcal{F}$.

The first purpose of the current paper is to write down the common properties and results shared by all such categories and provide proofs all at once. We call the category $(\mathcal{C}, \mathcal{F})$ a *category of n-sequences* and this is merely a working language for all aforementioned (and possible other) categories with similar distinguished collections (for example, category of N-complexes and category of precomplexes altogether). We do not claim this obvious collection of categories to have any other interesting properties, at least not in the current paper.

A sequence in $\mathcal{F}$ is called *good* and the morphisms *differentials* (although the morphisms don't need to form a complex). We only consider categories of n-sequences unless state otherwise so we will simply refer to these categories as *n-categories* and denote it by $\mathcal{C}$ (notice that our n-category is different from the notion of n-category in higher category theory). An *exact* functor $F : \mathcal{C} \to \mathcal{D}$ of n-categories is a functor that takes zero object to zero object and good sequences of $\mathcal{C}$ to good sequences of $\mathcal{D}$.

Several existing results of above categories in (higher) homological algebra only hold for categories where n is an odd integer because of the lack of an explicit description of an inverse element in the Grothendieck group. In fact the Grothendieck group can be obtained as a group completion of a corresponding monoid so there is a description of inverse elements but this naive description will not help in the classification so results in the current paper also require n to be an odd integer.

Finally, in the last section, we take Grayson's algebraic description of K-groups [4] as our definition of K-groups for n-categories. Notice that Grayson's approach is the first complete, pure algebraic description of the K-theory of exact categories. We study basic properties of these K-groups and prove the even Additivity Theorem for K-groups of these categories where n is an odd integer following the idea of Harris [5]. An integer $n \geq 1$ is fixed throughout this paper.

2. Grothendieck group

The Grothendieck group of an n-category is defined and studied following ideas of Bergh-Thaule [1]. As usual, we only talk about small categories in this paper as we are discussing about the Grothendieck group. Any pointed category $\mathcal{C}$ comes with two trivial n-category structures, $\mathcal{F}$ to be the collection of all n-sequences or $\mathcal{F}$ contains the minimal n-sequences generated from all three conditions. These two structures are not necessarily the same. For example, a classical module category with the collection of short exact sequences is abelian (hence a category of 1-sequences) and it contains the minimal 1-sequences structure. However, not all 1-sequences are short exact so the two 1-sequences structures are not the same. The Grothendieck group of the first n-structure is trivial as long as $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq \emptyset$ for any objects $X, Y \in \mathcal{C}$.

2.1. Grothendieck group

As pointed out in [2, Definition 1.1.3.1], one can define the Grothendieck group of any given small category $\mathcal{C}$ together with a collection of diagrams $c' \to c \to c''$ in $\mathcal{C}$. Usual properties like $[0] = 0$, $[A] = [B]$ if A and B are isomorphic and any object of $K_0(\mathcal{C})$ is represented as a difference of two elements $[A] - [B]$, all hold if the collection satisfies certain conditions.

Definition 2.1. Given an n-category $\mathcal{C}$, let $F(\mathcal{C})$ be the free abelian group on the set of isomorphism classes $\langle X \rangle$ of objects $X \in \mathcal{C}$. For any good sequence $X^\bullet : X^0 \to \cdots \to X^{n+1} \in \mathcal{F}$, the *Euler relation* in $F(\mathcal{C})$ is

$$\chi(X^\bullet) := \langle X^0 \rangle - \langle X^1 \rangle + \langle X^2 \rangle - \cdots + (-1)^{n+1} \langle X^{n+1} \rangle.$$

Let $R(\mathcal{C})$ be the subgroup of $F(\mathcal{C})$ generated by the following set of elements

$$\{\chi(X^\bullet) : X^\bullet \text{ a good sequence}\} \quad \text{if } n \text{ is odd,}$$

$$\{\langle 0 \rangle\} \cup \{\chi(X^\bullet) : X^\bullet \text{ a good sequence}\} \quad \text{if } n \text{ is even.}$$

The *Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is the quotient group $F(\mathcal{C})/R(\mathcal{C})$. Given an object $X \in \mathcal{C}$, the residue class in $K_0(\mathcal{C})$ is denoted by $[X]$.

Definition 2.2. Let $\mathcal{C}$ be an n-category, an *additive function* from $\mathcal{C}$ to an abelian group G is a function f from the objects of $\mathcal{C}$ to G such that

$$f(X^0) - f(X^1) + f(X^2) - \cdots + (-1)^{n+1} f(X^{n+1}) = 0 \quad \text{if } n \text{ is odd,}$$

$$f(0) = 0, f(X^0) - f(X^1) + f(X^2) - \cdots + (-1)^{n+1} f(X^{n+1}) = 0 \quad \text{if } n \text{ is even}$$

for any good sequence $X^0 \to X^1 \to \cdots \to X^{n+1}$ in $\mathcal{C}$.

Note that $K_0(\mathcal{C})$ is abelian (See Proposition 2.4(2)) and the function $\varphi : \mathcal{C} \to K_0(\mathcal{C})$ where $X \mapsto [X]$ is by definition an additive function.

Proposition 2.3 (Universal property)**.** *The Grothendieck group $K_0(\mathcal{C})$ satisfies the following universal property: for any additive function $\psi : \mathcal{C} \to G$, there is a unique morphism of abelian groups $\theta : K_0(\mathcal{C}) \to G$ such that $\theta \circ \varphi = \psi$, i.e., the following diagram holds:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\psi"] \arrow[d, "\varphi"'] & G \\ K_0(\mathcal{C}) \arrow[ur, dashed, "\exists! \theta"'] & \end{tikzcd}$$

Proof. First of all, define a map

$$\hat{\theta} : F(\mathcal{C}) \to G$$

$$\langle X \rangle \mapsto \psi(X)$$

and extended by linearity.

This is a well-defined map. Indeed, suppose $\langle X \rangle = \langle Y \rangle$ in $F(\mathcal{C})$, i.e., $X \cong Y$ in $\mathcal{C}$, there is a good sequence $X \xrightarrow{\cong} Y \to 0 \to \cdots \to 0$, so $\psi(X) = \psi(Y)$ (ψ is an additive function). Thus $\hat{\theta}$ induces a map on $K_0(\mathcal{C})$: for any good sequence $X^0 \to X^1 \to \cdots \to X^{n+1}$, we have $\hat{\theta}(\langle X^0 \rangle - \langle X^1 \rangle + \cdots + (-1)^{n+1} \langle X^{n+1} \rangle) = \psi(X^0) - \psi(X^1) + \cdots + (-1)^{n+1} \psi(X^{n+1}) = 0$ (again, because ψ is an additive function). Denote the induced map on $K_0(\mathcal{C})$ by $\theta : K_0(\mathcal{C}) \to G$ and it is clear from the definition that the diagram commutes.

Uniqueness is immediate: suppose θ' also makes the diagram commute ($\theta' \circ \varphi = \psi$), then

$$\theta'([X]) = \theta'(\varphi(X)) = (\theta' \circ \varphi)(X) = \psi(X) = (\theta \circ \varphi)(X) = \theta([X]). \qquad \square$$

We have the following proposition.

Proposition 2.4. *Let $\mathcal{C}$ be an n-category and $K_0(\mathcal{C})$ its Grothendieck group.*

1. *The element $[0]$ is the zero element in $K_0(\mathcal{C})$.*
2. *If X and Y are objects in $\mathcal{C}$, then $[X \oplus Y] = [X] + [Y]$.*
3. *Every element in $K_0(\mathcal{C})$ is of the form $[X] - [Y]$ for some objects $X, Y \in \mathcal{C}$.*

Proof. (1) If n is even, then $[0] = 0$ by definition. If n is odd, notice that we have a good sequence

$$0 \to 0 \to \cdots \to 0 \to 0$$

with odd number of zeros. The Euler relation of this sequence gives $[0] = 0$ in $K_0(\mathcal{C})$.

