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CONSTRUCTION FOR SELF-ORTHOGONAL CODES OVER A CERTAIN NON-CHAIN FROBENIUS RING

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Abstract. We present construction methods for free self-orthogonal (self-dual or Type II) codes over $\mathbb{Z}_4[v]/\langle v^2+2v\rangle$ which is one of the finite commutative local non-chain Frobenius rings of order 16. By considering their Gray images on $\mathbb{Z}_4$, we give a construct method for a code over $\mathbb{Z}_4$. We have some new and optimal codes over $\mathbb{Z}_4$ with respect to the minimum Lee weight or minimum Euclidean weight.

1. Introduction

Coding theory has been many developments with many related areas such as combinatorics, quantum information theory, and number theory (for instance [1, 3, 4, 6, 10, 11, 14, 15, 18, 19, 21]). In coding theory, one of the central problems is finding a code with the best parameter. This leads to the optimality of minimum weight for a code; we call a linear code *optimal* if it has the highest minimal weight of any linear code of that length. Many linear codes over $\mathbb{Z}_4$ have critical aspects in coding theory. A certain Gray maps image of a linear code over $\mathbb{Z}_4$ is a non-linear binary code with larger length. Also, the minimum weight of a non-linear binary code can be found from the minimum Lee weight of the linear code over $\mathbb{Z}_4$; this code over $\mathbb{Z}_4$ is the pre-image of the Gray map. From these reasons, linear codes over $\mathbb{Z}_4$ are still studied, and the information for the codes have been updating [7]; for finding new optimal code over $\mathbb{Z}_4$, this database is used normally. Furthermore, self-orthogonal codes have significance to research of quantum communications and quantum computations (see [2], [3]).

A *Frobenius ring* is one of the most interesting parts in coding theory since the ring is related to the MacWilliams identity. A generator matrix of a linear code is useful for researching in this area. Especially, over a finite commutative

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local non-chain Frobenius rings of order 16, the standard generator matrix of a linear code is introduced in [8]. In that respect, these are our motivations for looking at new optimal codes over $\mathbb{Z}_4$ via codes over the ring $\mathbb{Z}_4[v]/\langle v^2+2v\rangle$; this ring is one of the finite commutative local non-chain Frobenius rings of order 16.

In [9], N. Han et al. study α-constacyclic codes over a finite commutative Frobenius ring $\mathbb{Z}_4[u]/\langle u^2-1\rangle$. They also obtain new MDR cyclic codes over $\mathbb{Z}_4$ via α-constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2-1\rangle$. In [5], Y. Cao and Y. Cao classify all cyclic codes of odd length n over $\mathbb{Z}_4[v]/\langle v^2+2v\rangle$, and give all self-dual cyclic codes over the ring. In [16], S. Ling and P. Solé study a Gray map, construction of lattice and invariant for Type II codes over a finite commutative local chain Frobenius ring $\mathbb{F}_4[u]/\langle u^2\rangle$. Recently, B. Kim et al. give invariants and Jacobi forms via linear codes over $\mathbb{F}_4[u]/\langle u^2\rangle$ [12]. In general, a certain type of code over $\mathbb{F}_p[u]/\langle u^m\rangle$ is investigated by M. Shi et al. [20]. The Galois ring $GR(2^2,2)$ is a finite commutative local Frobenius ring of order 16, and B. Kim and Y. Lee suggest Lee weights for cyclic self-dual codes over an extended ring $GR(p^2,m)$, where p is prime and $m\geq 1$ [13].

In this paper, we focus on the ring $R:=\mathbb{Z}_4[v]/\langle v^2+2v\rangle$. First, we present construction methods for free self-orthogonal (self-dual or Type II) codes over $\mathbb{Z}_4[v]/\langle v^2+2v\rangle$ which is one of the finite commutative local non-chain Frobenius rings of order 16 (Theorems 1 and 3). We define the Euclidean weight in R for preserving the weight by a Gray map from R to $\mathbb{Z}_4^2$. By considering their Gray images on $\mathbb{Z}_4$, we construct codes over $\mathbb{Z}_4$ (Theorem 4). In Tables 1 and 2, we give some new and optimal codes over $\mathbb{Z}_4$ with respect to the minimum Lee weight or minimum Euclidean weight.

