J. Korean Math. Soc. **59** (2022), No. 1, pp. 105–127
https://doi.org/10.4134/JKMS.j210201
pISSN: 0304-9914 / eISSN: 2234-3008

MAXIMAL INVARIANCE OF TOPOLOGICALLY ALMOST CONTINUOUS ITERATIVE DYNAMICS

Byungik Kahng

Abstract. It is known that the maximal invariant set of a continuous iterative dynamical system in a compact Hausdorff space is equal to the intersection of its forward image sets, which we will call the *first minimal image set*. In this article, we investigate the corresponding relation for a class of discontinuous self maps that are on the verge of continuity, or *topologically almost continuous endomorphisms*. We prove that the iterative dynamics of a topologically almost continuous endomorphisms yields a chain of minimal image sets that attains a unique transfinite *length*, which we call the *maximal invariance order*, as it stabilizes itself at the maximal invariant set. We prove the converse, too. Given ordinal number ξ, there exists a topologically almost continuous endomorphism f on a compact Hausdorff space X with the maximal invariance order ξ. We also discuss some further results regarding the maximal invariance order as more layers of topological restrictions are added.

1. Introduction

1.1. General overview

Let X be a nonempty set and $f : X \to X$. We say $S \subset X$ is *invariant* under f if $f(S) = S$. We define the *maximal invariant set* $\mathcal{M}(X)$ of $f : X \to X$ as the union of all invariant subsets of X, that is, $\mathcal{M}(X) = \bigcup\{S \subset X : f(S) = S\}$. Clearly, a maximal invariant set is indeed *maximal* in that it includes all the invariant sets. Also, one can easily prove that it is indeed invariant, or $f(\mathcal{M}(X)) = \mathcal{M}(X)$, and the following result is known [2].

Theorem 1.1. *If X is a compact Hausdorff space and $f : X \to X$ is continuous, then the maximal invariant set $\mathcal{M}(X)$ satisfies the following equality.*

$$\mathcal{M}(X) = \bigcap_{k=0}^{\infty} f^k(X). \tag{1.1}$$

Received March 24, 2021; Accepted August 19, 2021.
2010 *Mathematics Subject Classification.* Primary 37B99, 37E99, 03E10; Secondary 93C10, 93C65, 93B05.
Key words and phrases. Maximal invariant set, discontinuous dynamics, ordinal numbers, transfinite induction and recursion.

The goal of the present research is to investigate what happens when the continuity condition in Theorem 1.1 is relaxed. The idea behind this generalization is inspired by two tracks of research that are apparently unrelated to each other, at least in a superficial point of view. One is the reachability/controllability problem for nonlinear control dynamical systems with singular disturbance, where sudden large disturbance displaces the continuity condition of Theorem 1.1 (Subsection 2.1). The other is the invariant decomposition and ergodicity problem for the iterative dynamics of piecewise continuous and/or piecewise isometric maps, which was, in turn, inspired by digital signal processing in electric engineering and kicked oscillators in nonlinear physics (Subsection 2.3).

The conclusion of the present research, which began from the applications to automatic control, digital signal processing and nonlinear physics, came down to transfinite set theory and logic. Neither of these is commonly associated to applied dynamical systems, even less to physics and engineering. As we will see in Subsection 2.2, however, this seemingly far-fetched connection not only exists, but also useful in terms of applications in engineering. The algorithmic approach of the present research was discussed in [29–32, 34]. See also, [25–27, 35] for the preliminary setup that led to the transfinite controllability problem.

The focal point of this paper is a topological invariant, which we call the *maximal invariance order*. The precise definition will be done in Theorem 5.1. Here, we begin with some background information.

1.2. Background information

As we pointed out in the previous subsection, the present research begins from reviewing Theorem 1.1 with more scrutiny. Let us call the intersection in the right hand side of the equality (1.1) of Theorem 1.1, the *first minimal image set* of $f : X \to X$, and denote it X_1^+. That is, $X_1^+ = \bigcap_{k=0}^{\infty} f^k(X)$. It is *minimal* in that the image set gets smaller each time f is applied, as we will see in the following chain inequality.

$$X \supseteq f(X) \supseteq f^2(X) \supseteq \cdots \supseteq X_1^+ \supseteq \mathcal{M}(X). \tag{1.2}$$

It is easy to prove that the chain inequality (1.2) always holds. If the conclusion of Theorem 1.1 holds at the last semi-inclusion of (1.2), we get the following stronger result, which serves a fundamental role in establishing some of the most important problems in nonlinear control and automation theory, both in analytical and computational. For more detail, see the survey articles, [9, 37] and the references therein.

$$X \supseteq f(X) \supseteq f^2(X) \supseteq \cdots \supseteq X_1^+ = \mathcal{M}(X). \tag{1.3}$$

The last equality of (1.3) is a result of Theorem 1.1, and thus subordinate to the continuity condition. In the presence of the discontinuity, however, the last equality does not hold in general [35, 46]. Consequently, the finite-step approximate control problem of the maximal invariant set fails.

The failure was amended (under a suitable condition) by [30–32] as follows. Because the first minimal image set is quasi-invariant, or $X_1^+ \supseteq f(X_1^+)$, we can repeat the iteration process identical to that of the chain inequality (1.2) with respect to the iterative dynamics of $f : X_1^+ \to X_1^+$. This process leads up to $X_2^+ = \bigcap_{k=0}^{\infty} f^k(X_1^+)$, which we call the *second minimal image set*. Repeating this process, we get the chain of minimal image sets (X_n^+), defined by $X_n^+ = \bigcap_{k=0}^{\infty} f^k(X_{n-1}^+)$ and $X_0^+ = X$, which descends down to the maximal invariant set as follows.

$$X_0^+ \supseteq X_1^+ \supseteq X_2^+ \supseteq \cdots \supseteq \mathcal{M}(X). \tag{1.4}$$

The descending chain inequality (1.4) brings about the following question. *How far do we have to go until we reach $\mathcal{M}(X)$?* The answer to this question is the main result of this paper. It can be quantified by a topological invariant, which we will name, the *maximal invariant order* (Theorem 5.1). Roughly speaking, the maximal invariant order corresponds to the transfinite length of the descending chain (1.4). This provides the theoretical foundation of the algorithmic research [29,31,34] and its optimization [31,33], consequently resolving the transfinite controllability problem opened up by [30, 32].

1.3. Questions and Main Theorems

When X is compact and f is continuous, Theorem 1.1 applies, and thus the chain of minimal image sets stops at the very first step. That is,

$$X = X_0^+ \supseteq X_1^+ = \mathcal{M}(X). \tag{1.5}$$

Moreover, it is known that the equality (1.5) also holds when f is finite-to-one [35,46,47]. This fact had been exploited successfully in nonlinear physics [8,13], as we will describe in more detail in Subsection 2.3.

In general, however, the maximal invariance order can be bigger. Indeed, in [46], Mendes provided an example of 1-dimensional discontinuous iterative dynamical system that satisfies the following descending chain condition.

$$X_0^+ \supsetneq X_1^+ \supsetneq X_2^+ \supsetneq \cdots \supsetneq X_\infty^+ = \mathcal{M}(X), \qquad X_\infty^+ = \bigcap_{n=0}^{\infty} X_n^+. \tag{1.6}$$

He then, goes ahead with the following challenge, which we will resolve through this paper.

Question 1 (Mendes, 2001)**.** Let X be a nonempty set and $f : X \to X$ be an endomorphism. Do we always have $X_\infty^+ = \mathcal{M}(X)$? If not, under what condition do we have $X_\infty^+ = \mathcal{M}(X)$?

Question 1 was raised by Mendes initially in [46]. Later, he conjectured that the identity $X_\infty^+ = \mathcal{M}(X)$ to be false in general [47]. We will confirm his conjecture by proving that a chain of the minimal image sets can attain an arbitrary *length*.

Main Theorem 1 (Theorem 5.1)**.** *Let X be a nonempty set and $f : X \to X$. Then, there is a unique ordinal number ξ such that*

$$X_0^+ \supsetneq X_1^+ \supsetneq \cdots \supsetneq X_\xi^+ = \mathcal{M}(X).$$

We call such ξ, the maximal invariance order of $f : X \to X$.

We prove the converse as well, that is, given ordinal number ξ, we can find an iterative dynamical system, $f : X \to X$ with the maximal invariance order ξ. In fact, we can do much better (Main Theorem 2). We need, however, the following preparatory step.