(2) The direct sum of good sequences

$$X \to X \to 0 \to 0 \to \cdots \to 0,$$

$$0 \to Y \to Y \to 0 \to \cdots \to 0$$

results to the good sequence

$$X \to X \oplus Y \to Y \to 0 \to \cdots \to 0$$

so it immediate gives $[X \oplus Y] = [X] + [Y]$.

(3) Let x be an element in $K_0(\mathcal{C})$. If $x = 0$, then $x = [0]$ by (1) and we are done as $[0] = [X] - [X]$ for any X. If x is nonzero, then there are non-negative integers $a_1, \ldots, a_r, b_1, \ldots, b_t$ and objects $X_1, \ldots, X_r, Y_1, \ldots, Y_t$ such that $x = a_1[X_1] + \cdots + a_r[X_r] - b_1[Y_1] - \cdots - b_t[Y_t]$ as the Grothendieck group is the free abelian group generated by isomorphism classes of objects in $\mathcal{C}$. Therefore by (2), we can combine these summands into $x = [X_1^{a_1} \oplus \cdots \oplus X_r^{a_r}] - [Y_1^{b_1} \oplus \cdots \oplus Y_t^{b_t}]$. □

2.2. Basic properties of the Grothendieck group

Results we are about to prove are slight generalizations of those discussed in Bergh-Thaule [1]. Once we proved everything, similar results follow immediately for n-categories.

Definition 2.5. $\mathcal{C}$ an n-category, define relations $\sim_i$ on the set of objects (one for each i, between 0 and $n+1$) to be: for any two objects $X, Y \in \mathcal{C}$, $X \sim_i Y$ if there are two good sequences

$$A^0 \to \cdots \to X \oplus A^i \to \cdots \to A^{n+1},$$

$$A^0 \to \cdots \to Y \oplus A^i \to \cdots \to A^{n+1}$$

for some objects $A^0, \ldots, A^{n+1}$.

Note that if $X \cong Y$, we have $X \sim_i Y$ because of the good sequences

$$0 \to \cdots \to X \xrightarrow{\cong} Y \to \cdots \to 0,$$

$$0 \to \cdots \to Y \to Y \to \cdots \to 0.$$

Also, recall for any commutative monoid S, elements of its group completion $G(S)$ are equivalence classes of pairs (s, s') with $s, s' \in S$ and the equivalence relation is defined to be $(s, s') \sim (t, t')$ if and only if there is some $u \in S$ such that $s + t' + u = s' + t + u$ in S. Denote the equivalence class of (s, s') by $[(s, s')]$ and the group structure of $G(S)$ is defined by the rule $[(s, s')] + [(t, t')] = [(s + s', t + t')]$.

Proposition 2.6 (cf. [1, Proposition 2.3])**.** *Let $\mathcal{C}$ be an n-category. Then*

(1) *The relation defined above is an equivalence relation.*
(2) *The set π_i of equivalence classes $\{X\}_i$ of objects in $\mathcal{C}$ forms a commutative monoid with addition $\{X\}_i + \{Y\}_i := \{X \oplus Y\}_i$.*
(3) *Denote the group completion of π_i by $G(\pi_i)$, then the groups $G(\pi_i)$ and $K_0(\mathcal{C})$ are isomorphic.*

Proof. (1) $X \sim_i X$ and $X \sim_i Y \Rightarrow Y \sim_i X$ are clear. For the condition: $X \sim_i Y$ and $Y \sim_i Z \Rightarrow X \sim_i Z$, note that $X \sim_i Y$ means there are objects $A^0, \dots, A^{n+1}$ and good sequences

$$A^0 \to \cdots \to X \oplus A^i \to \cdots \to A^{n+1},$$

$$A^0 \to \cdots \to Y \oplus A^i \to \cdots \to A^{n+1}.$$

Similarly, $Y \sim_i Z$ means there are objects $B^0, \dots, B^{n+1}$ and good sequences

$$B^0 \to \cdots \to Y \oplus B^i \to \cdots \to B^{n+1},$$

$$B^0 \to \cdots \to Z \oplus B^i \to \cdots \to B^{n+1}.$$

Take the direct sum of the 1st and the 3rd, and similarly direct sum of the 2nd and the 4th sequences we get:

$$\begin{aligned} &A^0 \oplus B^0 \to \cdots \to (X \oplus A^i) \oplus (Y \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \\ = &A^0 \oplus B^0 \to \cdots \to X \oplus (A^i \oplus Y \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \end{aligned}$$

and

$$\begin{aligned} &A^0 \oplus B^0 \to \cdots \to (Y \oplus A^i) \oplus (Z \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \\ = &A^0 \oplus B^0 \to \cdots \to Z \oplus (A^i \oplus Y \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \end{aligned}$$

and this shows $X \sim_i Z$.

(2) The addition $\{X\}_i + \{Y\}_i := \{X \oplus Y\}_i$ is well-defined. Indeed, for $X \sim_i X'$ and $Y \sim_i Y'$, we have $X \oplus Y \sim_i X' \oplus Y'$ if we re-group the above two good sequences in the following way:

$$\begin{aligned} &A^0 \oplus B^0 \to \cdots \to (X \oplus A^i) \oplus (Y \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \\ = &A^0 \oplus B^0 \to \cdots \to (X \oplus Y) \oplus (A^i \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \end{aligned}$$

and

$$\begin{aligned} &A^0 \oplus B^0 \to \cdots \to (X' \oplus A^i) \oplus (Y' \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1} \\ = &A^0 \oplus B^0 \to \cdots \to (X' \oplus Y') \oplus (A^i \oplus B^i) \to \cdots \to A^{n+1} \oplus B^{n+1}. \end{aligned}$$

The part that π_i is a commutative monoid is immediate.

(3) First, we will show that the Euler relations hold for the monoid π_i. This is a slight modification of Bergh-Thaule's proof. For any good sequence

$$X^0 \to X^1 \to \cdots \to X^{n+1}$$

we can obtain two good sequences

$$X^0 \to \bigoplus_{k\le 1} X^k \to \bigoplus_{k\le 2} X^k \to \cdots \to \bigoplus_{k\le i-1} X^k \to \bigoplus_{odd} X^k \to \bigoplus_{k\ge i+1} X^k \to \cdots \to \bigoplus_{k\ge n} X^k \to X^{n+1},$$

$$X^0 \to \bigoplus_{k\le 1} X^k \to \bigoplus_{k\le 2} X^k \to \cdots \to \bigoplus_{k\le i-1} X^k \to \bigoplus_{even} X^k \to \bigoplus_{k\ge i+1} X^k \to \cdots \to \bigoplus_{k\ge n} X^k \to X^{n+1}$$

by following: suppose i is odd and n is also odd (these assumptions are only needed so that we have a more convenient way to phrase the argument), consider following good sequences:

$$0 \to X^* \to X^* \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to \cdots \to 0 \to 0 \qquad * = 0$$

$$0 \to 0 \to X^* \to X^* \to 0 \to 0 \to \cdots \to 0 \to 0 \to \cdots \to 0 \to 0 \qquad * = 1$$

$$0 \to 0 \to 0 \to X^* \to X^* \to 0 \to \cdots \to 0 \to 0 \to \cdots \to 0 \to 0 \qquad * = 0, 2$$

$$0 \to 0 \to 0 \to 0 \to X^* \to X^* \to \cdots \to 0 \to 0 \to \cdots \to 0 \to 0 \qquad * = 1, 3$$

$$\vdots \qquad\qquad \vdots$$

$$0 \to 0 \to 0 \to \cdots \to X^* \to X^* \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \qquad * = 0, 2, 4, \ldots, i-3$$

$$0 \to 0 \to 0 \to \cdots \to 0 \to X^* \to X^* \to 0 \to 0 \to \cdots \to 0 \to 0 \qquad * = 1, 3, \ldots, i-2$$

$$X^0 \to X^1 \to X^2 \to \cdots \to X^{i-2} \to X^{i-1} \to X^i \to X^{i+1} \to X^{i+2} \to \cdots \to X^n \to X^{n+1}$$

$$0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to X^* \to X^* \to 0 \to \cdots \to 0 \to 0 \qquad * = i+2, i+4, \ldots, n$$

$$0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to X^* \to X^* \to \cdots \to 0 \to 0 \qquad * = i+3, i+5, \ldots, n+1$$

$$\vdots \qquad\qquad \vdots$$

$$0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to \cdots \to X^* \to X^* \to 0 \qquad * = n+1.$$