2. Preliminaries

A *linear code* C of length n over a ring $\mathfrak{R}$ is an $\mathfrak{R}$-submodule of $\mathfrak{R}^n$; from now on, we call a linear code by a *code* for simplicity. Any element $c=(c_1,\ldots,c_n)$ in C is called a *codeword*. The *dual code* $C^\perp$ of C is $\{c\in\mathfrak{R}^n : c\cdot\hat{c}=0 \text{ for all } \hat{c}\in C\}$ with respect to the usual inner product. If $C\subseteq C^\perp$ (resp. $C=C^\perp$), then C is a *self-orthogonal* (resp. *self-dual*) code.

For a finite commutative ring $\mathfrak{R}$, if the $\mathfrak{R}$-module is injective, then $\mathfrak{R}$ is *Frobenius*. A finite commutative local Frobenius non-chain ring of order 16 has a unique non-principal maximal ideal $\langle u,v\rangle$ and the socle $Soc(\mathfrak{R})$ of a $\mathfrak{R}$-module is $\langle\omega\rangle=\{0,\omega\}$ for some elements u,v,ω in $\mathfrak{R}$; the $Soc(\mathfrak{R})$ is defined as a sum of its minimal submodules. By the following proposition, we get a generator matrix for a code over a finite commutative local Frobenius non-chain ring of order 16.

Proposition 1 ([8, Theorem 4.1])**.** *Let $\mathfrak{R}$ be a finite commutative local Frobenius non-chain ring of order* 16*. Any code C over $\mathfrak{R}$ has the following generator matrix*:

$$(1)\qquad \left(\begin{array}{c|c|c|c|c|c|c}
I_{k_0} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ \hline
0 & uI_{k_1} & B_1 & B_2 & B_3 & B_4 & B_5 \\ \cline{1-2}
0 & vI_{k_1} & & & & & \\ \hline
0 & 0 & uI_{k_2} & 0 & 0 & & \\ \cline{1-5}
0 & 0 & 0 & vI_{k_3} & 0 & C_1 & C_2 \\ \cline{1-5}
0 & 0 & 0 & 0 & (u+v)I_{k_4} & & \\ \hline
0 & 0 & 0 & 0 & 0 & \omega I_{k_5} & D
\end{array}\right),$$

where

- I_{k_i} *is the* $k_i \times k_i$ *identity matrix,*
- A_i *consists of any elements in* $\mathfrak{R}$*,*
- B_i *consists of the elements from the unique maximal ideal of* $\mathfrak{R}$*,*
- *each column of* C_i *have elements of only one ideal of order* 4*,*
- *the elements of* D *are from* $Soc(\mathfrak{R})$*.*

In (1), if $k_0 \neq 0$ and $k_i = 0$ with $1 \leq i \leq 5$, then a code C is called a *free code* over $\mathfrak{R}$. Here, the value k_0 is called the *free rank* of the free code C.

The ring $R := \mathbb{Z}_4[v]/\langle v^2 + 2v\rangle$ is one of the finite commutative local non-chain Frobenius rings of order 16 (see [8]). The following set $\{1, 3, 1+v, 3+v, 1+2v, 3+2v, 1+3v, 3+3v\}$ is the set of all units in the ring R. By simple calculation, we get $\mathfrak{u}^2 = 1$ for any unit $\mathfrak{u}$ in R. This plays a key role in constructing for self-orthogonal codes over R.

3. Construction methods for self-orthogonal codes over R

In this section, we present construction methods for finding free self-orthogonal codes over R (Theorems 1 and 3). We recall that a code means a linear code in this paper. Denote the $k \times k$ identity matrix by I_k.

