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GLOBAL NONEXISTENCE FOR THE WAVE EQUATION WITH BOUNDARY VARIABLE EXPONENT NONLINEARITIES

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Abstract. This paper deals with a nonlinear wave equation with boundary damping and source terms of variable exponent nonlinearities. This work is devoted to prove a global nonexistence of solutions for a nonlinear wave equation with nonnegative initial energy as well as negative initial energy.

1. Introduction

In this paper, we consider the following the wave equation:

$$\begin{cases} u_{tt} - \mu(t)\Delta u + h(u) = 0 & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ \mu(t)\frac{\partial u}{\partial \nu} + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u & \text{on} \quad \Gamma_1 \times (0,T), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where, Ω is a bounded open domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint with $meas(\Gamma_0) > 0$. Let ν be the outward normal to Γ and $T > 0$, a real number, and $m(x)$, $p(x)$ be given functions.

This type of model arises in electro-rheological fluids or fluids with temperature dependent viscosity, viscoelasticity, filtration processes through a porous media and image processing (cf. [1,20]).

The problem of proving the nonexistence or blow-up of solutions for the wave equation has been widely studied (see [6,8–13,17,18,21–23]). Recently, many papers have treated problems with variable exponents. For the variable

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exponent problems, the main tool is based on the Lebesgue and Sobolev spaces with variable exponents, which was introduced in [3, 4] and has been widely used in the literature, see [2, 7, 14–16] and the list of references therein. For example, in [16], the authors proved the local existence of a unique weak solution for the nonlinear damped wave equation and the finite time blow-up of solutions for negative initial energies. Recently, in [7], the authors studied the global existence of solution for (1) using the stable-set method and proved the exponential or polynomial energy decay rate. However, the above mentioned references was only considered interior variable exponent nonlinearities.

On the other hand, there are very few results for the boundary variable-exponent-nonlinearity problems. In [19], the author proved the existence and asymptotic stability for the semilinear wave equation with boundary variable exponent nonlinearities. However, the blow-up was not considered.

Motivated by previous works, the goal of this paper is to prove a finite time blow-up for the solution for (1) under suitable condition on the initial data and the positive initial energy. As far as we know, there is no blow-up result concerning the boundary variable-exponent nonlinearities.

This paper is organized as follows: In Section 2, we recall the notation, hypotheses and some necessary preliminaries and introduce our main result. In Section 3, we prove the blow-up of solutions for (1) with nonnegative initial energy as well as negative initial energy.

2. Preliminaries

We begin this section by introducing some hypotheses and our main result. Throughout this paper, $||\cdot||_p$ and $||\cdot||_{p,\Gamma_1}$ denote the $L^p(\Omega)$ norm and $L^p(\Gamma_1)$ norm, respectively.

($\mathbf{H_1}$) Hypotheses on $\boldsymbol{\Omega}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint with $meas(\Gamma_0) > 0$, satisfying the following conditions:

$$\begin{aligned} &w(x) \cdot \nu(x) \geq \sigma > 0 \quad \text{on} \quad \Gamma_1, \quad w(x) \cdot \nu(x) \leq 0 \quad \text{on } \Gamma_0, \\ &w(x) = x - x^0 (x^0 \in \mathbb{R}^n) \quad \text{and} \quad R = \max_{x \in \overline{\Omega}} |w(x)|, \end{aligned} \tag{2}$$

where ν represents the unit outward normal vector to Γ. We assume that

$$\mu(0) \frac{\partial u_0}{\partial \nu} + |u_1|^{m(x)-2} u_1 = |u_0|^{p(x)-2} u_0 \quad \text{on} \quad \Gamma_1. \tag{3}$$

($\mathbf{H_2}$) Hypotheses on $\boldsymbol{m(x)}$, $\boldsymbol{p(x)}$.

Let $m(x)$ and $p(x)$ be given measurable functions on $\bar{\Omega}$ satisfying the following conditions:

$$\begin{cases} 2 \leq q^- \leq q(x) \leq q^+ < \frac{2(n-1)}{n-2} & \text{if} \quad n \geq 3, \\ q^- > 2 & \text{if} \quad n = 1, 2, \end{cases} \tag{4}$$

where

$$q^- = ess \inf_{x\in\bar{\Omega}} q(x), \quad \text{and} \quad q^+ = ess \sup_{x\in\bar{\Omega}} q(x).$$

Furthermore, $m(x)$ and $p(x)$ satisfy the log-Hölder continuity condition as follows

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|} \quad \text{for all} \quad x, y \in \Omega, \tag{5}$$

with $|x - y| < \delta$, $A > 0$, $0 < \delta < 1$.