The direct sum of above gives

$$X^0 \to \bigoplus_{k\leq 1} X^k \to \bigoplus_{k\leq 2} X^k \to \cdots \to \bigoplus_{k\leq i-1} X^k \to \bigoplus_{odd} X^k \to \bigoplus_{k\geq i+1} X^k \to \cdots \to \bigoplus_{k\geq n} X^k \to X^{n+1}.$$

Similarly, the direct sum of the following good sequences

$$X^* \to X^* \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \qquad * = 0$$

$$0 \to X^* \to X^* \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \qquad * = 1$$

$$0 \to 0 \to X^* \to X^* \to \cdots \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \qquad * = 0, 2$$

$$\vdots \qquad\qquad \vdots$$

$$0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail \cdots \rightarrowtail X^* \rightarrowtail X^* \to 0 \longrightarrow 0 \to \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \qquad * = 1, 3, \ldots, i-2$$

$$0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail \cdots \to 0 \to X^* \rightarrowtail X^*_{(i)} \to 0 \to \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \qquad * = 0, 2, \ldots, i-1$$

$$0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail \cdots \to 0 \longrightarrow 0 \to X^*_{(i)} \rightarrowtail X^* \rightarrowtail \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \qquad * = i+1, i+3, \ldots, n+1$$

$$0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \to X^* \rightarrowtail \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \qquad * = i+2, i+4, \ldots, n$$

$$\vdots \qquad\qquad \vdots$$

$$0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \to \cdots \rightarrowtail X^* \rightarrowtail X^* \to 0 \qquad * = n$$

$$0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail 0 \rightarrowtail \cdots \to 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \to \cdots \to 0 \to X^* \rightarrowtail X^* \qquad * = n+1$$

results to

$$X^0 \to \bigoplus_{k\leq 1} X^k \to \bigoplus_{k\leq 2} X^k \to \cdots \to \bigoplus_{k\leq i-1} X^k \to \bigoplus_{even} X^k \to \bigoplus_{k\geq i+1} X^k \to \cdots \to \bigoplus_{k\geq n} X^k \to X^{n+1}.$$

The only difference of the above two direct sums is the ith term and by definition this means exactly $(X^1 \oplus X^3 \oplus \cdots) \sim_i (X^0 \oplus X^2 \oplus \cdots)$ hence

$$\begin{aligned} &\{X^1 \oplus X^3 \oplus \cdots\}_i = \{X^0 \oplus X^2 \oplus \cdots\}_i \\ \Leftrightarrow\ &\{X^1\}_i + \{X^3\}_i + \cdots = \{X^0\}_i + \{X^2\}_i + \cdots \\ \Leftrightarrow\ &\{X^0\}_i - \{X^1\}_i + \{X^2\}_i - \{X^3\}_i + \cdots = 0 \end{aligned}$$

so the Euler relation holds for any good sequence.

It remains to prove that $K_0(\mathcal{C})$ and $G(\pi_i)$ are isomorphic groups. Notice that there is a well-defined map $\{-\}_i : \mathcal{C} \to \pi_i$ where $X \mapsto \{X\}_i$ ($X \cong Y \implies \{X\}_i = \{Y\}_i$), compose this with the structure map of group completion:

$$l : \pi_i \to G(\pi_i)$$

$$\{X\}_i \mapsto [(\{X\}_i, 0)].$$

The resulting composition is an additive function as

$$[(\{X^0\}_i, 0)] + [(\{X^2\}_i, 0)] + \cdots = [(\{X^1\}_i, 0)] + [(\{X^3\}_i, 0)] + \cdots.$$

Indeed, $LHS = [(\{X^0\}_i + \{X^2\}_i + \cdots, 0)] = [(\{X^1\}_i + \{X^3\}_i + \cdots, 0)] = RHS$, where the middle equality is exactly the Euler relation, that is,

$$\{X^0\}_i + \{X^2\}_i + \cdots = \{X^1\}_i + \{X^3\}_i + \cdots$$

for any good sequence $X^0 \to \cdots \to X^{n+1}$. Therefore, there is a unique group homomorphism $g : K_0(\mathcal{C}) \to G(\pi_i)$ by Proposition 2.3.

Define a map

$$l' : \pi_i \to K_0(\mathcal{C})$$

$$\{X\}_i \mapsto [X].$$

It is well-defined and a monoid homomorphism. Indeed, for $\{A\}_i = \{A'\}_i$, we have good sequences

$$X^0 \to \cdots \to A \oplus X^i \to \cdots \to X^{n+1},$$
$$X^0 \to \cdots \to A' \oplus X^i \to \cdots \to X^{n+1},$$

then in $K_0(\mathcal{C})$, Euler relations of the above two good sequences result to

$$[A \oplus X^i] = [A' \oplus X^i] \implies [A] = [A'].$$

It's a monoid homomorphism because $l'(\{A\}_i + \{B\}_i) = l'(\{A \oplus B\}_i) = [A \oplus B] = [A] + [B]$ and $l'(\{0\}_i) = [0]$.

Consider the abelian group $K_0(\mathcal{C})$ and monoid homomorphism l', there is a unique group homomorphism $h : G(\pi_i) \to K_0(\mathcal{C})$ by the universal property of group completion.

In the following commutative diagram:

$$\begin{tikzcd}
 & & K_0(\mathcal{C}) \arrow[d, dashed, "g"] \\
\mathcal{C} \arrow[r, "\{-\}_i"'] \arrow[urr] & \pi_i \arrow[r, "l"'] \arrow[ur, "l'"] \arrow[dr, "l'"'] & G(\pi_i) \arrow[d, dashed, "h"] \\
 & & K_0(\mathcal{C})
\end{tikzcd}$$

by the universal property (Proposition 2.3), we deduce that $hg = 1$. Similarly, from the following commutative diagram

$$\begin{tikzcd}
\pi_i \arrow[r, "l"] \arrow[dr, "l'"] \arrow[ddr, "l"'] & G(\pi_i) \arrow[d, "h"] \\
 & K_0(\mathcal{C}) \arrow[d, "g"] \\
 & G(\pi_i)
\end{tikzcd}$$

we deduce that $gh = 1$ so $G(\pi_i)$ and $K_0(\mathcal{C})$ are isomorphic. □

Corollary 2.7 (cf. [1, Corollary 2.4])**.** *Let $\mathcal{C}$ be an n-category. Then the following are equivalent*:

(1) $[X] = [(\{X_1\}_i, \{X_2\}_i)] = [(\{Y_1\}_i, \{Y_2\}_i)] = [Y]$ *in* $K_0(\mathcal{C})$.