Theorem 1. *Let* M *be a* $k_1 \times k_2$*-matrix over* R *with* $4 \mid k_1$*, where* r_i *is an* i*-th row vector of* M $(1 \leq i \leq k_1)$*. Let* $(I_{k_1} \mid M)$ *be a generator matrix for a free self-orthogonal (or self-dual) code* C *of length* $k_1 + k_2$ *over* R*. Let* u_i *be a unit in* R $(1 \leq i \leq k_1)$*. For a fixed integer* ℓ *with* $1 \leq \ell \leq k_1$*, the following matrix* $\tilde{M}$ *generates a free self-orthogonal code* $\tilde{C}$ *of length* $2k_1 + k_2$ *over* R *with free rank* $k_1 + 1$*:*

$$(2)\qquad \tilde{M} := \left(\begin{array}{c|c|c}
 & & u_1 r_1 \\
2E_{i,j} & I_{k_1} & \vdots \\
 & & u_{k_1} r_{k_1} \\ \hline
v_1 & v_2 & r_\ell
\end{array}\right)$$

such that

- r_ℓ *is the* ℓ*-th row vector of the matrix* M *for the fixed integer* $1 \leq \ell \leq k_1$*,*
- *the matrix* E_ℓ *is a* $k_1 \times k_1$*-matrix, where* (ℓ, ℓ)*-th component is equal to* 1*. The other components are all equal to* 0 $(1 \leq i, j \leq k_1)$*,*

– *the vector* v_1 *(resp.* v_2*) has length* k_1*, where* ℓ*-th component is equal to* u_ℓ *(resp.* $3u_\ell$*). The other components are all equal to* 1 *(resp.* 0*).*

Proof. Let $\tilde{m}_i$ be an i-th row vector of the matrix $\tilde{M}$. First, except for $\tilde{m}_\ell$, the inner product value $\tilde{m}_i \cdot \tilde{m}_{k_1+1} = 0$ for $1 \le i \le k_1$ by the orthogonality of the code C. We also have that $\tilde{m}_i \cdot \tilde{m}_j = 0$ for all $1 \le i, j \le k_1$. Moreover, we have $\tilde{m}_{k_1+1} \cdot \tilde{m}_{k_1+1} = v_1 \cdot v_1 + v_2 \cdot v_2 + r_\ell \cdot r_\ell = 0$; in detail, $v_1 \cdot v_1 = 0$ since $4 \mid k_1$. Clearly, $v_2 \cdot v_2 = 1$ and $r_\ell \cdot r_\ell = 3$. Finally, $\tilde{m}_\ell \cdot \tilde{m}_{k_1+1} = 2u_\ell + 3u_\ell + u_\ell(r_\ell \cdot r_\ell) = 0$ in R. Thus, the matrix $\tilde{M}$ generates a self-orthogonal code $\tilde{C}$ of length $2k_1 + k_2$ over R. The code $\tilde{C}$ is a free code since the nonzero components of the vectors v_1 and v_2 are units in R. Hence the result is proved. □

We give an example for Theorem 1.

Example 1. Let

$$M = \begin{pmatrix} 0 & 1+2v & 3+2v & 3+2v \\ 1+2v & 1+2v & 1+2v & 0 \\ 1+2v & 0 & 3+2v & 1+2v \\ 3+2v & 1+2v & 0 & 1+2v \end{pmatrix}$$

be a 4×4-matrix over R. Then the matrix $(I_4 \mid M)$ generates a free self-dual code of length 8 over R. By Theorem 1, we can construct the following matrix

$$\tilde{M} = \left(\begin{array}{c|c|cccc} & & 0 & 1+v & 3+3v & 3+3v \\ & & 1+v & 1+v & 1+v & 0 \\ 2E_{i,j} & I_4 & 1+v & 0 & 3+3v & 1+v \\ & & 3+3v & 1+v & 0 & 1+v \\ \hline v_1 & v_2 & 0 & 1+2v & 3+2v & 3+2v \end{array}\right),$$

where $2E_\ell = \begin{cases} 2 & \text{if } i = j = 1, \\ 0 & \text{otherwise,} \end{cases}$ $v_1 = (1+3v, 1, 1, 1)$, and $v_2 = (3+v, 0, 0, 0)$; here, set $\ell = 1$ and $u_i = 1 + 3v$ $(1 \le i \le 4)$ in Theorem 1. Then the matrix $\tilde{M}$ generates a free self-orthogonal code of length 12 over R with free rank 5.

We recall the *Euclidean weight* wt_E (resp. *Lee weight* wt_L) of elements in $\mathbb{Z}_4$ is defined as

$$wt_E(0) = 0, \; wt_E(1) = wt_E(3) = 1, \text{ and } wt_E(2) = 4,$$

$$(\text{resp. } wt_L(0) = 0, \; wt_L(1) = wt_L(3) = 1, \text{ and } wt_L(2) = 2).$$

In this paper, we define the Euclidean weight of an element in R as follows.