($\mathbf{H_3}$) Hypotheses on $\boldsymbol{\mu}$, h.

Let $\mu \in W^{1,\infty}(0,\infty) \cap W^{1,1}(0,\infty)$ satisfy the following conditions:

$$\mu(t) \geq \mu_0 > 0 \quad \text{and} \quad \mu'(t) < 0 \quad \text{a.e. in} \quad [0,\infty), \tag{6}$$

where μ_0 is a positive constant. Moreover, we assume that

(7) $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, and $2H(s) \geq h(s)s \geq 0$ for all $s \in \mathbb{R}$,

where $H(s) = \int_0^s h(\tau)d\tau$.

In order to treat the variable-exponent nonlinearities $m(x)$ and $p(x)$, we need some preliminary facts about the Lebesgue and Sobolev spaces with variable exponents (see [3, 4]). For the reader's convenience, we will repeat some of them here.

Let $q : \Omega \to [1,\infty]$ be a measurable function. We define the Lebesgue space with a variable exponent $q(\cdot)$ by

$$L^{q(\cdot)}(\Omega) := \{u \,|\, u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |\lambda u(x)|^{q(x)} dx < \infty \text{ for some } \lambda > 0\},$$

equipped the with following Luxembourg-type norm

$$||u||_{q(\cdot),\Omega} = ||u||_{q(\cdot)} := \inf\{\lambda > 0 \,|\, \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{q(x)} dx \leq 1\}.$$

$L^{q(\cdot)}(\Omega)$ is a Banach space. Next, we define the Sobolev space $W^{1,q(\cdot)}(\Omega)$ as follows:

$$W^{1,q(\cdot)}(\Omega) := \{u \in L^{q(\cdot)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{q(\cdot)}(\Omega)\}.$$

This is a Banach space with respect to the norm $||u||_{W^{1,q(\cdot)}(\Omega)} = ||u||_{q(\cdot)} + ||\nabla u||_{q(\cdot)}$. Furthermore, we set $W_0^{1,q(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,q(\cdot)}(\Omega)$.

Lemma 2.1 (Poincaré's Inequality [3, 4])**.** *Let Ω be a bounded domain of $\mathbb{R}^n$ and $q(\cdot)$ satisfies* (5)*. Then*

$$||u||_{q(\cdot)} \leq C_1 ||\nabla u||_{q(\cdot)} \quad \textit{for all} \quad u \in W_0^{1,q(\cdot)}(\Omega),$$

where C_1 is a positive constant which depends on $q^\pm$ and Ω. In particular, $||\nabla u||_{q(\cdot)}$ defines an equivalent norm on $W_0^{1,q(\cdot)}(\Omega)$.

Lemma 2.2 ([3,4]). *Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary Γ. Assume that $r : \Omega \to (1, \infty)$ is a measurable function such that*

$$1 < r^- \le r(x) \le r^+ < +\infty \quad \text{for a.e. } x \in \Omega.$$

If $q(x), r(x) \in C(\bar{\Omega})$ and $q(x) < r^*(x)$ in $\bar{\Omega}$ with

$$r^*(x) = \begin{cases} \frac{nr(x)}{n-r(x)} & \text{if } r^+ < n, \\ \infty & \text{if } r^+ \ge n. \end{cases}$$

Then the embedding $W^{1,r(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.3 ([5]). *Let Ω be a bounded domain in $\mathbb{R}^n$, $q \in C^{0,1}(\bar{\Omega})$, $1 < q^- \le q(x) \le q^+ < n$. Then for any $r \in C(\Gamma)$ with $1 \le r(x) \le \frac{(n-1)q(x)}{n-q(x)}$, there is a continuous trace $W^{1,q(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Gamma)$, and when $1 \le r(x) < \frac{(n-1)q(x)}{n-q(x)}$, the trace is compact, in particular, the continuous trace $W^{1,q(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Gamma)$ is compact.*

From Lemmas 2.2 and 2.3, we have the embedding $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Gamma_1)$, where

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$$

and

$$\begin{cases} 2 \le q^- \le q(x) \le q^+ < \frac{2(n-1)}{n-2} & \text{if } n \ge 3, \\ q^- > 2 & \text{if } n = 1, 2, \end{cases}$$

which satisfies the inequalities

$$(8) \quad ||u||_{q(\cdot)} \le C_2||\nabla u||_2 \quad \text{and} \quad ||u||_{q(\cdot),\Gamma_1} \le C_3||\nabla u||_2 \quad \text{for all} \quad u \in H_0^1(\Omega),$$

where C_2 and C_3 are for some positive constants.