(2) *There exist objects* $U^0, \ldots, U^{n+1}$ *and good sequences in* $\mathcal{C}$

$$U^0 \to \cdots \to X_1 \oplus Y_2 \oplus U^i \to \cdots \to U^{n+1},$$
$$U^0 \to \cdots \to Y_1 \oplus X_2 \oplus U^i \to \cdots \to U^{n+1}.$$

Proof. (1) $\implies$ (2): Given $[(\{X_1\}_i, \{X_2\}_i)] = [(\{Y_1\}_i, \{Y_2\}_i)]$, then by definition, there is an object $\{C\}_i \in \pi_i$ such that in π_i,

$$\{X_1\}_i + \{Y_2\}_i + \{C\}_i = \{Y_1\}_i + \{X_2\}_i + \{C\}_i,$$

i.e.,

$$\{X_1 \oplus Y_2 \oplus C\}_i = \{Y_1 \oplus X_2 \oplus C\}_i$$

so by definition, there are objects $U^0, \dots, U^{i-1}, U, U^{i+1}, \dots, U^{n+1} \in \mathcal{C}$ and good sequences

$$U^0 \to \cdots \to X_1 \oplus Y_2 \oplus C \oplus U \to \cdots \to U^{n+1},$$
$$U^0 \to \cdots \to Y_1 \oplus X_2 \oplus C \oplus U \to \cdots \to U^{n+1}.$$

Now define $U^i := U \oplus C$.

(2) $\implies$ (1): Given two good sequences as above

$$U^0 \to \cdots \to X_1 \oplus Y_2 \oplus U^i \to \cdots \to U^{n+1},$$
$$U^0 \to \cdots \to Y_1 \oplus X_2 \oplus U^i \to \cdots \to U^{n+1}.$$

In the proof of Proposition 2.6(3), we show that the Euler relation with respect to $\{-\}_i$ holds for any good sequence, therefore

$$\begin{aligned}
&\{U^0\}_i - \cdots + (-1)^i\{X_1 \oplus Y_2 \oplus U^i\}_i + \cdots + (-1)^{n+1}\{U^{n+1}\}_i = 0,\\
&\{U^0\}_i - \cdots + (-1)^i\{Y_1 \oplus X_2 \oplus U^i\}_i + \cdots + (-1)^{n+1}\{U^{n+1}\}_i = 0\\
\implies &\{X_1 \oplus Y_2 \oplus U^i\}_i = \{Y_1 \oplus X_2 \oplus U^i\}_i\\
\implies &\{X_1\}_i + \{Y_2\}_i + \{U^i\}_i = \{Y_1\} + \{X_2\} + \{U^i\}_i
\end{aligned}$$

hence $[(\{X_1\}_i, \{X_2\}_i)] = [(\{Y_1\}_i, \{Y_2\}_i)]$. □

3. Classification of dense subcategories

Let $\mathcal{C}$ be an n-category. Then a full additive subcategory $\mathcal{G}$ of $\mathcal{C}$ is an n-generator (resp. n-cogenerator) of $\mathcal{C}$ if for each object $A \in \mathcal{C}$, there is a good sequence

$$A' \to G_1 \to \cdots \to G_n \to A$$
$$(\text{resp. } A \to G_1 \to \cdots \to G_n \to A')$$

in $\mathcal{C}$ with $G_i \in \mathcal{G}$ for all i. This definition obviously comes from the one in [6] which is in turn motivated by the one in [9, 11]. A trivial example of an n-(co)generator is given by choosing $\mathcal{G}$ to be the entire category $\mathcal{C}$. Also, categories with enough projectives or injectives provide natural examples of n-(co)generators. The notion of *projective/injective objects* and what it means for an n-category to have *enough projective/injective objects* have an obvious extension to n-categories. Indeed, an object P is *projective* if for any good sequence $X_0 \xrightarrow{d_0} X_1 \to \cdots \to X_n \xrightarrow{d_n} X_{n+1}$ and morphism $f : P \to X_{n+1}$, there exists a morphism $g : P \to X_n$ such that $d_n \circ g = f$ and $\mathcal{C}$ is said to *has enough projectives* if for each object $A \in \mathcal{C}$, there exists a good sequence $A' \to P_1 \to \cdots \to P_n \to A$ in $\mathcal{C}$ with P_i projective for all i.

The subcategories to be classified are following:

Definition 3.1. Let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$. Then it's

(1) *dense* in $\mathcal{C}$ if each object in $\mathcal{C}$ is a direct summand of an object of $\mathcal{S}$.
(2) *complete* if given any good sequence in $\mathcal{C}$ with $n+1$ objects in $\mathcal{S}$, then so is the last object.

One checks directly that Haugland's arguments [6] work for n-categories so the classification result follows immediately.

Theorem 3.2. *Let $\mathcal{C}$ be an n-category with n is an odd integer, $\mathcal{G}$ an n-(co)generator and $\mathcal{S}$ a dense complete subcategory. Then there is a one-to-one correspondence*

$$\{\textit{subgroups of } K_0(\mathcal{C}) \textit{ containing } H_{\mathcal{G}}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{dense complete subcategories of } \mathcal{C} \textit{ containing } \mathcal{G}\}$$

$$H \longmapsto \{A \in \mathcal{C} : [A] \in H\}$$

$$\langle [A] \in K_0(\mathcal{C}) : A \in \mathcal{S} \rangle \longleftarrow\!\!\shortmid \mathcal{S},$$

where $H_{\mathcal{G}} := \langle [G] \in K_0(\mathcal{C}) : G \in \mathcal{G} \rangle$ is the subgroup of $K_0(\mathcal{C})$ generated by elements whose representations belongs to $\mathcal{G}$.

Define a relation $\sim$ on the set of isomorphism classes of objects in $\mathcal{C}$ by $\langle A \rangle \sim \langle B \rangle$ if and only if $A \oplus S_A \cong B \oplus S_B$ for some objects $S_A, S_B \in \mathcal{S}$. It is easy to see that this is an equivalence relation and denote by $G_{\mathcal{S}}$ the quotient of the isomorphism classes of objects in $\mathcal{C}$ by $\sim$. Elements in $G_{\mathcal{S}}$ are denoted by $\{A\}$.

Lemma 3.3.

(1) *An object $A \in \mathcal{C}$ is contained in $\mathcal{S}$ if and only if $\{A\} = \{0\}$ in $G_{\mathcal{S}}$.*
(2) *$G_{\mathcal{S}}$ is an abelian group with addition $\{A\} + \{B\} := \{A \oplus B\}$ and the identity element is $\{0\}$.*
(3) *There is a well-defined group isomorphism*
$$K_0(\mathcal{C})/g(\mathcal{S}) \to G_{\mathcal{S}}$$
$$[A] + g(\mathcal{S}) \mapsto \{A\}$$
In particular, $A \in \mathcal{C}$ is contained in $\mathcal{S}$ if and only if $[A] \in g(\mathcal{S})$.

Proof. It is straightforward to check that $\sim$ is an equivalence relation.

(1) If $A \in \mathcal{S}$, then $\{A\} = \{0\}$ as we have $A \oplus 0 \cong 0 \oplus A$ for $0, A \in \mathcal{S}$. Conversely, we have $A \oplus S_A \cong S_0$ for some $S_A, S_0 \in \mathcal{S}$ so there is the following good sequence
$$A \to A \oplus S_A \to S_A \to 0 \to \cdots \to 0$$
with the last $n+1$ terms objects in $\mathcal{S}$. Therefore $A \in \mathcal{S}$ as $\mathcal{S}$ is complete.

(2) Again, it's straightforward to show that $+$ is a well-defined operation and $+$ is commutative, associative with the $\{0\}$ being the identity element. For any $A \in \mathcal{C}$, there is an object $A' \in \mathcal{C}$ such that $A \oplus A' \in \mathcal{S}$ (as $\mathcal{S}$ is dense), therefore $\{A\} + \{A'\} = \{0\}$ and this means $\{A'\}$ is the inverse of $\{A\}$.