Definition 1.1. Let X be a compact Hausdorff space and let $f : X \to X$. We say f is a *topologically almost continuous endomorphism* if it is almost continuous with respect to the non-singular component of every strictly positive Borel measure on X. That is, for any a strictly positive measure μ on X such that

$$\mu = \mu_+ + \mu_0, \qquad supp(\mu_+) = X \quad \text{and} \quad \mu_+[supp(\mu_0)] = 0, \tag{1.7}$$

f must be almost continuous with respect to μ_+, or,

$$\mu_+\{x \in X : f \text{ is discontinuous at } x\} = 0.$$

The purpose of the "the non-singular component" condition, paraphrased by (1.7), is to prevent a *cheating.* That is, one can artificially create a strictly positive measure from another simply by adding a singular measure such as a point measure or a lower dimensional measure. The decomposition in (1.7) eliminates all effects of such *cheating* by trimming out the artificially added singular component.

We are now ready to state the stronger version of the converse of Main Theorem 1.

Main Theorem 2 (Theorem 5.3)**.** *Given ordinal number ξ, we can construct a compact Hausdorff space X and a topologically almost continuous endomorphism $f : X \to X$ with the maximal invariance order ξ.*

Note that Main Theorem 2 includes a condition far stronger than just discontinuity. It tells us that even a discontinuous map that is *on the verge of continuity* can still possess such a rich and complex chain structure of minimal image sets. This observation brings us to another question, a question that concerns the effect of the *level of discontinuity* on the maximal invariance order. One such *level* is the class of topologically almost continuous endomorphisms, as defined and mentioned in Definition 1.1 and in Main Theorem 2. The other *levels of discontinuity* this paper considers are as follows.

Definition 1.2. Let X be a compact Hausdorff space and $f : X \to X$. We define the *continuity partition* of $f : X \to X$ as the collection $\{P_i \subset X : i \in I\}$ of mutually disjoint connected sets that partitions the whole space, and that each P_i is a maximal connected set in which $f : X \to X$ is continuous with respect to the set-inclusion. We say f is *piecewise continuous with respect to a measure* μ in X, if there is a finite subset J of the index set I such that $\bigcup_{i\in(I\setminus J)} P_i$ includes the discontinuity set of f and $\mu(\bigcup_{i\in(I\setminus J)} P_i) = 0$. If f is piecewise continuous with respect to the non-singular component of every strictly positive Borel measure of X, as specified in (1.7), we call it a *topologically piecewise continuous endomorphism.* In particular, if its continuity partition is finite, we call it *piecewise continuous endomorphism with a finite partition*, or simply *finitely piecewise continuous.* If the continuity partition is countable, we say f is *countably piecewise continuous.*

Main Theorem 2 applies to the topologically almost continuous endomorphisms. Now, what will happen if we strengthen it to the topologically piecewise continuous endomorphisms, or still further to the finitely piecewise continuous endomorphisms? These are important questions to raise because most of the discontinuous dynamical systems that arise in control theory, signal processing and non-linear oscillation problems fall in these categories (plus some additional restrictions of topological and/or analytical nature).

It is known that the equality (1.1) holds for finitely piecewise isometries [8,13,47]. On the other hand, Mendes's 1-dimensional example that satisfies the descending chain condition (1.6) is based upon a countably piecewise continuous endomorphism [35, 46, 47]. Although the examples from the above references are far from exhaustive, they nonetheless present sufficient evidence to suspect that different conditions would generate different outcomes. Let us single out one of the possible "different outcomes" as follows.

Question 2. Let X be a compact subset of $\mathbb{R}^k$ and $f : X \to X$ be a piecewise continuous map with a finite partition. Do we always have $X_1^+ = \mathcal{M}(X)$? Or $X_\omega^+ = \mathcal{M}(X)$? Or possibly even bigger maximal invariance order?

In an intermediate step toward the proof of Main Theorem 1, we show that it is possible to construct a piecewise continuous map $f : X \to X$ with a finite partition in a compact set $X \subset \mathbb{R}^2$ such that $X_1^+ \supsetneqq \mathcal{M}(X)$ (Example 3.2, Example 3.3, Example 3.4). Whether it is possible to achieve a result as strong as Main Theorem 1 with piecewise continuous maps with finite partitions is by no means certain. The following is the strongest result we have thus far.

Main Theorem 3 (Example 4.1)**.** *Given $n \in \mathbb{N}$, there are a compact set $X \subset \mathbb{R}^k$ and a piecewise continuous endomorphism in $f : X \to X$ with a finite partition such that*

$$X = X_0^+ \supsetneqq X_1^+ \supsetneqq \cdots \supsetneqq X_{n-1}^+ \supsetneqq X_n^+ = \mathcal{M}(X). \tag{1.8}$$

The quest toward strengthening Main Theorem 3 and finding more complete answer to Question 2 is on-going, so are the corresponding problems regarding the other levels of discontinuity mentioned in Definition 1.2.

Taking the opposite viewpoint, we obtain another interesting question about the effect of different levels of discontinuity. Roughly speaking, the following question can be considered as a partial converse of Question 2.

Question 3. Let X be a compact Hausdorff space and let $f : X \to X$ be a topologically almost continuous endomorphism. Under what additional conditions on f (and/or on X) do we need to impose in order to get a finite (or countable) maximal invariance order?

In the application to control and automation theory, the maximal invariant set is controllable (reachable) in the traditional sense, only when the maximal invariance order ξ is equal to 1. That is, when the descending chain-inequality (1.3) holds, and thus the maximal invariant set $\mathcal{M}(X)$ is controllable in the first countably infinite steps. Through a series of diagonalization process, the controllability can be extended to the cases, $\xi = 2, 3, 4, \ldots ; \omega$, as discussed in [30–32]. Question 3 is asking when this would happen.

The piecewise continuity with finite partitions mentioned in Question 2 is a reasonable and practical candidate for Question 3 as well, but the research toward this goal is yet on-going. At this moment, we must be contend with the existence and the uniqueness of the maximal invariance order (Main Theorem 1 and Main Theorem 2), which which opened the door for the next series of research such as Question 2, Question 3 and possibly others.

Now what if the maximal invariance order ξ exceeds the first countable ordinal ω? Or worse, what is ξ is an uncountable ordinal? Particularly in the latter case, the maximal invariant set can never be countable step controllable, and thus no finite step approximate control problem is well posed. The partial answer to Question 3 can be used to avoid this situation.

The case with an uncountable maximal invariance order may best be avoided in automatic control theory, but it is far from trivial in other applications. The strange attractors that are sustained by continual feeding of the external orbits, not from within, are abundant in nature.

Establishing the existence and the uniqueness of the maximal invariance order opens the door for the next series of research. In control and automation theory, for instance, we can ask a question, "Under what conditions would the descending chain inequality (1.4) end in a finite or countable number of steps?" And then, we can consider, "When the maximal invariance order is finite or countable, how can we devise a computable descending chain of sets that converge to the maximal invariant set?" The second question is often called a controllability/reachability problem of the maximal invariant set. Once the controllability is established, we can consider an finite step approximate control problem. None of these is meaningful unless the existence and the uniqueness of the maximal invariance order is established (Theorem 5.1). Indeed, the author produced some results already regarding these questions [30–32, 35], citing this paper's conclusion as an intermediate step.

"What happens when the maximal invariance order is uncountable?" is another interesting question to ask, in non-linear physics. In this case, the steady state sets (based on the starting set) and the maximal invariant set (or the local maximal invariant sets) will mean two different sets in the phase space. The feasibility of the physical conditions that can lead to this situation is unclear at this moment

2. Motivation

2.1. Control and automation theory

The particular topic we deal with in this paper is inspired in part by the *reachability/controllability problem* and the *approximate control problem* in control and automation theory. Roughly speaking, a classical time-invariant nonlinear discrete-time control dynamical system can be modeled as the iterative dynamical system of a pair of maps,

$$\begin{cases} \phi : X \times U \to X \quad \text{and} \quad \psi : X \to U, \quad \text{where} \\ \phi : (x_k, u_k) \mapsto x_{k+1} \quad \text{and} \quad \psi : x_k \mapsto u_k. \end{cases} \tag{2.1}$$

Here, x_k stands for the k-th *state*, u_k is called the *control variable*, and ψ is called the *feedback control law* [37]. One important research topic in control and automation theory is how one attains and calculates the maximal invariant sets through the iterative dynamics of (ϕ, ψ) The importance of this problem is mentioned in variety of sources. See, for instance, the survey article [9] and the references therein.