Lemma 2.4 (Hölder's inequality [3,4]). *Let $q, r, s \ge 1$ be measurable functions defined on Ω such that*

$$\frac{1}{s(x)} = \frac{1}{q(x)} + \frac{1}{r(x)} \text{ for a.e. } x \in \Omega.$$

If $f \in L^{q(\cdot)}(\Omega)$ and $g \in L^{r(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$,

$$\int_\Omega |fg|^s(x)dx \le \int_\Omega |f|^q(x)dx + \int_\Omega |g|^r(x)dx$$

and

$$||fg||_{s(\cdot)} \le 2||f||_{q(\cdot)}||g||_{r(\cdot)}.$$

Lemma 2.5 ([3, 4]). *If $q : \bar{\Omega} \to [1, \infty)$ is a continuous function satisfying $2 \le q_1 \le q(x) \le q_2 < q^*$, where*

$$\begin{cases} q^* = \frac{2n}{n-2} & \text{for } n \ge 3, \\ q^* = \infty & \text{for } n = 1, 2. \end{cases}$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact, and we have*

$$\min\Big\{||u||^{q_1}_{q(\cdot),\bar{\Omega}}, ||u||^{q_2}_{q(\cdot),\bar{\Omega}}\Big\} \le \int_{\bar{\Omega}} |u|^{q(x)}dx \le \max\Big\{||u||^{q_1}_{q(\cdot),\bar{\Omega}}, ||u||^{q_2}_{q(\cdot),\bar{\Omega}}\Big\}. \tag{9}$$

The following theorem is the local existence of solution of problem (1), which can be established employing the Faedo-Galerkin method as in the work of [16,19].

Theorem 2.6. *Let the initial data* $\{u_0, u_1\}$ *belong to* $H_0^1(\Omega) \times L^2(\Omega)$ *and the hypotheses* (H_1)*-*(H_3) *hold. Additionally* $p(x)$ *satisfies*

$$2 \le p^- \le p(x) \le p^+ < \frac{2n-3}{n-2} \quad \text{if} \quad n \ge 3. \tag{10}$$

Then problem (1) *has a unique weak solution such that*

$$u \in L^\infty((0,T); H_0^1(\Omega)), \quad L^\infty((0,T); L^2(\Omega)) \cap L^{m(\cdot)}(\Gamma_1 \times (0,T)).$$

In order to formulate another result, it is convenient to introduce the energy associated with problem (1):

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{1}{2}\mu(t)||\nabla u||_2^2 + \int_\Omega H(u)dx - \int_{\Gamma_1} \frac{|u|^{p(x)}}{p(x)} d\Gamma, \tag{11}$$

where $H(s) = \int_0^s h(\tau)d\tau$. Then by (6),

$$E'(t) = \frac{1}{2}\mu'(t)||\nabla u||_2^2 - \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma \le 0,$$

which implies that $E(t)$ is a nonincreasing function.

Theorem 2.7. *Suppose that the hypotheses* (H_1)*-*(H_3) *hold. Moreover, we assume that* $m^+ < p^-$ *and,*

$$E(0) < d \quad \text{and} \quad \eta_1 < ||\nabla u_0||_2 \le C_4^{-1}, \tag{12}$$

where $d = \mu_0\Big(\frac{1}{2} - \frac{1}{p^-}\Big)\eta_1^2$, $\eta_1 = \big(\mu_0 C_4^{-p^-}\big)^{\frac{1}{p^- - 2}}$ *and* $C_4 = \max\{1, C_3\}$. *Then the solution of problem* (1) *cannot exist for all time.*

3. Blow-up

This section is devoted to prove Theorem 2.7. By similar arguments as in [16] and using (8), we get the following lemma.