(3) First, $\varphi : K_0(\mathcal{C}) \to G_{\mathcal{S}}$ $[A] \mapsto \{A\}$ is well-defined as it sends Euler relations to zero. Indeed, for any good sequence $X_0 \to X_1 \to \cdots \to X_{n+1}$ in $\mathcal{C}$, we can form a new good sequence

$$\bar{X} \to \bigoplus_{i=1}^{n+1}(X_i \oplus X_i') \to \bigoplus_{i=2}^{n+1}(X_i \oplus X_i') \to \cdots \to \bigoplus_{i=n}^{n+1}(X_i \oplus X_i') \to X_{n+1} \oplus X_{n+1}'$$

with X_i' objects in $\mathcal{C}$ that makes $X_i \oplus X_i' \in \mathcal{S}$ and $\bar{X} = X_0 \oplus X_1' \oplus X_2 \oplus \cdots \oplus X_{n+1}$ exactly like Haugland in [6, Lemma 5.4]. From this sequence we conclude that $\bar{X} \in \mathcal{S}$ so

$$\begin{aligned}\{0\} = \{\bar{X}\} &= \{X_0\} + \{X_1'\} + \{X_2\} + \cdots + \{X_{n+1}\} \\ &= \{X_0\} - \{X_1\} + \{X_2\} - \cdots + \{X_{n+1}\} \text{ in } G_{\mathcal{S}}\end{aligned}$$

so φ is well-defined.

It's obvious that φ is a surjection so the only thing left is to show $Ker(\varphi) = g(\mathcal{S})$. It is immediate that $g(\mathcal{S}) \subseteq Ker(\varphi)$ by (1). Notice that every element in $K_0(\mathcal{C})$ is of the form $[X] - [Y]$ for some objects $X, Y \in \mathcal{C}$ by Proposition 2.4(3). Since $\mathcal{G}$ is an n-generator, there is a good sequence $Y' \to G_1 \to \cdots \to G_n \to Y$ in $\mathcal{C}$ with $G_i \in \mathcal{G}$ so $[Y] = -[Y'] + [G_1] - [G_2] + \cdots - [G_{n-1}] + [G_n]$ as n is odd. Therefore any element can be written as

$$\begin{aligned}[X] - [Y] &= [X] + [Y'] - [G_1] + [G_2] - \cdots + [G_{n-1}] - [G_n] \\ &= [X \oplus Y' \oplus G_2 \oplus G_4 \oplus \cdots \oplus G_{n-1}] - [G_1 \oplus G_3 \oplus \cdots \oplus G_n] \\ &=: [A] - [G],\end{aligned}$$

where $A \in \mathcal{C}$ and $G \in \mathcal{G}$.

Now, for any element $[A] - [G] \in Ker(\varphi)$, we have $\{0\} = \varphi([A] - [G]) = \{A\} - \{G\} = \{A\}$ as $G \in \mathcal{G} \subseteq \mathcal{S}$ and this means $A \in \mathcal{S}$ as well, hence $[A] - [G] \in g(\mathcal{S})$.

This gives the isomorphism $K_0(\mathcal{C})/g(\mathcal{S}) \cong G_{\mathcal{S}}$. □

Now we are ready to prove Theorem 3.2.

Proof. By definition $g(S)$ is a subgroup containing $H_{\mathcal{G}}$ and $\mathcal{G} \subseteq f(H)$. To see that $f(H)$ is a dense subcategory containing $\mathcal{G}$, notice that there is a good sequence $A' \to G_1 \to \cdots \to G_n \to A$ for any $A \in \mathcal{C}$ and this implies that $[A \oplus A'] = [G_1] - [G_2] + \cdots - [G_{n-1}] + [G_n] \in H$ as n is odd. For completeness of $f(H)$, we have $[X_0] - [X_1] + \cdots + (-1)^{n+1}[X_{n+1}] = 0 \in H$ for any good sequence $X_0 \to \cdots \to X_{n+1}$ so any $n+1$ terms in this equation implies that the last one must also be an element in H.

The inclusions $gf(H) \subseteq H$ and $\mathcal{S} \subseteq fg(\mathcal{S})$ hold by definition so we only need to show the inverse directions. For $gf(H) \supseteq H$, take $[A] - [G] \in H$ (see the proof of (3) of last lemma), we have $[A] \in H$ as $[A] = ([A] - [G]) + [G]$ and both $[A] - [G]$ and $[G]$ are elements of H and this implies that $[A] - [G]$ belongs to the subgroup generated by elements $[X]$ with $[X] \in H$.

For $\mathcal{S} \subseteq fg(\mathcal{S})$, choose an element $A \in fg(\mathcal{S})$, that is, $[A] \in g(\mathcal{S})$ so (3) of the above lemma tells us that $A \in \mathcal{S}$. □

Therefore all Thomason type dense subcategories classification results are formal consequence of the distinguished collection of good sequences for n-categories with n an odd integer together with the existence of an n-(co)generator $\mathcal{G}$. In particular, for the category of N-complexes and precomplexes, 0 is always a (co)generator so there are similar classification results for these categories as well. Recall that for an additive category $\mathcal{C}$ and a fixed integer $N \geq 2$, an N-complex $X^\bullet$ is a diagram

$$\cdots \xrightarrow{d} X^i \xrightarrow{d} X^{i+1} \xrightarrow{d} \cdots$$

with $X^i \in \mathcal{C}$ and $d^N = 0$. A precomplex is nothing but a sequence of objects connected by morphisms of the underlying category.

4. Grayson's algebraic K-theory

Grayson's recent approach to algebraic K-theory [4] defines all K-groups of an exact category as the Grothendieck group of the associated exact category of acyclic binary multicomplexes modulo relationships coming from "short exact sequences" and diagonal acyclic binary multicomplexes. Therefore, if there is a well-defined notion of K_0 for a category with a "good" collection of diagrams, one can apply Grayson's construction in defining all K-groups for any such category.

We use Grayson's construction to give a definition of K-groups for n-categories and prove a naive even Additive Theorem at the end of this section.

Definition 4.1. Let $\mathcal{C}$ be an n-category, denote the category of bounded sequences of $\mathcal{C}$ by $S\mathcal{C}$. Objects of $S\mathcal{C}$ are ordinary sequences of $\mathcal{C}$ and morphisms are componentwise morphisms in $\mathcal{C}$ that make the obvious squares commute.

One can make $S\mathcal{C}$ into an n-category by declaring a sequence $S^0_\bullet \to S^1_\bullet \to \cdots \to S^{n+1}_\bullet$ to be good if $S^0_k \to S^1_k \to \cdots \to S^{n+1}_k$ is good in $\mathcal{C}$, for any k. It's easy to see that the sequence $0_\bullet \to \cdots \to 0_\bullet$ is good and two sequences $A_\bullet$ and $B_\bullet$ are isomorphic: $A_\bullet \xrightarrow[f]{\cong} B_\bullet$ if f_k is an isomorphism for each k, so condition (2) of Definition 2.1 is true. Finally, the direct sum of two good sequences

$$S^0_\bullet \to S^1_\bullet \to \cdots \to S^{n+1}_\bullet,$$
$$T^0_\bullet \to T^1_\bullet \to \cdots \to T^{n+1}_\bullet$$

in $S\mathcal{C}$ is good as the direct sum of

$$S^0_k \to S^1_k \to \cdots \to S^{n+1}_k,$$

$$T_k^0 \to T_k^1 \to \cdots \to T_k^{n+1}$$

is good in $\mathcal{C}$, for each k.

Definition 4.2. A bounded *acyclic* sequence, in an n-category $\mathcal{C}$ is a bounded sequence $N_\bullet$ whose differentials factor through good sequences of $\mathcal{C}$. That is, the differentials factor as

$$\begin{tikzcd}
\cdots \arrow[r] & N_i \arrow[rr, "d_i"] \arrow[ddd] & & N_{i-1} \arrow[r] & \cdots \\
& & \cdots \arrow[ur] & & \\
& & {} \arrow[u] & & \\
& Z_{i-1} \arrow[ur] & & &
\end{tikzcd}$$

such that each $Z_{i-1} \longrightarrow \ \longrightarrow \cdots \longrightarrow N_{i-1} \longrightarrow Z_{i-2}$ is a good sequence of $\mathcal{C}$.