Designing and investigating control systems and their feedback controllers are not the main concerns of this paper. Therefore, we reduce the classical discrete-time control dynamics model (ϕ, ψ) through $f(x) = \phi(x, \psi(x))$, yielding an iterative dynamical system of one map, $f : X \to X$. In this case, a standard scheme can be summarized by the following chain of set inequalities ending with an approximate equality [30].

$$X \supseteq f(X) \supseteq f^2(X) \supseteq \cdots \supseteq f^N(X) \approx \mathcal{M}(X). \tag{2.2}$$

When the control dynamics model is simple enough to be reduced to an iterative dynamics of one continuous map on a compact phase space, Theorem 1.1 yields, the following chain of set inequalities ending with an equality.

$$X \supseteq f(X) \supseteq \cdots \supseteq f^N(X) \supseteq \cdots \supseteq X_1^+ = \mathcal{M}(X). \tag{2.3}$$

The chain-inequality (2.3) is often referred as the *reachability* or the *controllability problem* of the maximal invariant set. Comparing the two chain-inequalities (2.2) and (2.3), one can easily see that the reachability/controllability (2.3) is a fundamental problem that must be resolved before discussing an approximate control problem (2.2). When a control system is more complicated due to disturbance, however, the reduction to an iterative dynamics of a continuous map on a compact space is not always possible, and thus Theorem 1.1 and the chain-inequality (2.3) are of no use. The the reachability/controllability problems under such a difficulty are important topics in the current research of nonlinear control and automation theory. See, for instance, [3, 25, 26, 29–32, 35–37, 45, 49, 51–54] for more detail.

2.2. Motivation for transfiniteness

The failure of the chain-inequality (2.3) prompted the adjustment of the classical model (2.1) in various regards. Examples include, adding in disturbance variables [36, 54], making the states and control variables time-dependent [49], and making the maps multiple-valued [3, 26, 30, 32]. All these adjustments work well under small non-singular disturbance such as white noise, but in general under the presence of a sudden large disturbance that is significant enough to change the qualitative properties of the dynamics.

Indeed, [29–32] discuss the controllability problems of the locally maximal invariant set, whose corresponding chain inequalities go beyond the order of the first countably infinite ordinal number. In such a case, the traditional methods in engineering fail to apply. In response, [29–32] provided an alternative approach. First, set up the transfinite chain-inequality. Second, through a series of diagonalization method, find an alternative path of descending chain that lead to the maximal invariant set in the first countably infinite steps. Third, establish the approximate control algorithm based on the new countably infinite descending chain.

However, none of the algorithmic solutions discussed in the previous paragraph would make sense (albeit useful), without the mathematical foundation that assures the problem itself is well-posed. This is the main contribution of the present paper. Here, we establish the existence of the transfinite chain and the uniqueness of its *length* (maximal invariance order).

2.3. Piecewise continuous dynamics

Another inspiration of the present research came from the iterative dynamics of piecewise continuous and piecewise isometric maps. This topic, in turn, was inspired by digital signal processing in electric engineering [4, 6, 7, 12, 38, 39, 48] and kicked oscillators in nonlinear physics [5, 40, 42]. The particular class of problems that is most directly connected to our present research is the one related to the invariant decomposition of the phase space. The structure of the invariant sets is a fundamental problem one must study in detail in order to achieve deeper understanding of the dynamics within the invariant sets, such as, ergodicity [1,10,11,16,17,19,22–24,44], chaotic dynamics [22,23,27,28,40,42,44], periodic and aperiodic orbit structure [1,18,19,41–44,55], steady state behavior [8, 13–15, 50], and so on.

This approach complements the other motivation discussed in the previous subsections from the opposite direction, putting more emphasis into the singularity itself than its influence to the reachability/controllability problem. In particular, this paper was most directly inspired by [13–15], which used the countable reachability (2.3) to study the invariant measure and the Lyapunov stability of a class of piecewise isometric and piecewise continuous iterative dynamical systems in their steady state sets.

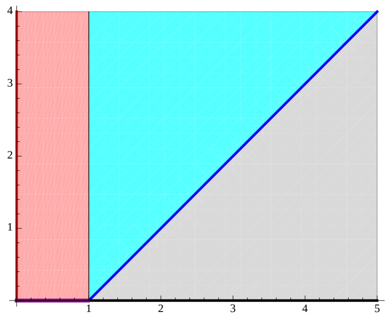

FIGURE 3.1. X.

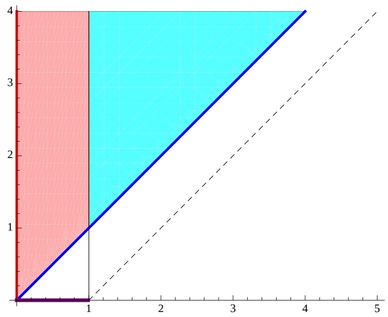

FIGURE 3.2. $f(X)$.

Indeed, this paper resolves an issue left behind by [13–15], proving that the maximal invariant set of a discontinuous iterative dynamical system is not necessarily countably reachable in general (Main Theorem 2). Consequently, our Main Theorems will necessitate some adjustment to the conventional way of studying the Lyapunov stability and the invariant measure theory on the steady state sets, when we go beyond piecewise isometries and more deeply into the discontinuous dynamics.

3. Constructing non-invariant first minimal image sets

Most of the technical content of this paper concerns the constructive proof of Main Theorem 1. The strategy of our choice is to begin with an example and then gradually refine it toward the proof of Main Theorem 1. In this section, we restrict ourselves only to the first minimal image sets.

3.1. The basic example: A piecewise affine map

We begin with the following elementary example.

Example 3.1. Let $X = [0,\infty)^2 \subset \mathbb{R}^2$, as depicted in Figure 3.1. Define $f : X \to X$ as follows.

$$f(x,y) = \begin{cases} (0,0), & \text{if } y < x-1 \text{ (gray region)}, \\ (x-1,y), & \text{if } 1 < x < y+1 \text{ (blue region)}, \\ (x,0), & \text{if } 0 \le x \le 1 \text{ and } y > 0 \text{ (red region)}, \\ (x/2,0), & \text{if } 0 \le x \le 1 \text{ and } y = 0 \text{ (purple line segment)}. \end{cases}$$

Then, the iterative dynamics of $f : X \to X$ has the property, $X_1^+ \supsetneq \mathcal{M}(X)$.

Proof. It is easy to prove that $X_1^+ = [0,1] \times \{0\}$ and $\mathcal{M}(X) = \{(0,0)\}$. Consequently, $X_1^+ \supsetneq \mathcal{M}(X)$. We leave the detail to the readers. □

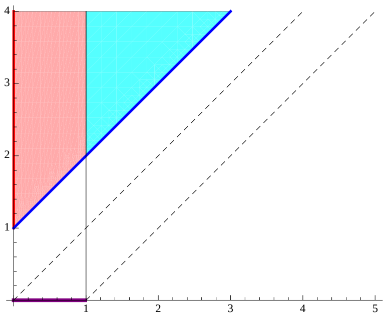

FIGURE 3.3. $f^2(X)$.

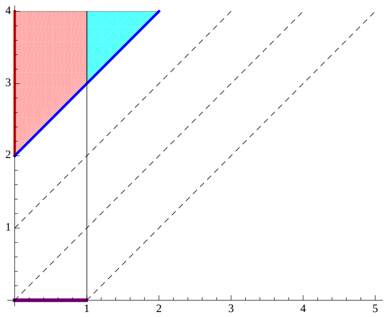

FIGURE 3.4. $f^3(X)$.

The first three image sets, $f(X)$, $f^2(X)$ and $f^3(X)$ are presented in Figure 3.2, Figure 3.3 and Figure 3.4, respectively. Note that each point in X_1^+ has backward orbits of arbitrary finite length, but only $(0, 0)$ has backward orbits of infinite length[1]. Note also that f is piecewise affine and it has a finite partition.

3.2. A piecewise continuous map on a compact space

Example 3.1 is somewhat incomplete in that X is unbounded. We can improve it with the following adjustment.

Example 3.2. Let $X = [0, 1]^2 \subset \mathbb{R}^2$. Let $h : [0, \infty)^2 \to [0, 1)^2$ be a homeomorphism, say,

$$h(x, y) = (1 - e^{-x},\ \ 1 - e^{-y}).$$

Let $f : [0, \infty)^2 \to [0, \infty)^2$ be the piecewise affine map in Example 3.1. Now, we define $g : X \to X$ as follows.

$$g(x, y) = \begin{cases} (0, 0), & \text{if } x = 1 \text{ or } y = 1, \\ h \circ f \circ h^{-1}(x, y), & \text{otherwise.} \end{cases}$$

Then, the iterative dynamics of $g : X \to X$ satisfies the property, $X_1^+ \supsetneq \mathcal{M}(X)$.

Proof. It is easy to see that $X_1^+ = [0, (e-1)/e] \times \{0\}$, the red line segment in the bottom of Figure 3.5 and Figure 3.6, and $\mathcal{M}(X) = \{(0, 0)\}$, the lower left hand side corner of Figure 3.5 and Figure 3.6. Clearly, therefore, $X_1^+ \supsetneq \mathcal{M}(X)$. $\square$

Figure 3.5 visualizes the domain X of Example 3.2. The red rectangular region, the blue triangular region and the white triangular region of Figure 3.5 are homeomorphic to the red, blue and white regions of Figure 3.1. g sends the top edge, the right hand side edge and the white triangular region to the lower left hand side vertex $(0, 0)$. On the colored regions, $g = h \circ f \circ h^{-1}$, so the dynamics of g follows from that of f.