Definition 2. For an element $\alpha = a + bv$ in R $(a, b \in \mathbb{Z}_4)$, the Euclidean weight $\hat{w}t_E(\alpha)$ of α is

$$\hat{w}t_E(\alpha) = wt_E(b) + wt_E(a+b),$$

where wt_E is the Euclidean weight in $\mathbb{Z}_4$. The Euclidean weight $\hat{w}t_E(\mathbf{u})$ of a vector $\mathbf{u} = (\mathbf{u_1}, \ldots, \mathbf{u_n})$ in R^n is equal to $\sum_{i=1}^{n} \hat{w}t_E(\mathbf{u_i})$.

Similarly, the Lee weight $\hat{w}t_L(\alpha)$ of α in R can be defined as $\hat{w}t_L(\alpha) = wt_L(b) + wt_L(a+b)$, where wt_L is the Lee weight in $\mathbb{Z}_4$ (see [4]).

In the next proposition, we introduce a Gray map ϕ from R^n to $\mathbb{Z}_4^{2n}$. This map ϕ preserves the Lee weight and orthogonality by [4].

Proposition 2. *Let ϕ be a map from R^n to $\mathbb{Z}_4^{2n}$ as follows:*

$$\phi : \begin{array}{ccc} R^n & \longrightarrow & \mathbb{Z}_4^{2n} \\ (a_1 + b_1 v, \ldots, a_n + b_n v) & \longmapsto & (b_1, a_1 + b_1, \ldots, b_n, a_n + b_n), \end{array}$$

where $u = (a_1 + b_1 v, \ldots, a_n + b_n v)$ and $a_i, b_i \in \mathbb{Z}_4$ for $1 \le i \le n$. This map is a Gray map which preserves the Euclidean weight and the Lee weight as $wt_E(\phi(u)) = \hat{w}t_E(u)$ and $wt_L(\phi(u)) = \hat{w}t_L(u)$. Furthermore, the map ϕ also preserves orthogonality.

For a self-dual code C over R, the code C is called *Type II code* if the Euclidean weight of every codeword is divisible by 8. If not, the code C is called a *Type I code*.

In Theorem 3, we present another construction method for (self-orthogonal, self-dual, or Type II) codes over R via (self-orthogonal, self-dual, or Type II) codes over $\mathbb{Z}_4$.

Theorem 3. *Let M be a $k_1 \times k_2$-matrix over $\mathbb{Z}_4$ with an i-th row vector r_i $(1 \le i \le k_1)$. Let $G = (I_{k_1} \mid M)$ be a generator matrix for a free (self-orthogonal or self-dual) code of length $k_1 + k_2$ over $\mathbb{Z}_4$. Then for every unit $u_i \in R$,*

$$\hat{G} = (I_{k_1} \mid \hat{M}) = \left(\begin{array}{c|c} & u_1 r_1 \\ I_{k_1} & \vdots \\ & u_{k_1} r_{k_1} \end{array} \right)$$

generates a free (self-orthogonal, self-dual) code of length $k_1 + k_2$ over R for $1 \le i \le k_1$. Especially, if G generates a free Type II code over $\mathbb{Z}_4$, then $\hat{G}$ generates a free Type II code over R, where $u_i = 1 + 3v$ or $3 + v$ in R $(1 \le i \le k_1)$.

Proof. Considering the lifting method from $\mathbb{Z}_4$ to R, the matrix G generates a free (self-orthogonal or self-dual) code over R; it means that all the elements of G have the form $a + bv$ with $b = 0$. Then the matrix $\hat{G}$ generates a free (self-orthogonal or self-dual) code over R; the orthogonality is preserving since u_i is a unit in R for all $1 \le i \le k_1$. We note that $u_i^2 = 1$ for any unit $u_i \in R$ $(1 \le i \le k_1)$ as we mentioned in Section 2. The result follows.
Especially, for an element α and a unit u in R,

$$\hat{w}t_E(\alpha) = \hat{w}t_E(u\alpha) \text{ if and only if } u = 1 + 3v \text{ or } 3 + v;$$

we can prove this by simple calculation. Notably, we say that Type II code also can be obtained in this theorem. □

The construction method in Theorem 3 is simple, but their Gray images give very meaningful database for linear codes over $\mathbb{Z}_4$ (see Section 4).