Definition 4.3. A *binary sequence* in $\mathcal{C}$ is a sequence with two independent differentials. More precisely, a binary sequence is a triple $(N_\bullet, d, d')$ such that (N, d) and (N, d') are sequences in $\mathcal{C}$. We call a binary sequence *acyclic* if each of the sequences (N, d) and (N, d') is acyclic in $\mathcal{C}$. A morphism between binary sequences is a morphism between the underlying graded objects that commutes with both differentials. A *good sequence* (of binary sequences) is a sequence of composable sequences of such morphisms that is good componentwise.

For $\mathcal{C}$ an n-category, $S\mathcal{C}$ the category of bounded sequences, denote by $S^q\mathcal{C}$ the category of bounded acyclic sequences. Similarly, we denote the category of bounded binary sequences by $B\mathcal{C}$ and the category of bounded binary acyclic sequences $B^q\mathcal{C}$. We proved before that $S\mathcal{C}$ is an n-category, similar results also hold for the other three categories (with similar collections of good sequences). Take the category $S^q\mathcal{C}$ for example, we have

(1) $0_\bullet \to 0_\bullet \to \cdots \to 0_\bullet$ is good as long as the zero sequence $0_\bullet$ is acyclic (i.e., an object of $S^q\mathcal{C}$). Indeed, we can factor it as

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[rr, "d_i"] \arrow[ddd] & & 0 \arrow[r] & \cdots \\
& & \cdots \arrow[ur] & & \\
& & 0 \arrow[u] & & \\
& 0 \arrow[ur] & & &
\end{tikzcd}$$

with $0 \to 0 \to \cdots \to 0$ a good sequence in $\mathcal{C}$.

(2) is straightforward.

(3) Given two acyclic sequences

$$S_\bullet^0 \longrightarrow S_\bullet^1 \longrightarrow \cdots \longrightarrow S_\bullet^{n+1},$$

$$T_\bullet^0 \longrightarrow T_\bullet^1 \longrightarrow \cdots \longrightarrow T_\bullet^{n+1}$$

in $S^q\mathcal{C}$. Notice that the direct sum of acyclic sequences is also acyclic. Indeed, given

$$\begin{tikzcd}
\cdots \arrow[r] & S_i^* \arrow[rr, "d_i"] \arrow[dr] & & S_{i-1}^* \arrow[r] & \cdots & \cdots \arrow[r] & T_i^* \arrow[rr, "d_i"] \arrow[dr] & & T_{i-1}^* \arrow[r] & \cdots \\
& & Z_{i-1}^* \arrow[ur, "\cdots"'] & & & & & Z'^*_{i-1} \arrow[ur, "\cdots"'] & &
\end{tikzcd}$$

with

$$Z_{i-1}^* \longrightarrow \quad \longrightarrow \cdots \longrightarrow S_{i-1}^* \longrightarrow Z_{i-2}^* \ ,$$

$$Z'^*_{i-1} \longrightarrow \quad \longrightarrow \cdots \longrightarrow T_{i-1}^* \longrightarrow Z'^*_{i-2}$$

good sequences in $\mathcal{C}$, the direct sum of these two good sequences is good in $\mathcal{C}$. Hence

$$S_\bullet^0 \oplus T_\bullet^0 \to S_\bullet^1 \oplus T_\bullet^1 \to \cdots \to S_\bullet^{n+1} \oplus T_\bullet^{n+1}$$

is an object of $S^q\mathcal{C}$. It is acyclic because

$$S_k^0 \oplus T_k^0 \to S_k^1 \oplus T_k^1 \to \cdots \to S_k^{n+1} \oplus T_k^{n+1}$$

is good (as the direct summand is).

Similarly, one can prove the same result for $B\mathcal{C}$ and $B^q\mathcal{C}$.

Definition 4.4. There is a diagonal functor $\Delta : S\mathcal{C} \to B\mathcal{C}$, sending $(N_\bullet, d)$ to $(N_\bullet, d, d)$. A binary sequence that is in the image of Δ is also called *diagonal*. The diagonal functor is split by the *top* and *bottom* functors $\top, \bot : B\mathcal{C} \to S\mathcal{C}$; it is clear that $\Delta, \top, \bot$ are all exact.

Since $B^q\mathcal{C}$ is an n-category, we can iteratively define an n-category $(B^q)^m\mathcal{C} = B^qB^q \cdots B^q\mathcal{C}$ for each $m \geq 0$. The objects of this category are bounded acyclic binary sequences of bounded acyclic binary sequences $\cdots$ of objects of $\mathcal{C}$. It is clear that the following is an equivalent definition of $(B^q)^m\mathcal{C}$.

Definition 4.5. The n-category $(B^q)^m\mathcal{C}$ of bounded acyclic binary multisequences of dimension m in $\mathcal{C}$ is defined as follows. A *bounded acyclic binary multisequence of dimension m* is a bounded (i.e., only finitely many non-zero),

$\mathbb{Z}^m$-graded collection of objects of $\mathcal{C}$ together with a pair of acyclic differentials (in each direction) such that any pair of differentials in different directions commute. A morphism $\varphi : N \to N'$ between such binary multisequences is a $\mathbb{Z}^m$-graded collection of morphisms of $\mathcal{C}$ that commutes with all of the differentials of N and N'. A good sequences in $(B^q)^m\mathcal{C}$ is a composable pair of such morphisms that is good componentwise.

In addition to $(B^q)^m\mathcal{C}$, for each $m \geq 1$ we have an n-category $S^q(B^q)^{m-1}\mathcal{C}$ of bounded acyclic sequences of objects of $(B^q)^{m-1}\mathcal{C}$. For each i with $1 \leq i \leq m$ there is a diagonal functor $\Delta_i : S^q(B^q)^{m-1}\mathcal{C} \to (B^q)^m\mathcal{C}$ that consists of "doubling up" the differential of the (non-binary) acyclic sequence and regarding it as direction i in the resulting acyclic binary multisequence. Any object of $(B^q)^m\mathcal{C}$ that is in the image of one of these Δ_i is called *diagonal*. The diagonal binary multisequences are those that have $d^i = \tilde{d}^i$ for at least one i.

Now we formally take Grayson's presentation of the algebraic K-theory of exact categories to be the definition of algebraic K-groups for n-categories. Note that the Grothendieck group K_0 defined below agrees with the Grothendieck group of n-categories discussed in previous sections (which is in turn a generalization of the Grothendieck group of n-angulated categories [1]).

Definition 4.6. For $\mathcal{C}$ an n-category and $m \geq 0$, the abelian group $K_m\mathcal{C}$ is presented as follows. There is one generator for each bounded acyclic binary multisequence of dimension m, and for $X^\bullet : X^0 \to \cdots \to X^{n+1}$ a good sequence in $(B^q)^m\mathcal{C}$, there are relations:

(1) $\chi(X^\bullet)$ when n is odd or $\{\langle 0\rangle\} \cup \chi(X^\bullet)$ when n is even.
(2) $[T] = 0$ if T is a diagonal acyclic binary multisequence.

Observe that if we only consider the first relation, this is then exactly the Grothendieck group of the category $(B^q)^m\mathcal{C}$. That is, $K_m\mathcal{C}$ is a quotient group of the Grothendieck group of the n-category $(B^q)^m\mathcal{C}$. Denote by $T^m_{\mathcal{C}}$ the subgroup of $K_0(B^q)^m\mathcal{C}$ generated by the classes of the diagonal binary multisequences in $(B^q)^m\mathcal{C}$. Then we may write $K_m\mathcal{C} \cong K_0(B^q)^m\mathcal{C}/T^m_{\mathcal{C}}$.