[1]See [47] for the implications of the backward orbits to the maximal invariant set.

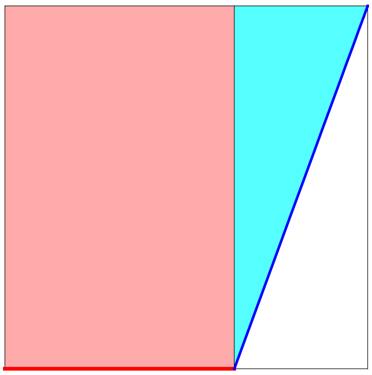

FIGURE 3.5. X.

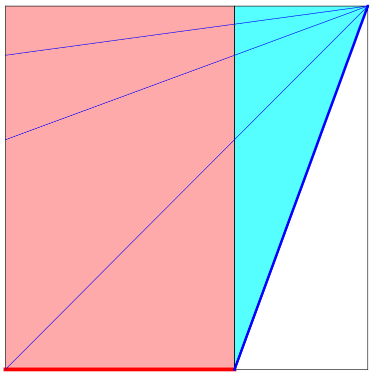

FIGURE 3.6. $\partial(g^k(X))$.

Note that X is compact and $g : X \to X$ is piecewise continuous with finite partition, but g is no longer piecewise affine. Indeed, one can see that the lower boundary edges of the first three image sets $g(X)$, $g^2(X)$ and $g^3(X)$ in Figure 3.6 (blue curves) are not linear.

3.3. The use of space filling curves

In this subsection, we improve Example 3.2 still further. Example 3.2 is still insufficient in our standard, because $\mathcal{M}(X)$ and X_1^+ are almost the same with respect to 2-dimensional Lesbesgue measure, or $\mu_2(X_1^+ \triangle \mathcal{M}(X)) = 0$. This is not much of an improvement over the equality (1.1) of Theorem 1.1.

Partly to resolve this issue, we construct a pair of examples of piecewise continuous maps with finite partitions in a compact subset of $\mathbb{R}^n$, whose first minimal image sets are polygons. Here is the first of the two.

Example 3.3. Let $X = [0,1]^2 \times \{0,1\} \subset \mathbb{R}^2$. Figure 3.7 and Figure 3.8 depict the top and the bottom layer, respectively. Let $g : [0,1]^2 \to [0,1]^2$ be the piecewise continuous map in Example 3.2. Finally, let $\phi : [0,(e-1)/e] \to [0,1]^2$ be a space-filling curve such that $\phi([0,(e-1)/e]) = [0,1]^2$. We define $f : X \to X$ as follows.

$$f(x,y,z) = \begin{cases} (x/2, y, 0), & \text{if } z = 0, \\ (\phi(x), 0), & \text{if } (x,y) \in [0,(e-1)/e] \times [0,1) \text{ and } z = 1, \\ (g(x,y), 1), & \text{otherwise.} \end{cases}$$

Then, $X_1^+ = [0,1]^2 \times \{0\}$, the bottom layer (the gray unit square in Figure 3.8), and $\mathcal{M}(X) = \{0\} \times [0,1] \times \{0\}$, the left hand side vertical edge of the bottom layer (the black line segment in Figure 3.8).

Proof. The dynamics in the bottom layer, $[0,1]^2 \times \{0\}$, is given by $(x,y,0) \mapsto (x/2, y, 0)$, as depicted by Figure 3.8. The dynamics in the top layer, $[0,1]^2 \times \{1\}$

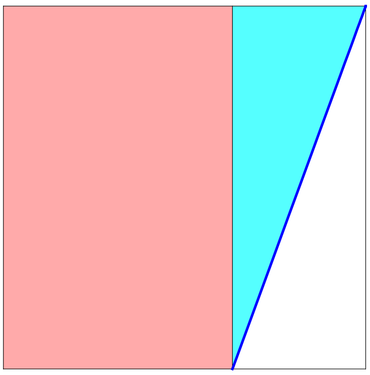

FIGURE 3.7. $[0,1]^2 \times \{1\}$.

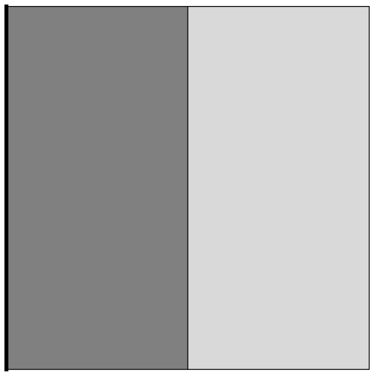

FIGURE 3.8. $[0,1]^2 \times \{0\}$.

(Figure 3.7), is somewhat similar to that of $g : [0,1]^2 \to [0,1]^2$ in Example 3.2 (Figure 3.5). The points in the blue region and white region of Figure 3.7 stay in the top layer by $(x, y, 1) \mapsto (g(x, y), 1)$. The points in the red region of Figure 3.7, on the other hand, are mapped to the bottom layer by $(x, y, 1) \mapsto (\phi(x), 0)$, via the space-filling curve, $\phi : [0, (e-1)/e] \to [0,1]^2$. Consequently, We get $X_1^+ = [0,1]^2 \times \{0\}$, the entire bottom layer, and $\mathcal{M}(X) = \{0\} \times [0,1] \times \{0\}$, the left hand side boundary edge of the bottom layer. $\square$

Note that we obtained Example 3.3 by replacing the vertical projection of Example 3.2 by the space-filling curve. Note also that we can also to put the gray layer X_1^+, beside the top layer and make the map 2-dimensional.

This time, let us construct an example where both $\mathcal{M}(X)$ and X_1^+ are polygons.

Example 3.4. Let $X = [0,1]^2 \times \{0,1\}$ as in Example 3.3 (Figure 3.7 and Figure 3.8). Let $g : [0,1]^2 \to [0,1]^2$ and $\phi : [0, (e-1)/e] \to [0,1]^2$ also be as in Example 3.3. Now, let $\psi : [0,1] \to [0,1/2] \times [0,1]$ be a space-filling curve. We define $f : X \to X$ as,

$$f(x,y,z) = \begin{cases} (x/2, y, 0), & \text{if } x > 0 \text{ and } z = 0, \\ (\psi(y), 0), & \text{if } x = 0 \text{ and } z = 0, \\ (\phi(x), 0), & \text{if } (x,y) \in [0, (e-1)/e] \times [0,1) \text{ and } z = 1, \\ (g(x,y), 1), & \text{otherwise.} \end{cases}$$

Then, $X_1^+ = [0,1]^2 \times \{0\}$ as before. However, $\mathcal{M}(X) = [0,1/2] \times [0,1] \times \{0\}$.

Proof. The only difference between the iterative dynamical system $f : X \to X$ of Example 3.4 and that of Example 3.3 is the use of another space-filling curve, $\psi : [0,1] \to [0,1/2] \times [0,1]$, to fill up left hand side half of the bottom layer (the dark gray rectangle in Figure 3.8). It is easy to see that this changes the maximal invariant set from the left hand side edge, $\{0\} \times [0,1] \times \{0\}$ to the left hand side rectangle, $[0,1/2] \times [0,1] \times \{0\}$, in Figure 3.8. □

Example 3.3 and Example 3.4 are notably better than the previous examples in that the two sets are not almost the same any more. Indeed, Example 3.3 satisfies

$$0 = \mu_2(\mathcal{M}(X)) < \mu_2(X_1^+) = 1,$$

while Example 3.4 has the property,

$$0 < \frac{1}{2} = \mu_2(\mathcal{M}(X)) < \mu_2(X_1^+) = 1$$

Example 3.4 is particularly strong in that $\mathcal{M}(X)$ is almost the same as neither X_1^+ nor $\emptyset$. We will not return to Example 3.4, however. The Example that we will use as the basic building block toward the main results of this paper is Example 3.3.

4. Constructing the chains of minimal image sets

Recall that all four examples discussed in the previous section have non-invariant first minimal image sets. Among them, we are particularly interested in Example 3.3. We will use it as the basic building block to construct chains of minimal image sets.

4.1. A finite chain of minimal image sets

In this subsection, we will construct a finite chain of minimal image sets by repeatedly applying the method we used in Example 3.3. As a consequence, we get a constructive proof of Main Theorem 3.

Example 4.1. Let $X = [0,1]^2 \times \{1, 1/2, \ldots, 1/n\} \subset [0,1]^3 \subset \mathbb{R}^3$, so that each layer, $[0,1] \times \{1/k\}$, is a unit square with an appropriate partition as illustrated in Figure 3.7. Let g and ϕ are as in Example 3.3. We define $f : X \to X$ as

(4.1)
$$f(x,y,z) = \begin{cases} (x,y,z), & \text{if } z = \frac{1}{n}, \\ (\phi(x), \frac{1}{k+1}), & \text{if } (x,y) \in [0,(e-1)/e] \times [0,1) \text{ and } z = \frac{1}{k},\ k < n, \\ (g(x,y), z), & \text{otherwise.} \end{cases}$$

Then we must have,

$$\begin{aligned} X_1^+ &= [0,1]^2 \times \{1/2, \ldots, 1/n\}, \\ X_2^+ &= [0,1]^2 \times \{1/3, \ldots, 1/n\}, \\ \cdots &= \cdots \\ X_n^+ &= [0,1]^2 \times \{1/n\} = \mathcal{M}(X), \end{aligned}$$

consequently yielding the descending chain (1.8) of Main Theorem 3.