Remark 1. We use the same notation as Theorem 3. Let $\tilde{M}$ be a matrix such that $u_i = 1$ for all i in $\hat{M}$; it means that the matrix M is regarded as matrix over R, namely $\tilde{M}$. By considering the Gray map ϕ for $\hat{M}$ and $\tilde{M}$, we have two $\mathbb{Z}_4$ codes $C_{\phi(\hat{G})}$ and $C_{\phi(\tilde{G})}$ generated by the matrices $\phi(\hat{G}) = (I_{k_1} \mid \phi(\hat{M}))$ and $\phi(\tilde{G}) = (I_{k_1} \mid \phi(\tilde{M}))$, respectively. In Proposition 3, we compare the minimum weights for the codes $C_{\phi(\hat{G})}$ and $C_{\phi(\tilde{G})}$ with respect to Lee and Euclidean weights. Taking a similar point of view, in Proposition 3, we focus on the codes $C_{\hat{G}}$ and $C_{\tilde{G}}$ over R which are generated by the matrices $\hat{G}$ and G, respectively.

Proposition 3. *We use the same notation as Theorem 3 and Remark 1.*

(i) *Let $C_{\phi(\hat{G})}$ (resp. $C_{\phi(\tilde{G})}$) be a code over $\mathbb{Z}_4$ generated by the matrix $\phi(\hat{G})$ (resp. $\phi(\tilde{G})$). Then*

$$wt_E(C_{\phi(\hat{G})}) \geq wt_E(C_{\phi(\tilde{G})}) \text{ and } wt_L(C_{\phi(\hat{G})}) \geq wt_L(C_{\phi(\tilde{G})}).$$

(ii) *Let $C_{\hat{G}}$ (resp. $C_{\tilde{G}}$) be a code over R generated by the matrix $\hat{G}$ (resp. $\tilde{G}$). Then*

$$\hat{w}t_E(C_{\hat{G}}) \geq \hat{w}t_E(C_{\tilde{G}}) \text{ and } \hat{w}t_L(C_{\hat{G}}) \geq \hat{w}t_L(C_{\tilde{G}}).$$

Proof. Let $\alpha = a + bv$ be an element in R, where $a, b \in \mathbb{Z}_4$. Set $b = 0$ since all the elements of $\tilde{M}$ have the form $a + bv$ with $b = 0$. Then, for a unit u in R, we get that $wt_E(\phi(u\alpha)) \geq wt_E(\phi(\alpha))$ by considering the Gray map ϕ; for any unit in $\{1, 3, 1+v, 3+v, 1+2v, 3+2v, 1+3v, 3+3v\}$ and arbitrary element α, we can check that the inequality is true through simple calculations. Hence, (i) is proved. Moreover, $wt_E(C_{\phi(\hat{G})}) = \hat{w}t_E(C_{\hat{G}})$ and $wt_E(C_{\phi(\tilde{G})}) = \hat{w}t_E(C_{\tilde{G}})$ since the Gray map ϕ preserves the Euclidean weight. For the Lee weight, we can prove it similarly. Thus (ii) follows. □

We close this section with some examples for Theorem 3 and Proposition 3.

Example 2. Let $(I_4 \mid M)$ be a 4×8-matrix which generates a free self-dual code C of length 8 over $\mathbb{Z}_4$, where

$$M = \begin{pmatrix} 0 & 1 & 3 & 3 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 1 & 3 \\ 1 & 3 & 0 & 3 \end{pmatrix};$$

the minimum Lee weight of C is 4 and the minimum Euclidean weight of C is 4.

(i) Set $u_i = 1 + 3v$ $(1 \leq i \leq 4)$ in Theorem 3. Then $(I_4 \mid (1 + 3v)M)$ generates a free self-dual code of length 8 over R with the minimum Lee weight 4 and the minimum Euclidean weight 4.

(ii) Set $u_1 = u_2 = 1 + 2v$ and $u_3 = u_4 = 3 + 2v$ in Theorem 3. Then the matrix

$$\left(\begin{array}{c|cccc} & 0 & 1+2v & 3+2v & 3+2v \\ & 1+2v & 1+2v & 1+2v & 0 \\ I_4 & 1+2v & 0 & 3+2v & 1+2v \\ & 3+2v & 1+2v & 0 & 1+2v \end{array}\right)$$

generates a free self-dual code of length 8 over R with the minimum Lee weight 6 and the minimum Euclidean weight 8.