Lemma 3.1. *Suppose that the assumption* (6) *holds and* u *is a solution of* (1). *Then we have*

$$\Big(\int_{\Gamma_1} |u|^{p(x)} d\Gamma\Big)^{\frac{s}{p^-}} \le C_5\Big(||\nabla u||_2^2 + \int_{\Gamma_1} |u|^{p(x)} d\Gamma\Big), \tag{13}$$

$$\int_{\Gamma_1} |u|^{p(x)} d\Gamma \ge C_6 ||u||_{p^-}^{p^-}, \tag{14}$$

$$\int_{\Gamma_1} |u|^{m(x)} d\Gamma \le C_7\Big(\Big(\int_{\Gamma_1} |u|^{p(x)} d\Gamma\Big)^{\frac{m^-}{p^-}} + \Big(\int_{\Gamma_1} |u|^{p(x)} d\Gamma\Big)^{\frac{m^+}{p^-}}\Big), \tag{15}$$

where $2 \le s \le p^-$ *and* $C_5, C_6, C_7 > 1$ *are positive constants depending only on* Ω.

The next lemma plays an essential role for the proof of the blow-up result.

Lemma 3.2. *Let the assumption in Theorem 2.7 be satisfied. Then there exists a positive constant* $\eta_* > \eta_1$ *such that*

$$||\nabla u(t)||_2 \ge \eta_* \quad \text{for all} \quad 0 < t < T_{\max}, \tag{16}$$

where $T_{\max}$ *is the maximal time of existence of the solution of* (1).

Proof. **Case 1 :** $0 \le E(0) < d$.

By using (11), (6), (9) and (8), we get that

$$\begin{aligned} E(t) &\ge \frac{1}{2}\mu(t)||\nabla u||_2^2 - \frac{1}{p^-}\int_{\Gamma_1} |u|^{p(x)} d\Gamma \\ &\ge \frac{1}{2}\mu_0||\nabla u||_2^2 - \frac{1}{p^-}\max\Big\{||u||^{p^-}_{p(\cdot),\Gamma_1}, ||u||^{p^+}_{p(\cdot),\Gamma_1}\Big\} \\ &\ge \frac{1}{2}\mu_0||\nabla u||_2^2 - \frac{1}{p^-}\max\Big\{C_3^{p^-}||\nabla u||_2^{p^-}, C_3^{p^+}||\nabla u||_2^{p^+}\Big\} \\ &:= f(||\nabla u(t)||_2) \end{aligned} \tag{17}$$

for any $t \in [0, T_{\max})$.

We note that $f(\eta) = g(\eta)$ for $0 \le \eta \le C_4^{-1}$, where $g(\eta) = \frac{1}{2}\mu_0\eta^2 - \frac{C_4^{p^-}}{p^-}\eta^{p^-}$. It is easy to verify that g is strictly increasing on $(0, \eta_1)$ and strictly decreasing on (η_1, ∞), where $\eta_1 = \big(\mu_0 C_4^{-p^-}\big)^{\frac{1}{p^- - 2}}$ is the maximum point of $g(\eta)$, and $g(\eta_1) = d$. Hence we have $g(\eta) \to -\infty$ as $\eta \to \infty$. Since $E(0) < d = g(\eta_1)$, there exists $\eta_2 > \eta_1$ such that $g(\eta_2) = E(0)$. Therefore we obtain from (17),

$$g(\eta_2) = E(0) \ge f(||\nabla u_0||_2) = g(||\nabla u_0||_2),$$

which implies that $\eta_2 \le ||\nabla u_0||_2$. From (12), we also have

$$\eta_2 \le C_4^{-1}. \tag{18}$$

Now we prove that

$$||\nabla u(t)||_2 \ge \eta_2 \quad \text{for all} \quad 0 < t < T_{\max} \tag{19}$$

by using the contradiction method. Suppose that (19) does not hold. Then there exists $t^* \in (0, T_{\max})$ such that

$$||\nabla u(t^*)||_2 < \eta_2. \tag{20}$$

If $||\nabla u(t^*)||_2 > \eta_1$, then we obtain from (17), (18) and (20),

$$E(t^*) \ge f(||\nabla u(t^*)||_2) = g(||\nabla u(t^*)||_2) > g(\eta_2) = E(0),$$

which is a contradiction with respect to the monotonicity of the energy.