Proof. The dynamics on the top $n-1$ layers have two parts. One of them, $(x,y,z)\mapsto(g(x,y),z)$ is confined to the same layer, and it is identical to the corresponding dynamics of Example 3.3 (Figure 3.7). The other part, $(x,y,\frac{1}{k})\mapsto(\phi(x),\frac{1}{k+1})$, *peels off the top layer* from $X^+_{k-1}=[0,1]^2\times\{1/k,\dots,1/n\}$. Repeating this argument, we get $X^+_1,\dots,X^+_n$ as claimed. On the bottom layer, f is the same as the identity map. Therefore, the maximal invariant set is the entire bottom layer.

Note that the map $f:X\to X$ we just developed here is a piecewise continuous map with a finite partition in a compact subset of $\mathbb{R}^3$. This completes the proof of Main Theorem 3. $\square$

4.2. Infinite chains of minimal image sets

Example 4.1 can be used to attain the chain of minimal image sets with arbitrary finite length. It is easy to modify it to get an infinite chain. All we have to do is to repeat the same process recursively.

Example 4.2. Let $X=[0,1]^2\times\{1,1/2,1/3,\dots;0\}\subset[0,1]^3\subset\mathbb{R}^3$. Let g and ϕ are as in Example 3.3. We define $f:X\to X$ as,
(4.2)

$$f(x,y,z)=\begin{cases}(x,y,0), & \text{if } z=0,\\ (\phi(x),\frac{1}{n+1}), & \text{if } z=\frac{1}{n} \text{ and } (x,y)\in[0,(e-1)/e]\times[0,1),\\ (g(x,y),z), & \text{otherwise.}\end{cases}$$

Then, we have $X^+_n=[0,1]^2\times\{\frac{1}{n+1},\frac{1}{n+2},\dots;0\}$ and $X^+_\infty=[0,1]^2\times\{0\}=\mathcal{M}(X)$. Consequently, the inequality (1.6) follows.

Proof. This is but a trivial generalization of Example 4.1. The detail is left to the readers. $\square$

Example 4.2 serves its purpose in a sense that we did get a chain of minimal image sets with infinite length. However, it does not help us to go beyond that. The problem with Example 4.2 is that the bottom layer, $[0,1]^2\times\{0\}$, is independent of the other layers as far as the dynamics of f is concerned. This difficulty and its implication will be clarified in the next example.

Example 4.3. Let $X=[0,1]^2\times\{1,1/2,\dots;0\}\subset[0,1]^3\subset\mathbb{R}^3$. Let $g:[0,1]^2\to[0,1]^2$ be the piecewise continuous map in Example 3.2. Now, let $\psi_1:[0,(e-1)/(2e)]\to[0,1]^2$ and $\psi_2:((e-1)/(2e),(e-1)/e]\to[0,1]^2$ be space-filling curves. We define $f:X\to X$ as

$$f(x,y,z)=\begin{cases}(x/2,y,0), & \text{if } z=0,\\ (\psi_2(x),0), & \text{if } (x,y)\in(\frac{e-1}{2e},\frac{1-e}{e}]\times[0,1) \text{ and } z=\frac{1}{n},\\ (\psi_1(x),\frac{1}{n+1}), & \text{if } (x,y)\in[0,\frac{e-1}{2e}]\times[0,1) \text{ and } z=\frac{1}{n},\\ (g(x,y),z), & \text{otherwise.}\end{cases}$$

Then, $X_n^+ = [0,1]^2 \times \{\frac{1}{n+1}, \frac{1}{n+2}, \ldots; 0\}$ and $X_\infty^+ = [0,1]^2 \times \{0\}$, but $\mathcal{M}(X) = \{0\} \times [0,1] \times \{0\}$. Hence, the inequality (1.6) does not hold. Note that we get the bottom layer, $[0,1]^2 \times \{0\}$ from ψ_2.

Proof. As far as the technical proof is concerned, this is another elementary generalization of the previous examples. It is tedious, but elementary. We leave the detail to the readers. □

The key improvement from Example 4.2 to Example 4.3 is the splitting of the space-filling curve. The part $(x, y, \frac{1}{n}) \mapsto (g(x), \frac{1}{n+1})$ peels off the top layer of the unit square from X_{n-1}^+, as in Example 4.1 and Example 4.2. The other part $(x, y, z) \mapsto (\psi(x), 0)$ fills up the bottom layer in order to ensure that the bottom layer $[0,1]^2 \times \{0\}$ is included in all minimal image sets, even though the bottom layer itself is not invariant. We will use this idea again when we prove Theorem 5.2 and in Theorem 5.3.

The key advantage of Example 4.3 is that it allows us to go beyond the first countably infinite ordinal, ω. In order to do this, we must revise the definition of the minimal image sets according to the ordinal numbers, using the transfinite induction and recursion. We will do this in the next section.

Finally, note that the resulting map $f : X \to X$ is no longer piecewise continuous with a finite partition. It is now a discontinuous map with a countably infinite partition, for both Example 4.2 and Example 4.3. At this moment, we do not know if we can accomplish the same result with a piecewise continuous map with a finite partition. From now on, therefore, we will abandon our effort to keep piecewise continuity with a finite partition[2]. We will implicitly accept a discontinuous map with an infinite partition.

5. Transfinite chains of minimal image sets

Now that we successfully established the examples of infinite chains of minimal image sets, let us turn our attention to get beyond X_∞^+. We will now define and investigate the chains of minimal image sets of arbitrary ordinal order.

5.1. Minimal image sets and maximal invariance order

Let us first revise the definition of the minimal image sets.

Definition 5.1. Let X be a set and $f : X \to X$. Given ordinal number, ξ, we define the *minimal image set of order* ξ as follows.

$$\begin{cases} X_0^+ = X, & \\ X_\xi^+ = \displaystyle\bigcap_{k=0}^{\infty} f^k(X_{\xi-1}^+), & \text{if } \xi \text{ is a successor ordinal,} \\ X_\xi^+ = \displaystyle\bigcap_{\eta<\xi} X_\eta^+, & \text{if } \xi \text{ is a limit ordinal.} \end{cases} \tag{5.1}$$

[2]In fact, we do not completely abandon it. In Theorem 5.3, we make a conscious effort to construct a piecewise continuous map, albeit it has infinite partition.

The well-definedness of Definition 5.1 follows immediately from the transfinite induction and recursion principle in axiomatic set theory. See [20] or [21], for instance, for more detail. Here, a successor ordinal means an ordinal with an immediate predecessor, or an ordinal ξ that can be written as $\xi = \eta + 1$. In contrast, a limit ordinal means that the ordinal that is not a successor ordinal [20, 21].

Now that the precise definition of minimal images sets is done, we will no longer use the ambiguous notation, X^+_∞, from now on. It will be replaced by X^+_ω. That is,

$$X^+_\omega = \bigcap_{n<\omega} X^+_n = \bigcap_{n\in\mathbb{Z}} X^+_n = \bigcap_{n=0}^{\infty} X^+_n.$$

Here is the list of some of the basic properties of minimal image sets of arbitrary ordinal order.

Lemma 5.1. *Let X be a set and $f : X \to X$. Let $\mathcal{M}(X)$ be the maximal invariant set and X^+_ξ be the minimal image set of order ξ. Then,*

(a) $X^+_\xi \supset \mathcal{M}(X)$.
(b) $X^+_\xi \supset f(X^+_\xi)$.
(c) $X^+_\xi \supset X^+_{\xi+1}$.
(d) $X^+_\xi = f(X^+_\xi)$ *if and only if* $X^+_\xi = \mathcal{M}(X)$.
(e) $X^+_\xi = X^+_{\xi+1}$ *if and only if* $X^+_\xi = \mathcal{M}(X)$.
(f) $X^+_\xi \supset X^+_\eta$ *if* $\xi < \eta$.
(g) *Let* $\xi < \eta$. *Then,* $X^+_\xi = X^+_\eta$ *if and only if* $X^+_\xi = \mathcal{M}(X)$.