By (i) and (ii), Proposition 3 can be checked.

Example 3. (i) Let

$$M = \left(\begin{array}{c|ccccc} & 0 & 3 & 3 & 3 & 2 \\ & 3 & 0 & 1 & 3 & 2 \\ I_4 & 1 & 1 & 0 & 3 & 2 \\ & 2 & 2 & 1 & 3 & 1 \end{array}\right)$$

be a 4×9-matrix which generates a free code C of length 9 over $\mathbb{Z}_4$ with the minimum Lee weight 6 and the minimum Euclidean weight 6. By Theorem 3, we obtain the following matrix

$$\hat{M} := (I_4|M_s) := \left(\begin{array}{c|ccccc} & 0 & 3+3v & 3+3v & 3+3v & 2+2v \\ & 3+3v & 0 & 1+v & 3+3v & 2+2v \\ I_4 & 1+v & 1+v & 0 & 3+3v & 2+2v \\ & 2+2v & 2+2v & 1+v & 3+3v & 1+v \end{array}\right).$$

The matrix $\hat{M}$ generates a free code of length 9 over R with the minimum Lee weight 8 and the minimum Euclidean weight 12; we consider the Gray map's image $\phi(M_s)$ of M_s

$$\phi(M_s) = \begin{pmatrix} 0 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 2 & 0 \\ 3 & 2 & 0 & 0 & 1 & 2 & 3 & 2 & 2 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 3 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix}. \tag{3}$$

Then the matrix $(I_4|\phi(M_s))$ gives a linear code of length 14 over $\mathbb{Z}_4$ with the minimum Lee weight 8 and the minimum Euclidean weight 12.

(ii) Now, we will find a new code over $\mathbb{Z}_4$ by using our construction method in Theorem 3. Set the matrix M_s as

$$\begin{pmatrix} 0 & 0 & 2 & 3 & 2 & 1 & 2 & 1 \\ 2 & 3 & 2 & 3 & 2 & 3 & 0 & 0 \\ 2 & 3 & 0 & 0 & 2 & 1 & 2 & 3 \\ 2 & 1 & 2 & 3 & 0 & 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \tag{4}$$

over $\mathbb{Z}_4$ (the matrix $(I_4 \mid M_s)$ gives a self-orthogonal code of length 12 over $\mathbb{Z}_4$ with the minimum Lee weight 6). By using the Gray map ϕ, we obtain the following matrix $\hat{M_s}$

$$\phi\begin{pmatrix} (1+v)r_1 \\ (3+v)r_2 \\ (3+3v)r_3 \\ (1+v)r_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 3 & 2 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 2 \\ 2 & 0 & 3 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 2 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 2 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 2 \end{pmatrix}. \tag{5}$$

The matrix $(I_4 \mid \hat{M}_s)$ generates a self-orthogonal code of length 20 over $\mathbb{Z}_4$ with the minimum Lee weight 8; it is one of the meaningful results because this is a new code over $\mathbb{Z}_4$ by [7].

4. New optimal codes over $\mathbb{Z}_4$ from codes over R

In this section, we give a method for finding codes over $\mathbb{Z}_4$ by using codes over R and the Gray map ϕ. From this method, we can find many new optimal codes over $\mathbb{Z}_4$. We recall that a code is optimal if it has the highest minimal weight of any linear code of that length. In contrast, a code to be *extremal* if it meets the applicable bounds. It means that if a code is extremal, then it is optimal; the reverse is not true (see [17]).

The following theorem is a construction method for codes over $\mathbb{Z}_4$ via Gray map's images of codes over R.