If $||\nabla u(t^*)||_2 \leq \eta_1$, then since $\eta_1 < \eta_2$, there exists η_3 which verifies

$$||\nabla u(t^*)||_2 \leq \eta_1 < \eta_3 < \eta_2 \leq ||\nabla u_0||_2.$$

From the continuity of the function $||\nabla u(\cdot)||_2$, there exists $\bar{t} \in (0, t^*)$ verifying $||\nabla u(\bar{t})||_2 = \eta_3$. Therefore from (17) and (18) we deduce

$$E(\bar{t}) \geq f(||\nabla u(\bar{t})||_2) = g(||\nabla u(\bar{t})||_2) > g(\eta_2) = E(0),$$

which is also contradiction.

Case 2 : $E(0) < 0$.

There is $\eta_4 > \eta_1$ such that $g(\eta_4) = E(0)$, consequently, by (17) we have

$$g(\eta_4) = E(0) \geq f(||\nabla u_0||_2) = g(||\nabla u_0||_2).$$

From the fact $g(\eta)$ is decreasing for $\eta_1 < \eta$, we get

$$||\nabla u_0||_2 \geq \eta_4.$$

By the same argument as in Case 1, we obtain

$$||\nabla u(t)||_2 \geq \eta_4 \quad \text{for all} \quad 0 < t < T_{\max}.$$

Let $\eta_* = \max\{\eta_2, \eta_4\}$. Then the proof of Lemma 3.2 is completed. □

Now we will prove the blow-up result in finite time. We set

$$G(t) = E_1 - E(t), \tag{21}$$

where E_1 is a constant lying in $(E(0), d)$. Then

$$G'(t) = -E'(t) \geq \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma \geq 0, \tag{22}$$

which implies that $G(t)$ is a nondecreasing function, consequently, from Lemma 3.2 and the definition of d, and using the fact that $\eta_1 < \eta_*$ and $\mu_0 > 0$,

$$0 < G(0) \leq G(t) < d - \frac{1}{2}\mu_0 \eta_*^2 + \frac{1}{p^-} \int_{\Gamma_1} |u|^{p(x)} d\Gamma \leq \frac{1}{p^-} \int_{\Gamma_1} |u|^{p(x)} d\Gamma. \tag{23}$$

We define

$$L(t) = G^{1-\delta}(t) + \epsilon N(t), \; N(t) = \int_\Omega u_t u dx, \tag{24}$$

where $\epsilon > 0$ will be chosen later and

$$0 < \delta \leq \min\Big\{\frac{p^- - 2}{2p^-}, \frac{p^- - m^+}{p^-(m^+ - 1)}\Big\}. \tag{25}$$

Then we have

$$L'(t) = (1 - \delta) G^{-\delta}(t) G'(t) + \epsilon N'(t). \tag{26}$$

We are now going to analyze the last term on the right hand side of (26). From the definition of $E(t)$, Lemma 3.2 and (7), we have

$$\begin{aligned} N'(t) &= ||u_t||_2^2 - \mu(t)||\nabla u||_2^2 - \int_\Omega h(u)udx + \int_{\Gamma_1} |u|^{p(x)}d\Gamma \\ &\quad - \int_{\Gamma_1} |u_t|^{m(x)-2}u_tud\Gamma + \theta E(t) - \theta E(t) \\ &\geq \Big(1+\frac{\theta}{2}\Big)||u_t||_2^2 + \underbrace{\mu_0\Big(\frac{\theta}{2}-1\Big)||\nabla u||_2^2 - \theta E_1}_{:=I_1} + \theta G(t) \\ &\quad + \Big(1-\frac{\theta}{p^-}\Big)\int_{\Gamma_1}|u|^{p(x)}d\Gamma - \underbrace{\int_{\Gamma_1}|u_t|^{m(x)-2}u_tud\Gamma}_{:=I_2}, \end{aligned} \tag{27}$$

provided that $\theta = p^- - \epsilon$ with $0 < \epsilon < p^- - 2$.

Estimate for I_1.