Proof. (a) We use the transfinite induction. When $\xi = 0$, there is nothing to prove. If ξ is a successor ordinal, then

$$X^+_\xi = \bigcap_{k=0}^{\infty} f^k(X^+_{\xi-1}) \supset \bigcap_{k=0}^{\infty} f^k(\mathcal{M}(X)) = \bigcap_{k=0}^{\infty} \mathcal{M}(X) = \mathcal{M}(X).$$

If X^+_ξ is a limit ordinal, then $X^+\xi = \bigcap_{\eta<\xi} X^+_\eta \supset \bigcap_{\eta<\xi} \mathcal{M}(X) = \mathcal{M}(X)$. This proves the part (a).

(b) Again, there is nothing to prove when $\xi = 0$. When X^+_ξ is a successor ordinal, $X^+_\xi = \bigcap_{k=0}^\infty f^k(X^+_{\xi-1})$. That is, $x \in X^+_\xi$ means $x = x_0 = f(x_1) = f^2(x_2) = \cdots$, $x_k \in X^+_\xi$, $k \in \mathbb{W}$. Therefore, $f(x) = f(x_0) = f(f(x_1)) = f^2(f(x_2)) = \cdots$, $f(x_k) \in f(X^+_{\xi-1}) \subset X^+_\xi$. Hence, $f(x) \in X^+_\xi$ for all $x \in X^+_\xi$, or $X^+_\xi \supset f(X^+_\xi)$.

When ξ is a limit ordinal, $X^+_\xi = \bigcap_{\eta<\xi} X^+_\eta$. That is, $x \in X^+_\xi$ means $x = x_\eta \in X^+_\eta$ for all ordinal η such that $0 \le \eta < \xi$. Therefore, $f(x) = f(x_\eta) \subset f(X^+_\eta) = X^+_\eta$ for all $\eta < \xi$. Hence, $f(x) \in \bigcap_{\eta<\xi} X^+_\eta$, or $X^+_\eta \supset f(X^+_\eta)$.

(c) From (b), we get $X^+_\xi \supset f(X^+_\xi) \supset f^2(X^+_\xi) \supset \cdots \supset \bigcap_{k=0}^\infty f^k(X^+_\xi) = X^+_{\xi+1}$.

(d) $\mathcal{M}(X)$ is the union of all invariant subsets of X, so $f(X_\xi^+) = X_\xi^+$ implies $X_\xi^+ \subset \mathcal{M}(X)$, which gives rise to $X_\xi^+ = \mathcal{M}(X)$ from (a). On the other hand, if $X_\xi^+ = \mathcal{M}(X)$, then $X_\xi^+ = \mathcal{M}(X) = f(\mathcal{M}(X)) = f(X_\xi^+)$. This proves (d).

(e) If $X_{\xi+1}^+ = \bigcap_{k=0}^\infty f^k(X_\xi^+) = X_\xi^+$, then from (b) and (c) we get $X_\xi^+ \supset f(X_\xi^+) \supset \bigcap_{k=0}^\infty f^k(X_\xi^+) = X_\xi^+$. Thus, $f(X_\xi^+) = X_\xi^+$. From (d), therefore, we conclude $X_\xi^+ = \mathcal{M}(X)$. On the other hand, if $X_\xi^+ = \mathcal{M}(X)$, then $X_{\xi+1}^+ = \bigcap_{k=0}^\infty f^k(X_\xi^+) = \bigcap_{k=0}^\infty f^k(\mathcal{M}(X)) = \bigcap_{k=0}^\infty \mathcal{M}(X) = \mathcal{M}(X) = X_\xi^+$. This proves (e).

(f) If there is no limit ordinal between ξ and η, then $\eta = \xi + n$ for some $n \in \mathbb{N}$. Therefore, (f) follows from the repeated application of (c). Otherwise, suppose that ζ is the largest limit ordinal less than or equal to η, that is, $\eta = \zeta + k$, $k \in \mathbb{W}$. Then, by the third line of the equality (5.1) in Definition 5.1, $X_\xi^+ \supset X_\zeta^+$. Thus, $X_\xi^+ \supset X_\zeta^+ \supset X_{\zeta+k}^+ = X_\eta^+$. This proves (f).

(g) If $X_\xi^+ = X_\eta^+$, then from (c) we get $X_\xi^+ \supset X_{\xi+1}^+ \supset X_\eta^+ = X_\xi^+$. Therefore, $X_\xi^+ = X_{\xi+1}^+$, and from (e), we get $X_\xi^+ = \mathcal{M}(X)$. On the other hand, if $X_\xi^+ = \mathcal{M}(X)$, then from (a) and (f), $\mathcal{M}(X) = X_\xi^+ \supset X_\eta^+ \supset \mathcal{M}(X)$. Hence, $X_\eta = \mathcal{M}(X)$ for every $\eta > \xi$. This proves (g). $\square$

Lemma 5.1(f) allows us to form a transfinite chain of the minimal image sets. Note that Lemma 5.1(c) alone is not enough, because of the existence of the limit ordinals and the initial ordinals. However, it is important to note that every chain of minimal image sets *eventually* does reach the maximal invariant set, even though it could be *after uncountable steps later.* The following theorem explains this phenomenon.

Theorem 5.1. *Let X be a nonempty set and $f : X \to X$. Then, there is a unique ordinal number ξ such that*

$$X_0^+ \supsetneq X_1^+ \supsetneq \cdots \supsetneq X_\xi^+ = \mathcal{M}(X). \tag{5.2}$$

We call such ξ, the maximal invariance order.

Proof. Let α be an ordinal whose cardinality is bigger than that of the power set $\mathcal{P}(X)$ of X. That is, $|\alpha| > |\mathcal{P}(X)|$. Let $\mathcal{X} = \{X_\beta^+ : 0 \le \beta < \alpha\}$. Then, since every $X_\beta^+ \subset X$, we must have $\mathcal{X} \subset \mathcal{P}(X)$. Therefore, $|\mathcal{X}| \le |\mathcal{P}(X)| < |\alpha| = |\{\beta : 0 \le \beta < \alpha\}|$.

Therefore, the map $T : \{\beta : 0 \le \beta < \alpha\} \to \mathcal{X}$, $T : \beta \mapsto X_\beta^+$ is surjective but not bijective. Therefore, $\mathcal{X}_T = \{\beta : X_\beta^+ = X_\gamma^+ \text{ for some } \gamma,\ \beta < \gamma < \alpha\} \ne \emptyset$. Let ξ be the smallest element of $\mathcal{X}_T$. Then, because of Lemma 5.1(g), we must have $X_\xi^+ = \mathcal{M}(X)$.

Also, because ξ is the smallest of $\mathcal{X}_T$, each ordinal number less than ξ corresponds to distinct minimal image set. Hence, the inequality (5.2) follows. $\square$

The maximal invariance order measures the *length* of the chain of minimal image sets.

5.2. Arbitrary maximal invariance order

In the previous subsection, we proved that every self-map $f : X \to X$ has a unique ordinal ξ which measures the length of the chain of minimal image sets (Theorem 5.1). We wish to prove the converse as well. That is, given ordinal number ξ, can we find a self-map with the maximal invariance order ξ? The following theorem answers this question.

Theorem 5.2. *Given ordinal number ξ, we can construct a discontinuous map $f : X \to X$ in a set X, with the maximal invariance order ξ.*

Proof. Let $X = [0,1]^2 \times [0, \xi)$, where $[0, \xi) = \{\eta : \eta$ is an ordinal number such that $0 \le \eta < \xi\}$. Let ψ_1, ψ_2 and g be as in Example 4.3, and let $n(\eta)$ denote the least limit ordinal greater than η. We define the discontinuous map $f : X \to X$ as follows.

$$f(x, y, \eta) = \begin{cases} (\psi_2(x), n(\eta)), & \text{if } (x, y) \in (\frac{e-1}{2e}, \frac{e-1}{e}] \times [0, 1) \text{ and } n(\eta) < \xi, \\ (\psi_2(x), \eta + 1), & \text{if } (x, y) \in (\frac{e-1}{2e}, \frac{e-1}{e}] \times [0, 1) \text{ and } n(\eta) \ge \xi, \\ (\psi_1(x), \eta + 1), & \text{if } (x, y) \in [0, \frac{1-e}{2e}] \times [0, 1) \text{ and } \eta + 1 < \xi, \\ g(x, y, \eta), & \text{otherwise.} \end{cases}$$

This construction is analogous to that of Example 4.3. The part $(x, y, \eta) \mapsto (\psi_1(x), \eta+1)$ handles the successor ordinal, peeling off one layer of unit square away from X^+_η to get $X^+_{\eta+1}$.

To use the transfinite induction properly, we need to take care of the limit ordinals also, The part $(x, y, \eta) \mapsto (\psi_2(x), n(\eta))$ does the work, creating an ∞-to-one correspond to the next limit ordinal $n(\eta)$, so that we can go on with peeling off the layers again. If there is no next limit ordinal before ξ, we duplicate the *peeling-off* map discussed in the previous paragraph through $(x, y, \eta) \mapsto (\psi_2(x), \eta + 1)$.