4.1. Even additivity theorem

We start by presenting the necessary materials needed for the even additivity theorem.

Definition 4.7. Let $\mathcal{B}$ be an n-category. Then an *n-subcategory* $\mathcal{A}$ of $\mathcal{B}$ is a full subcategory (containing the zero object) with the collection of good sequences being all the good sequences in $\mathcal{B}$ whose entries are objects of $\mathcal{A}$.

Suppose $\mathcal{B}$ is an n-category, $\mathcal{A}, \mathcal{C}$ and $\mathcal{X}^2, \mathcal{X}^4, \dots$ n-subcategories of $\mathcal{B}$ (the sequence ends with $\mathcal{X}^n$ if n is even, or $\mathcal{X}^{n-1}$ if n is odd). Then the *even extension category* $\mathcal{E}(\mathcal{A}, \mathcal{X}^{even}, \mathcal{C})$ is the category whose objects are good sequences of $\mathcal{B}$ of the form

$$A \to X^1 \to \cdots \to X^n \to C$$

with $A \in \mathcal{A}, C \in \mathcal{C}, X^{even} \in \mathcal{X}^{even}$ (i.e., X^i is an object of $\mathcal{X}^i$ for all even index i and this convention will also be applied in the remaining of the article) and morphisms are commuting rectangles. Denote the above object by (A, X^{even}, C).

Lemma 4.8. *$\mathcal{E}^{even} = \mathcal{E}(\mathcal{A}, \mathcal{X}^{even}, \mathcal{C})$ is an n-category, with $\mathcal{F}_{\mathcal{E}^{even}}$ being the collection of sequences of the form*

$$(A^0, X^{even,0}, C^0) \xrightarrow{f^0} (A^1, X^{even,1}, C^1) \xrightarrow{f^1} \cdots \xrightarrow{f^n} (A^{n+1}, X^{even,n+1}, C^{n+1}),$$

where

$$A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} \cdots \xrightarrow{f^n} A^{n+1},$$

$$X^{i,0} \xrightarrow{f^0} X^{i,1} \xrightarrow{f^1} \cdots \xrightarrow{f^n} X^{i,n+1},$$

$$C^0 \xrightarrow{f^0} C^1 \xrightarrow{f^1} \cdots \xrightarrow{f^n} C^{n+1}$$

good sequences in $\mathcal{A}, \mathcal{X}^i$ (i even) and $\mathcal{C}$.

Proof. (1) $0 \in \mathcal{A}, \mathcal{X}^i, \mathcal{C}$ and $0_\bullet = (0 \to \cdots \to 0)$ is good in $\mathcal{B}$, so $0_\bullet \in \mathcal{E}^{even}$ and $0_\bullet \to \cdots \to 0_\bullet \in \mathcal{F}_{\mathcal{E}^{even}}$ as the sequences $0 \to \cdots \to 0$ are good.

(2) Suppose

$$(A, X^{even}, C) \xrightarrow{\cong} (A', (X')^{even}, C') \,.$$

$$\begin{tikzcd}
C \arrow[r, "\cong"] & C' \\
\vdots \arrow[u] \arrow[r, "\cong"] & \vdots \arrow[u] \\
X^{even} \arrow[u] \arrow[r, "\cong"] & (X')^{even} \arrow[u] \\
\vdots \arrow[u] \arrow[r, "\cong"] & \vdots \arrow[u] \\
A \arrow[u] \arrow[r, "\cong"] & A' \arrow[u]
\end{tikzcd}$$

Then

$$0_\bullet \longrightarrow \cdots \longrightarrow (A, X^{even}, C) \xrightarrow{\cong} (A', (X')^{even}, C') \longrightarrow \cdots \longrightarrow 0_\bullet$$

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & C \arrow[r, "\cong"] & C' \arrow[r] & \cdots \arrow[r] & 0 \\
\vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r, "\cong"] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[u] \\
\vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & X^{even} \arrow[r, "\cong"] \arrow[u] & (X')^{even} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & \cdots \arrow[u] \\
\vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r, "\cong"] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[u] \\
0 \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & A \arrow[r, "\cong"] \arrow[u] & A' \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

all squares commute and this is a good sequence in $\mathcal{E}$ as the required sequences are good in $\mathcal{A}, \mathcal{X}^{even}$ and $\mathcal{C}$.

(3) The direct sum of two good sequences in $\mathcal{E}^{even}$ is good as the required sequences are direct sums of good sequences in $\mathcal{A}, \mathcal{X}^{even}$ and $\mathcal{C}$. □

Remark 4.9. One can impose an n-category structure on $\mathcal{E}^{even}$ by defining a sequence to be good if some components are good. The reason for considering the n-category structure as defined in Lemma 4.8 will become clear in proofs of the following.

Later n will be assume to be an odd integer, so for the classic case (i.e., $n = 1$), the good (i.e., exact) structure on $\mathcal{E}^{even}$ we defined here is not the classic definition for extension category of exact categories (classically, the exact structure on extension categories is a sequence exact at all spots). However, as the reader will see in the proof of Theorem 4.13, the requirement of being good (i.e., exact) in the middle ($n = 1$ spot) is not necessary in proving the even additive theorem.

We now start the discussion for the even additivity theorem.

Lemma 4.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor of n-categories. Then there is an induced group homomorphism*

$$F_* : K_0\mathcal{C} = G(\pi_i^{\mathcal{C}}) \to G(\pi_i^{\mathcal{D}}) = K_0\mathcal{D}$$
$$[(\{X_1\}_i, \{X_2\}_i)] \mapsto [(\{F(X_1)\}_i, \{F(X_2)\}_i)].$$

Proof. Use the definition of $\sim_i$ and equality in the group completion $G(\pi_i)$ to obtain two good sequences, apply F to these sequences. □

Lemma 4.11. *Let $\mathcal{B}$ be an n-category with n an odd integer, $\mathcal{A}, \mathcal{X}^{even}, \mathcal{C}$ n-subcategories of $\mathcal{B}$, then for any $A \in \mathcal{A}, X^{even} \in \mathcal{X}^{even}, C \in \mathcal{C}$, there is a good* *sequence*

$$\begin{array}{cccccccccccccccccccccccc}
\mathcal{A} & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & \cdots & & n-2 & & n-1 & & n & & \mathcal{C} \\
A & \to & A \oplus X^2 & \to & X^2 & \to & X^4 & \to & X^4 & \to & X^6 & \to & X^6 & \to & \cdots & \to & X^{n-1} & \to & X^{n-1} & \to & C & \to & C
\end{array}$$

Proof. Given $A \in \mathcal{A}, X^{even} \in \mathcal{X}^{even}, C \in \mathcal{C}$, we have following good sequences

$$\begin{array}{cccccccccccccccccccccccc}
\mathcal{A} & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & \cdots & & n-2 & & n-1 & & n & & \mathcal{C} \\
A & \to & A & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & X^2 & \to & X^2 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & X^4 & \to & X^4 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & X^6 & \to & X^6 & \to & \cdots & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
 & & & & & & \cdots & & & & & & \cdots & & & & & & \cdots & & & & \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & X^{n-1} & \to & X^{n-1} & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & C & \to & C
\end{array}$$

the direct sum of all above sequences is the desired sequence so it is indeed a good sequence and therefore an object of the category $\mathcal{E}^{even}$. $\square$

Given finitely many n-categories $\mathcal{A}, \mathcal{X}^{even}$ and $\mathcal{C}$, we can make the product category $\mathcal{A} \times \mathcal{X}^{even} \times \mathcal{C}$ into an n-category by declaring good sequences to be sequences of the form $A^0 \times X^{even,0} \times C^0 \to \cdots \to A^{n+1} \times X^{even,n+1} \times C^{n+1}$ where $A^0 \to \cdots \to A^{n+1}$, $X^{even,0} \to \cdots \to X^{even,n+1}$ and $C^0 \to \cdots \to C^{n+1}$ are good sequences in $\mathcal{A}$, $\mathcal{X}^{even}$ and $\mathcal{C}$. Notice that here we denote objects of $\mathcal{A} \times \mathcal{X}^{even} \times \mathcal{C}$ as $A \times X^{even} \times C$ to distinguish them from objects of $\mathcal{E}^{even}$ discussed before. From now on, n denotes an odd integer.