From Lemma 3.2 and the definition of θ, we obtain

$$\begin{aligned} \mu_0\Big(\frac{\theta}{2}-1\Big)||\nabla u||_2^2 - \theta E_1 &> \mu_0\Big(\frac{p^- - \epsilon}{2}-1\Big)\eta_1^2 - (p^- - \epsilon)E_1 \\ &= \Big(E_1 - \frac{\mu_0\eta_1^2}{2}\Big)\epsilon + \mu_0\Big(\frac{p^-}{2}-1\Big)\eta_1^2 - p^-E_1 := F(\epsilon). \end{aligned}$$

We note that

$$E_1 - \frac{\mu_0\eta_1^2}{2} < d - \frac{\mu_0\eta_1^2}{2} = -\frac{1}{p^-}\mu_0\eta_1^2 < 0$$

and

$$\mu_0\Big(\frac{p^-}{2}-1\Big)\eta_1^2 - p^-E_1 > \mu_0\Big(\frac{p^-}{2}-1\Big)\eta_1^2 - p^-d = 0.$$

Thus, $F(\epsilon)$ represent a decreasing line connecting vertical and horizontal axes points $v_\epsilon := \mu_0\Big(\frac{p^-}{2}-1\Big)\eta_1^2 - p^-E_1$ and $h_\epsilon := v_\epsilon\big(\frac{\mu_0\eta_1^2}{2} - E_1\big)^{-1}$, respectively. Hence, we get that

$$\mu_0\Big(\frac{\theta}{2}-1\Big)||\nabla u||_2^2 - \theta E_1 > F(\epsilon) > 0 \quad \text{for} \quad 0 < \epsilon < h_\epsilon. \tag{28}$$

Estimate for I_2.

By multiplying by $1 = \xi\xi^{-1}$ for $\xi > 0$, and by using Lemma 2.4 with $s = 1$, $q(x) = \frac{m(x)}{m(x)-1}$ and $r(x) = m(x)$, it holds that

$$\Big|\int_{\Gamma_1}|u_t|^{m(x)-2}u_tud\Gamma\Big| \leq \int_{\Gamma_1}\xi^{m(x)}|u|^{m(x)}d\Gamma + \int_{\Gamma_1}\xi^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}d\Gamma.$$

We take $\xi^{-\frac{m(x)}{m(x)-1}} = kG^{-\delta}(t)$, for a large constant k to be chosen later. The choice of ξ is allowed since $G(t) > 0$ for every t as (23) holds true. Hence the

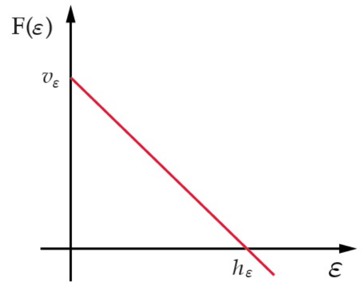

FIGURE 1. The figure of $F(\epsilon)$

above inequality takes the form:

$$\int_{\Gamma_1} |u_t|^{m(x)-1}|u|d\Gamma \leq kG^{-\delta}(t)G'(t) + k^{1-m^-}G^{\delta(m^+-1)}(t)\int_{\Gamma_1} |u|^{m(x)}d\Gamma. \tag{29}$$

Applying (15) and (23), and then using (13) with $s = m^- + \delta p^-(m^+ - 1) \leq p^-$ and $s = m^+ + \delta p^-(m^+ - 1) \leq p^-$ we deduce that

$$\begin{aligned}
&k^{1-m^-}G^{\delta(m^+-1)}(t)\int_{\Gamma_1} |u|^{m(x)}d\Gamma \\
&\leq k^{1-m^-}\Big(\frac{1}{p^-}\int_{\Gamma_1} |u|^{p(x)}d\Gamma\Big)^{\delta(m^+-1)}\int_{\Gamma_1} |u|^{m(x)}d\Gamma \\
&\leq k^{1-m^-}(p^-)^{\delta(1-m^+)}C_7\Big\{\Big(\int_{\Gamma_1} |u|^{p(x)}d\Gamma\Big)^{\frac{m^-}{p^-}+\delta(m^+-1)} \\
&\qquad + \Big(\int_{\Gamma_1} |u|^{p(x)}d\Gamma\Big)^{\frac{m^+}{p^-}+\delta(m^+-1)}\Big\} \\
&= k^{1-m^-}(p^-)^{\delta(1-m^+)}C_7\Big\{\Big(\int_{\Gamma_1} |u|^{p(x)}d\Gamma\Big)^{\frac{m^-+\delta p^-(m^+-1)}{p^-}} \\
&\qquad + \Big(\int_{\Gamma_1} |u|^{p(x)}d\Gamma\Big)^{\frac{m^++\delta p^-(m^+-1)}{p^-}}\Big\} \\
&\leq 2k^{1-m^-}(p^-)^{\delta(1-m^+)}C_5C_7\Big(||\nabla u||_2^2 + \int_{\Gamma_1} |u|^{p(x)}d\Gamma\Big).
\end{aligned} \tag{30}$$