Consequently, we get

$$\begin{aligned} X^+_k &= [0,1]^2 \times [k, \xi), \\ X^+_\omega &= [0,1]^2 \times [\omega, \xi), \\ X^+_{\omega+k} &= [0,1]^2 \times [\omega + k, \xi), \\ X^+_{n\omega} &= [0,1]^2 \times [n\omega, \xi), \\ \cdots &= \cdots . \end{aligned} \tag{5.3}$$

This continues transfinitely. Now, if ξ is a successor ordinal, by looking at the last layer of X, we get

$$X^+_{\xi-1} = [0,1]^2 \times \{\xi - 1\}, \qquad X^+_\xi = [0, (e-1)/e] \times \{0\} \times \{\xi\} = \mathcal{M}(X).$$

Note that the line segment $[0, (e-1)/e]$ in the expression of X^+_ξ comes from the first minimal image set of g.

If ξ is a limit ordinal, on the other hand,

$$X_\xi^+ = \bigcap_{k=0}^{\infty} X_{p(\xi)+k}^+ = \bigcap_{k=0}^{\infty} [0,1]^2 \times [p(\xi)+k, \xi) = \emptyset = \mathcal{M}(X).$$

Hence, the map $f : X \to X$ satisfies all the conditions of Theorem 5.2. $\square$

We are now ready to prove Main Theorem 2. The technical improvement from Theorem 5.2 is rather slight, but the implication is significant.

Theorem 5.3. *Given ordinal number ξ, we can construct a compact topological space X and a piecewise continuous map $f : X \to X$ with the maximal invariance order ξ, which is almost continuous with respect to the non-singular component of any strictly positive Borel measure on X.*

Proof. Let Z be a compact topological space such that $|Z| = |\xi|$, such a set must exist due to Tychnoff's Theorem. Let $\mathcal{F} : [0, \xi) \to Z$ be a bijection. Let $X = [0,1]^2 \times Z$. Now, let $g : [0,1]^2 \to [0,1]^2$ be the piecewise continuous map in Example 3.2 and let $\phi : [0, (e-1)/e] \to [0,1]^2$ be a space-filling curve. Finally, let $\psi_1 : [0, (e-1)/(2e)] \to [0,1]^2$ and $\psi_2 : ((e-1)/(2e)], (e-1)/e] \to [0,1]^2$ be space-filling curves, too.

We define $f : X \to X$ as,

$$f(x, y, \mathcal{F}(\eta)) = \begin{cases} (\phi(x), \mathcal{F}(\eta)), & \text{if } 0 \le x \le \frac{e-1}{e},\ y = 0 \text{ and } n(\eta) \ge \xi, \\ (\psi_2(x), \mathcal{F}(n(\eta))), & \text{if } \frac{e-1}{2e} < x \le \frac{e-1}{e},\ y = 0 \text{ and } n(\eta) < \xi, \\ (\psi_1(x), \mathcal{F}(\eta+1)), & \text{if } 0 \le x \le \frac{e-1}{2e},\ y = 0 \text{ and } \eta + 1 < \xi, \\ g(x, y, \mathcal{F}(\eta)), & \text{otherwise.} \end{cases}$$

Then, by similar argument as in Theorem 5.2, the maximal invariance order of f is ξ as before. The difference is only slight. Instead of (5.3), we get

$$X_\eta^+ = \{0\} \times [0, (e-1)/e] \times \{\eta - 1\} \cup [0,1]^2 \times [\eta, \xi).$$

Let us now turn our attention to the discontinuity. From the definition of f, we see that the discontinuity set of f follows from the discontinuity set of g on each layer, because the change of z-coordinate makes no difference unless $y = 0$. That is, the discontinuity set of f is $\{(x, y, z) : g \text{ is discontinuous at } (x, y)\}$. The discontinuity set of g in a typical layer and the discontinuity set of f are illustrated in Figure 5.1 and Figure 5.2, respectively.

Even though f has an infinite partition, only four sets in the partition have non-empty interior (Figure 5.2). The other pieces of the partition belong to the boundary surfaces of the four visible sets. Hence, the discontinuity set of f is the union of the boundary surfaces of the four dominant regions (Figure 5.2), and therefore, it must have measure zero with respect to the non-singular component of any strictly positive Borel measure of X. $\square$

Note that Theorem 5.3 remains true for $\xi = \omega$ as well. In this case, we can take $Z = \{1, 1/2, 1/3, \ldots ; 0\}$ as in Example 4.3, and define $F(0) = 0$,

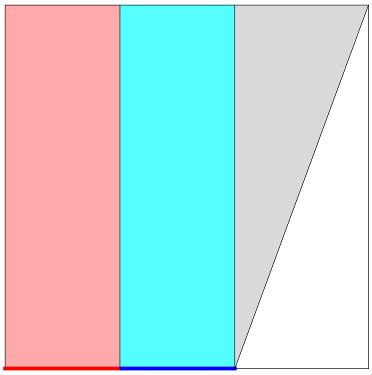

FIGURE 5.1.

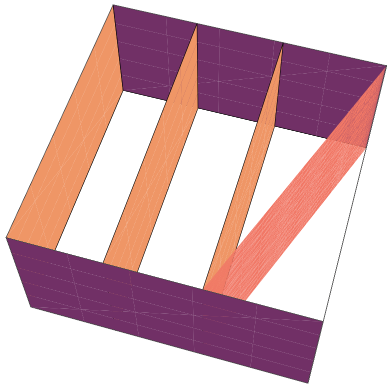

FIGURE 5.2.

$F(n) = 1/n$ for $n \in \mathbb{N}$. This case can be visualized by Figure 5.1 and Figure 5.2, too, except that there are only countably many layers.

References

[1] R. Adler, B. Kitchens, and C. Tresser, *Dynamics of non-ergodic piecewise affine maps of the torus*, Ergo. The. & Dynam. Sys. **21** (2001), 959–999.

[2] E. Akin, *The general topology of dynamical systems*, Graduate Studies in Mathematics, 1, American Mathematical Society, Providence, RI, 1993. `https://doi.org/10.1090/gsm/001`

[3] Z. Artstein and S. V. Raković, *Feedback and invariance under uncertainty via set-iterates*, Automatica J. IFAC **44** (2008), no. 2, 520–525. `https://doi.org/10.1016/j.automatica.2007.06.013`

[4] P. Ashwin, *Non-smooth invariant circles in digital overflow oscillations*, in Proceedings of the 4th International Workshop on Nonlinear Dynamics and Electronic Systems, pp. 417–422, Seville, Spain, 1996.

[5] P. Ashwin, *Elliptic behaviour in the sawtooth standard map*, Phys. Lett. A **232** (1997), no. 6, 409–416. `https://doi.org/10.1016/S0375-9601(97)00455-6`

[6] P. Ashwin, W. Chambers, and G. Petrov, *Lossless digital filter overflow oscillations: approximation of invariant fractals*, Int. J. Bifur. Chaos Appl. Sci. Eng. **7** (1997), 2603–2610.

[7] P. Ashwin, J. H. B. Deane, and X. Fu, *Dynamics of a bandpass sigma-delta modulator as a piecewise isometry*, in Proc. IEEE Int. Symp. Circuit Sys., pp. 811–814, Sydney, Australia, 2001.

[8] P. Ashwin, X. Fu, T. Nishikawa, and K. Życzkowski, *Invariant sets for discontinuous parabolic area-preserving torus maps*, Nonlinearity **13** (2000), no. 3, 819–835. `https://doi.org/10.1088/0951-7715/13/3/317`

[9] F. Blanchini, *Set invariance in control*, Automatica J. IFAC **35** (1999), no. 11, 1747–1767. `https://doi.org/10.1016/S0005-1098(99)00113-2`

[10] J. Buzzi, *Piecewise isometries have zero entropy*, Ergo. The. & Dynam. Sys. **21** (2001), 1371–1377.

[11] J. Buzzi and P. Hubert, *Piecewise monotone maps without periodic points: rigidity, measures and complexity*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 383–405. `https://doi.org/10.1017/S0143385703000488`

[12] A. C. Davis, *Nonlinear oscillations and chaos from digital filter overflow*, Phil. Trans. Royal Soc. Lond. **A-353** (1995), 85–99.

[13] X.-C. Fu, F.-Y. Chen, and X.-H. Zhao, *Dynamical properties of 2-torus parabolic maps*, Nonlinear Dynam. **50** (2007), no. 3, 539–549. https://doi.org/10.1007/s11071-006-9179-9

[14] X.-C. Fu and J. Duan, *Global attractors and invariant measures for non-invertible planar piecewise isometric maps*, Phys. Lett. A **371** (2007), no. 4, 285–290. https://doi.org/10.1016/j.physleta.2007.06.033

[15] X.-C. Fu and J. Duan, *On global attractors for a class of nonhyperbolic piecewise affine maps*, Phys. D **237** (2008), no. 24, 3369–3376. https://doi.org/10.1016/j.physd.2008.07.012

[16] A. Goetz, *Dynamics of a piecewise rotation*, Discrete Contin. Dynam. Systems **4** (1998), no. 4, 593–608. https://doi.org/10.3934/dcds.1998.4.593

[17] A. Goetz, *Sofic subshifts and piecewise isometric systems*, Ergo. The. & Dynam. Sys. **19** (1999), 1485–1501.