Theorem 4. *Let $(I_k \mid \tilde{M}_i)$ be a generator matrix for a free code of length $k + \ddot{m}_i$ over R constructed by Theorems 1 or 3 with $1 \leq i \leq n_1$. Let $(I_k \mid M_j)$ be a generator matrix for an arbitrary free code of length $k + m_j$ over $\mathbb{Z}_4$ with $1 \leq j \leq n_2$. We consider the following matrix over $\mathbb{Z}_4$:*

$$\hat{M} := (I_k \mid \phi(\tilde{M}_1) \mid \cdots \mid \phi(\tilde{M}_{n_1}) \mid M_1 \mid \cdots \mid M_{n_2}).$$

Then the matrix $\hat{M}$ generates a free code of length $k + 2\sum_{i=1}^{n_1} \ddot{m}_i + \sum_{j=1}^{n_2} m_j$ over $\mathbb{Z}_4$ with free rank k.

Table 1. New free (optimal) codes over $\mathbb{Z}_4$ with free rank 4

length	generator matrix	min. Lee weight	min. Euclidean weight	L -opt	E -opt
66	$(D_{58} \mid M_8)$	46*	73*	○	○
68	$(D_{60} \mid M_8)$	50*	65*	○	○
70	$(D_{62} \mid M_8)$	50*	78*	○	○
70	$(D_{60} \mid M_{10})$	50*	67*	○	×
72	$(D_{64} \mid M_8)$	50*	73*	○	×
72	$(D_{62} \mid M_{10})$	50*	74*	○	○
74	$(D_{58} \mid M_8 \mid M_8)$	48*	78*	×	○
74	$(D_{64} \mid M_{10})$	50*	75*	○	×
76	$(D_{60} \mid M_8 \mid M_8)$	52*	78*	○	×
76	$(D_{58} \mid M_8 \mid M_{10})$	48*	88*	×	○
78	$(D_{62} \mid M_8 \mid M_8)$	52*	88*	○	○
78	$(D_{60} \mid M_8 \mid M_{10})$	52*	80*	○	×
80	$(D_{64} \mid M_8 \mid M_8)$	52*	86*	○	×
80	$(D_{62} \mid M_8 \mid M_{10})$	52*	88*	○	○
82	$(D_{74} \mid M_8)$	52*	70*	○	×

82	$(D_{64} \mid M_8 \mid M_{10})$	52*	88*	◯	◯
86	$(D_{78} \mid M_8)$	52	86*	×	◯
86	$(D_{76} \mid M_{10})$	54*	78*	◯	×
88	$(D_{78} \mid M_{10})$	52*	86*	×	◯
92	$(D_{82} \mid M_{10})$	58*	86*	◯	◯
92	$(D_{74} \mid D_8 \mid M_{10})$	56	84*	◯	×
94	$(D_{76} \mid D_8 \mid M_{10})$	60*	92*	◯	◯
96	$(D_{80} \mid M_{16})$	56*	85*	×	×
96	$(D_{78} \mid D_8 \mid M_{10})$	56*	100*	×	◯
98	$(D_{82} \mid M_{16})$	59*	87*	×	×
98	$(D_{88} \mid M_{10})$	60	94*	◯	◯
98	$(D_{80} \mid M_8 \mid M_{10})$	60	92	◯	×
100	$(D_{84} \mid M_{16})$	59*	86*	×	×
100	$(D_{82} \mid M_8 \mid M_{10})$	64*	100*	◯	◯
102	$(D_{84} \mid M_8 \mid M_{10})$	64*	92*	◯	◯
104	$(D_{86} \mid M_8 \mid M_{10})$	68*	100*	◯	◯
106	$(D_{88} \mid M_8 \mid M_{10})$	64	108*	◯	◯

It is possible to get many free new and optimal codes by using Theorem 4; our results are compared with database in [7]. In Table 1, we let M_8 (resp. M_{10}, M_{16}) be a matrix which is obtained in (4) (resp. (3), (5)). In Table 2, the matrix $M_{10,3}$ is a 3×10-matrix over $\mathbb{Z}_4$ obtained by Theorem 3;

$$M_{10,3} = \phi \begin{pmatrix} 0 & 3+2v & 3+2v & 3+2v & 2 \\ 2+3v & 0 & 2+v & 2+3v & 2v \\ 1+2v & 1+2v & 0 & 3+2v & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 & 1 & 3 & 3 & 1 & 2 & 2 \\ 2 & 3 & 2 & 3 & 0 & 0 & 2 & 1 & 0 & 2 \end{pmatrix}.$$

Let D_n be a generator matrix of a linear code of length n over $\mathbb{Z}_4$ given in [7]. The *L-opt* (resp. *E-opt*) means that a code is an optimal code with respect to the minimum Lee weight (resp. Euclidean weight). The $*$-marked weight presents that the weight is new in each case.