Lemma 4.12. *The functors $\varphi : \mathcal{E}^{even} \to \mathcal{A} \times \mathcal{X}^{even} \times \mathcal{C}$, where one sends $(A \to X^1 \to \cdots \to X^n \to C)$ to $A \times X^{even} \times C$ and $\psi : \mathcal{A} \times \mathcal{X}^{even} \times \mathcal{C} \to \mathcal{E}^{even}$*

that maps $A \times X^{even} \times C$ to $(A \to A \oplus X^2 \to X^2 \to \cdots \to X^{n-1} \to C \to C)$ are exact functors of n-categories.

Theorem 4.13 (Even additivity)**.** *For $\mathcal{B}$ an n-category, $\mathcal{A}$, $\mathcal{X}^{even}$, $\mathcal{C}$ n-subcategories of $\mathcal{B}$. Moreover, we require that the composition of any two consecutive maps of any good sequence is zero. Then $K_m\mathcal{E}^{even} \cong K_m\mathcal{A} \times K_m\mathcal{X}^{even} \times K_m\mathcal{C}$ for any $m \geq 0$.*

Proof. First, the theorem holds for $m = 0$: $K_0\mathcal{E}^{even} \cong K_0\mathcal{A} \times K_0\mathcal{X}^{even} \times K_0\mathcal{C}$. The maps φ, ψ induce maps of the Grothendieck groups. We obviously have $\varphi \circ \psi = 1$ so φ_* is a left inverse of ψ_*. Thus it suffices to show that $\psi_* \circ \varphi_* = 1$. For every $E : A \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} C$ in $\mathcal{E}^{even}$ we have

$$\psi \circ \varphi(E) = (A \to A \oplus X^2 \to X^2 \to \cdots \to X^{n-1} \to C \to C)$$

so we need to show

$$[E] = [(A \to A \oplus X^2 \to X^2 \to \cdots \to X^{n-1} \to C \to C)]$$

in $K_0\mathcal{E}^{even}$.

This relation follows from the fundamental relation of K_0, given that the following are both good sequences (commutativity is straightforward, they are good as the even spots and two boundary sequences are good)

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "1"] & C \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & C \arrow[r] \arrow[u, "1"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & X^{n-1} \arrow[r, "1"] \arrow[u] & X^{n-1} \arrow[r] \arrow[u, "0"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & X^{n-1} \arrow[r] \arrow[u] & X^{n-1} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] \arrow[u] & X^2 \arrow[r, "1"] \arrow[u] & X^2 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
A \arrow[r] \arrow[u] & A \oplus X^2 \arrow[r] \arrow[u] & X^2 \arrow[r] \arrow[u, "1"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
A \arrow[r, "1"'] \arrow[u, "1"] & A \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

$$L^0_\bullet \to \psi \circ \varphi(E) \to L^1_\bullet \to 0 \to \cdots \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "1"] & C \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & X^n \arrow[r] \arrow[u] & C \arrow[r] \arrow[u, "1"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & X^{n-1} \arrow[r, "1"] \arrow[u] & X^{n-1} \arrow[r] \arrow[u, "0"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & X^{n-2} \arrow[r, "d"] \arrow[u, "d"] & X^{n-1} \arrow[r] \arrow[u, "1"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
0 \arrow[r] \arrow[u] & X^2 \arrow[r, "1"] \arrow[u] & X^2 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
A \arrow[r, "d^0"] \arrow[u] & X^1 \arrow[r, "d^1"] \arrow[u, "d^1"] & X^2 \arrow[r] \arrow[u, "1"'] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u] \\
A \arrow[r, "1"'] \arrow[u, "1"] & A \arrow[r] \arrow[u, "d^0"] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

$$L^0_\bullet \to E \to L^1_\bullet \to 0 \to \cdots \to 0.$$

The commutativity of the right diagram is because the composition of any two consecutive maps of E is zero and it is representing a good sequence in $\mathcal{E}^{even}$ as the even components are given by isomorphisms $0 \to X^{even} \xrightarrow{1} X^{even} \to 0 \to \cdots \to 0$. Therefore we have $K_0\mathcal{E}^{even} \cong K_0\mathcal{A} \times K_0\mathcal{X}^{even} \times K_0\mathcal{C}$.

Now, for any $m \geq 0$, $(B^q)^m\mathcal{A}, (B^q)^m\mathcal{X}^i$ and $(B^q)^m\mathcal{C}$ are subcategories with good collections of $(B^q)^m\mathcal{B}$. Define the mth extension category $(\mathcal{E}^{even})^m(\mathcal{A}, \mathcal{X}^{even}, \mathcal{C}) := \mathcal{E}^{even}((B^q)^m\mathcal{A}, (B^q)^m\mathcal{X}^{even}, (B^q)^m\mathcal{C})$ and we have $(\mathcal{E}^{even})^m \cong (B^q)^m\mathcal{E}^{even}$ because any binary multisequence of good sequences is the same as a good sequence of binary multisequences, just like the (binary multicomplexes) situation as Harris's proof of Theorem 2.9 [5]. This implies that $K_0(\mathcal{E}^{even})^m \cong K_0(B^q)^m\mathcal{A} \times K_0(B^q)^m\mathcal{X}^{even} \times K_0(B^q)^m\mathcal{C}$ as before. Similarly, under the identification of $(\mathcal{E}^{even})^m$ and $(B^q)^m\mathcal{E}^{even}$, a binary multisequence in $(B^q)^m\mathcal{E}^{even}$ is diagonal in a direction if and only if its constituent binary multisequences in $(B^q)^m\mathcal{A}, (B^q)^m\mathcal{X}^{even}$, and $(B^q)^m\mathcal{C}$ are also diagonal in the same direction; and if $A^\cdot \in (B^q)^m\mathcal{A}, X^{even} \in (B^q)^m\mathcal{X}^{even}$ and $C^\cdot \in (B^q)^m\mathcal{C}$ are diagonal, then so are the binary multisequences corresponding to $A^\cdot \to A^\cdot \to \cdots \to 0$, $0 \to \cdots \to 0 \to C \to C$ and $0 \to \cdots \to (X^{2k})^\cdot \to (X^{2k})^\cdot \to 0 \to \cdots \to 0$ (with the second $(X^{2k})^\cdot$ placed at position $2k$), so ψ preserves diagonal binary multisequences and this means the isomorphism $K_0(\mathcal{E}^{even})^m \cong K_0(B^q)^m\mathcal{A} \times K_0(B^q)^m\mathcal{X}^{even} \times K_0(B^q)^m\mathcal{C}$ restricts to an isomorphism $T^m_{\mathcal{E}^{even}} \cong T^m_{\mathcal{A}} \times T^m_{\mathcal{X}^{even}} \times T^m_{\mathcal{C}}$. Passing to the quotient groups finishes the proof. □

Corollary 4.14. *The even additive theorem holds for K groups of n-exangulated categories, where n is odd.*

Proof. The condition that the composition of any two consecutive maps of a good sequence is zero is automatic for n-exangulated categories as any distinguished n-exangle is a complex satisfying some other conditions. □

Remark 4.15. Our additive theorem does not hold for the category of N-complexes and precomplexes as the technical assumption of zero composition for good sequences in Theorem 4.13 fails.

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