[18] A. Goetz and M. Mendes, *Piecewise rotations*: *bifurcations, attractors and symmetries*, in Bifurcation, symmetry and patterns (Porto, 2000), 157–165, Trends Math, Birkhäuser, Basel, 2003.

[19] A. Goetz and G. Poggiaspalla, *Rotations by* $\pi/7$, Nonlinearity **17** (2004), no. 5, 1787–1802. https://doi.org/10.1088/0951-7715/17/5/013

[20] K. Hrbáček and T. Jech, *Introduction to Set Theory*, second edition, Monographs and Textbooks in Pure and Applied Mathematics, 85, Marcel Dekker, Inc., New York, 1984.

[21] T. Jech, *Set theory*, the third millennium edition, revised and expanded., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[22] B. Kahng, *Dynamics of symplectic piecewise affine elliptic rotations maps on tori*, Ergod. Th. & Dynam. Sys. **22** (2002), 483–505.

[23] B. Kahng, *Dynamics of kaleidoscopic maps*, Adv. Math. **185** (2004), no. 1, 178–205. https://doi.org/10.1016/S0001-8708(03)00170-1

[24] B. Kahng, *The unique ergodic measure of the symmetric piecewise toral isometry of rotation angle* $\theta = k\pi/5$ *is the Hausdorff measure of its singular set*, Dyn. Syst. **19** (2004), no. 3, 245–264. https://doi.org/10.1080/14689360410001729595

[25] B. Kahng, *Maximal invariant sets of multiple valued iterative dynamics in disturbed control systems*, Int. J. Circ. Sys. Signal Proc. **2** (2008), 113–120.

[26] B. Kahng, *Positive invariance of multiple valued iterative dynamical systems in disturbed control models*, in Proceedings of the 17th IEEE Mediterranean Conference on Control and Automation, pp. 664–668, Thessaloniki, Greece, 2009.

[27] B. Kahng, *Redifining chaos: Devaney-chaos for piecewise isometric dynamical systems*, International J. Math. Models and Methods in Applied Sciences **4** (2009), 317–326.

[28] B. Kahng, *Singularities of 2-dimensional invertible piecewise isometric dynamics*, Chaos **19** (2009), 023115.

[29] B. Kahng, *The approximate control problems of the maximal invariant sets of non-linear discrete-time disturbed control dynamical systems*: *an algorithmic approach*, in Proceedings of the 4th International Conference on Control, Automation and Systems, pp. 1513–1518, KINTEX, Gyeonggi-do, Korea, 2010.

[30] B. Kahng, *Multiple valued iterative dynamics models of nonlinear discrete-time control dynamical systems with disturbance*, J. Korean Math. Soc. **50** (2013), no. 1, 17–39. https://doi.org/10.4134/JKMS.2013.50.1.017

[31] B. Kahng, *An optimization of maximal invariance in a class of multiple valued iterative dynamics models of nonlinear disturbed control systems*, Fractals **24** (2016), 1650044.

[32] B. Kahng, *Theory and implementation of the multiple valued iterative dynamics algorithms for a class of singularly disturbed nonlinear control dynamical systems*, Appl. Sci. **1** (2019), 1:1061.

[33] B. Kahng and J. Davis, *Maximal dimensions of uniform Sierpinski fractals*, Fractals **18** (2010), no. 4, 451–460. https://doi.org/10.1142/S0218348X10005135

[34] B. Kahng, M. Gomez, and E. Padilla, *Visualization algorithms for the steady state sets of a class of singularly disturbed nonlinear control dynamical systems*, Int. J. Math. Models Methods **10** (2016), 237–243.

[35] B. Kahng and M. Mendes, *The characterization of maximal invariant sets of non-linear discrete-time control dynamical systems*, Discrete Contin. Dyn. Syst. 2013, Dynamical systems, differential equations and applications. 9th AIMS Conference. Suppl., 393–406, 2013. https://doi.org/10.3934/proc.2013.2013.393

[36] E. Kerrigan, J. Lygeros, and J. M. Maciejowski, *A geometric approach to reachability computations for constrained discrete-time systems*, in IFAC World Congress, Barcelona, Spain, 2002.

[37] E. Kerrigan and J. M. Maciejowski, *Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control*, in Proc. 39th IEEE Conf. on Decision and Control, Sydney, Australia, 2000.

[38] L. Kocarev, C. W. Wu, and L. O. Chua, *Complex behavior in digital filters with overflow nonlinearity: analytical results*, IEEE Trans. Circuits Systems II **43** (1996), 234–246.

[39] T. Lin and L. O. Chua, *A new class of pseudo-random generators based on chaos in digital filters*, Int. J. Cir. Th. Appl., **21** (1993), 473–480.

[40] J. H. Lowenstein, *Fixed-point densities for a quasiperiodic kicked oscillator map*, Chaos **5** (1995), 566–577.

[41] J. H. Lowenstein, *Aperiodic orbits of piecewise rational rotations of convex polygons with recursive tiling*, Dyn. Syst. **22** (2007), no. 1, 25–63. https://doi.org/10.1080/14689360601028100

[42] J. Lowenstein, S. Hatjispyros, and F. Vivaldi, *Quasi-periodicity, global stability and scaling in a model of Hamiltonian round-off*, Chaos **7** (1997), no. 1, 49–66. https://doi.org/10.1063/1.166240

[43] J. H. Lowenstein, K. L. Kouptsov, and F. Vivaldi, *Recursive tiling and geometry of piecewise rotations by* $\pi/7$, Nonlinearity **17** (2004), no. 2, 371–395. https://doi.org/10.1088/0951-7715/17/2/001

[44] J. H. Lowenstein, G. Poggiaspalla, and F. Vivaldi, *Sticky orbits in a kicked-oscillator model*, Dyn. Syst. **20** (2005), no. 4, 413–451. https://doi.org/10.1080/14689360500167611

[45] D. Q. Mayne, M. M. Seron, and S. V. Raković, *Robust model predictive control of constrained linear systems with bounded disturbances*, Automatica J. IFAC **41** (2005), no. 2, 219–224. https://doi.org/10.1016/j.automatica.2004.08.019

[46] M. Mendes, *Dynamics of piecewise isometric systems with particular emphasis to the g map*, Ph. D. Thesis, University of Surrey, 2001.

[47] M. Mendes, *On maximal invariant sets*, preprint, (August, 2006).

[48] M. J. Ogorzałek, *Chaos and complexity in nonlinear electronic circuits*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 22, World Scientific Publishing Co. Pte. Ltd., Singapore, 1997. https://doi.org/10.1142/9789812798626

[49] C. J. Ong and E. G. Gilbert, *Constrained linear systems with disturbances*: *enlargement of their maximal invariant sets by nonlinear feedback*, in Proc. Amer. Control Conf. Minneapolis, pp. 5246–5251, MN, 2006.

[50] M. Pollicott and H. Weiss, *The dimensions of some self-affine limit sets in the plane and hyperbolic sets*, J. Statist. Phys. **77** (1994), no. 3-4, 841–866. https://doi.org/10.1007/BF02179463

[51] S. V. Rakovic and M. Fiacchini, *Invariant Approximations of the Maximal Invariant Set or "Encircling the Square*, in IFAC World Congress, Seoul, Korea, July 2008.

[52] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, *Invariant approximation of the minimal robust positively invariant set*, IEEE Trans. Automat. Control **50** (2005), no. 3, 406–410. https://doi.org/10.1109/TAC.2005.843854

[53] S. V. Raković, E. C. Kerrigan, D. Q. Mayne, and K. I. Kouramas, *Optimized robust control invariance for linear discrete-time systems: theoretical foundations*, Automatica J. IFAC **43** (2007), no. 5, 831–841. https://doi.org/10.1016/j.automatica.2006.11.006

[54] S. V. Raković, E. C. Kerrigan, D. Q. Mayne, and J. Lygeros, *Reachability analysis of discrete-time systems with disturbances*, IEEE Trans. Automat. Control **51** (2006), no. 4, 546–561. https://doi.org/10.1109/TAC.2006.872835

[55] M. Trovati and P. Ashwin, *Tangency properties of a pentagonal tiling generated by a piecewise isometry*, Chaos **17** (2007), no. 4, 043129, 11 pp. https://doi.org/10.1063/1.2825291

Byungik Kahng
Department of Mathematics and Information Sciences
University of North Texas at Dallas
7400 University Hills Boulevard
Dallas, TX 75241, USA
Email address: byungik.kahng@untdallas.edu