TABLE 2. New free (optimal) codes over $\mathbb{Z}_4$ with free rank 3

length	generator matrix	min. Lee weight	min. Euclidean weight	L -opt	E -opt
69	$(D_{59} \mid M_{10,3})$	54*	76*	◯	◯
71	$(D_{61} \mid M_{10,3})$	58*	84*	◯	◯

73	$(D_{63} \mid M_{10,3})$	60^*	84^*	◯	◯
75	$(D_{65} \mid M_{10,3})$	50	66^*	◯	◯
77	$(D_{67} \mid M_{10,3})$	50	72^*	◯	◯
79	$(D_{69} \mid M_{10,3})$	54^*	70^*	◯	◯
81	$(D_{71} \mid M_{10,3})$	52	73^*	×	◯
83	$(D_{73} \mid M_{10,3})$	54^*	76^*	◯	◯
85	$(D_{75} \mid M_{10,3})$	58^*	74^*	◯	◯
87	$(D_{77} \mid M_{10,3})$	56^*	77^*	×	◯
89	$(D_{79} \mid M_{10,3})$	58^*	80^*	◯	◯
91	$(D_{81} \mid M_{10,3})$	62^*	78^*	◯	◯
93	$(D_{83} \mid M_{10,3})$	60^*	81^*	×	◯
95	$(D_{85} \mid M_{10,3})$	62^*	84^*	◯	◯
97	$(D_{87} \mid M_{10,3})$	66^*	82^*	◯	◯
99	$(D_{89} \mid M_{10,3})$	64^*	85^*	×	◯
101	$(D_{91} \mid M_{10,3})$	66^*	88^*	◯	◯
103	$(D_{93} \mid M_{10,3})$	70^*	86^*	◯	◯
105	$(D_{95} \mid M_{10,3})$	70^*	86^*	◯	◯
107	$(D_{97} \mid M_{10,3})$	70^*	92^*	◯	◯
109	$(D_{99} \mid M_{10,3})$	74^*	90^*	◯	◯
111	$(D_{101} \mid M_{10,3})$	72^*	93^*	×	◯
113	$(D_{103} \mid M_{10,3})$	74	96^*	◯	◯
115	$(D_{105} \mid M_{10,3})$	78^*	94^*	◯	◯
117	$(D_{107} \mid M_{10,3})$	76^*	97^*	×	◯
119	$(D_{109} \mid M_{10,3})$	78^*	100^*	◯	◯
121	$(D_{111} \mid M_{10,3})$	82^*	98^*	◯	◯
123	$(D_{113} \mid M_{10,3})$	80^*	101^*	×	◯

Remark 2. Let D_{62} be a generator matrix for a linear code C over $\mathbb{Z}_4$ of length 62 and free rank 4 with the minimum Lee weight 38; the code C is not an optimal code. By using Theorem 4 and the matrix M_{10}, we consider the matrix $(D_{62} \mid M_{10})$ over $\mathbb{Z}_4$; this matrix generates a linear code $\hat{C}$ over $\mathbb{Z}_4$ of length 72, where the free rank is 4 and the minimum Lee weight is 44. This code $\hat{C}$ is an optimal code over $\mathbb{Z}_4$ by [7]. This means that even though a code C is not optimal over $\mathbb{Z}_4$, a code $\hat{C}$ generated by the code C can be an optimal code over $\mathbb{Z}_4$ by using Theorem 4.

5. Conclusion

In this paper, we present new construction methods for self-orthogonal, self-dual, or Type II codes over $R = \mathbb{Z}_4[v]/\langle v^2 + 2v\rangle$. We find new optimal codes over $\mathbb{Z}_4$ by using the Gray map's images of the codes suggested by our methods over R. Except for the codes given in this paper, we can construct more new optimal codes over $\mathbb{Z}_4$. This is significant in this area since linear codes over $\mathbb{Z}_4$ can give many applications such as non-linear binary codes with many quantum codes. Later, our results for self-dual codes or Type II codes also can be used in number theory for finding invariants, new modular forms or Jacobi forms over certain number fields.

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