We note that from the definition of $E(t)$ and Lemma 3.2,

$$\begin{aligned}
\frac{1}{p^-}\int_{\Gamma_1} |u|^{p(x)}d\Gamma \geq \int_{\Gamma_1} \frac{|u|^{p(x)}}{p(x)}d\Gamma &\geq \frac{1}{2}\mu_0||\nabla u||_2^2 - E(t) \\
&\geq \frac{1}{2}\mu_0\eta_*^2 - d > \frac{1}{2}\mu_0\eta_1^2 - d = d\Big(\frac{2}{p^- - 2}\Big)
\end{aligned}$$

by the definition of d. Hence, from the above inequality and the definition of $E(t)$, we have

$$\begin{aligned} \mu_0||\nabla u||_2^2 &\le \mu(t)||\nabla u||_2^2 \\ &= 2E(t) - ||u_t||_2^2 - 2\int_\Omega H(u)dx + 2\int_{\Gamma_1} \frac{|u|^{p(x)}}{p(x)} d\Gamma \\ &= 2E_1 - 2G(t) - ||u_t||_2^2 - 2\int_\Omega H(u)dx + 2\int_{\Gamma_1} \frac{|u|^{p(x)}}{p(x)} d\Gamma \\ &< 2d + \frac{2}{p^-}\int_{\Gamma_1} |u|^{p(x)} d\Gamma \\ &= \int_{\Gamma_1} |u|^{p(x)} d\Gamma. \end{aligned} \tag{31}$$

Combining (30) and (31), we obtain

$$\begin{aligned} &k^{1-m^-} G^{\delta(m^+-1)}(t) \int_{\Gamma_1} |u|^{m(x)} d\Gamma \\ &\le 2k^{1-m^-}(p^-)^{\delta(1-m^+)} C_5 C_7 (\mu_0^{-1}+1) \int_{\Gamma_1} |u|^{p(x)} d\Gamma. \end{aligned} \tag{32}$$

Therefore (29) and (32) yield

$$\begin{aligned} &\int_{\Gamma_1} |u_t|^{m(x)-1}|u| d\Gamma \\ &\le kG^{-\delta}(t)G'(t) + 2k^{1-m^-}(p^-)^{\delta(1-m^+)} C_5 C_7 (\mu_0^{-1}+1) \int_{\Gamma_1} |u|^{p(x)} d\Gamma. \end{aligned} \tag{33}$$

Combining (26), (27), (28) and (33), we have for $0 < \epsilon < h_\epsilon$,

$$\begin{aligned} L'(t) \ge\ & (1-\delta-\epsilon k)G^{-\delta}(t)G'(t) + \epsilon\Big(1+\frac{\theta}{2}\Big)||u_t||_2^2 + \epsilon\theta G(t) \\ &+ \epsilon\Big(1 - \frac{\theta}{p^-} - 2k^{1-m^-}(p^-)^{\delta(1-m^+)} C_5 C_7 (\mu_0^{-1}+1)\Big) \int_{\Gamma_1} |u|^{p(x)} d\Gamma. \end{aligned}$$

We now choose k large enough such that

$$1 - \frac{\theta}{p^-} - 2k^{1-m^-}(p^-)^{\delta(1-m^+)} C_5 C_7 (\mu_0^{-1}+1) \ge 0.$$

Once k is fixed, we take ϵ small enough such that $1-\delta-\epsilon k \ge 0$ and $L(0) = G^{1-\delta}(0) + \epsilon N(0) > 0$, consequently, we conclude that from (14)

$$L'(t) \ge C_8\big(||u_t||_2^2 + ||u||_{p^-}^{p^-} + G(t)\big),$$

where C_8 is a positive constant, which implies that $L(t)$ is a positive increasing function. By the same arguments as in [16], page 3036, we have

$$L'(t) \ge C_9 L^{\frac{1}{1-\delta}}(t) \quad \text{for all} \quad t \in (0, T_{\max}),$$

where C_9 is a positive constant. Hence we conclude that $L(t)$ blows up in finite time and u also blows up in finite time. Thus the proof of Theorem 2.7 is completed.

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